International Journal of Difference Equations (IJDE).
ISSN 0973-6069 Volume 18, Number 1 (2023), pp. 11-34

https://www.ripublication.com/ijde.htm

Generalized Fractional Cosine Family

Abdelmjid Benmerrous*[1], Lalla saadia Chadli †[1], Abdelaziz Moujahid ‡[1], M'hamed Elomari §[1] and Said Melliani ¶[1]

[1]*Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, PO Box 532,23000 Beni Mellal, Morocco.*

Abstract

This paper, deal with the following problem

$$\begin{cases} D^{\alpha}u(x,t) = Au(x,t) + F\left(t, u(x,t)\right), \quad t \in [0,T] \\ u(x,0) = a_0(x) \\ u'(x,0) = b_0(x) \end{cases}$$

Where a_0 and b_0 are singular generalized functions, D^{α} is Caputo derivative of order α, $1 < \alpha < 2$ and A is an operator defined from the Colombeau algebra into itself, F satisfies L^{∞} logarithmic type. The outcome of the following problem is studied in two cases this research. In the typical example, we provided the integral solution and demonstrated existence and uniqueness using Banach's fixed point theorem. In Colombeau's algebra, the Gronwall lemma is used to prove the existence and uniqueness of the results.
keywords: Fractional cosine family, Mettg-Leffler function, Caputo fractional derivative, Laplace transform
MSC Classification: 46F05; 35D05; 35G29; 35G25; 35Q55

1. INTRODUCTION

The fractional operator is very important and it often appears in Analysis [9]. As an illustration, it is used to simulate natural phenomena that depict the dynamics of specific phenomena that ordinary phenomena cannot. They provide an alternative to nonlinear ordinary differential models by having solutions that are independent of any ordinary differential equation [17]. In recent years many researchers have focused on the study of phenomena whose modeling is given by nonlinear differential equations with a singularity, To do this, it is necessary to define the multiplication of two distributions in a manner that is consistent with the standard multiplication. The thing that led us automatically to do the study in Colombeau's algebra. This algebra which is

*Corresponding author :abdelmajid.benmerrous@usms.ma
†:chadli@fstbm.ac.ma
‡:moujahid.aziz.usms@gmail.com
§:m.elomari@usms.ma
¶:melliani@fstbm.ac.ma

commutative, associative, differential in which we can imbed the space of distributions so that the product of the infinitely differentiable functions and the usual derivative are respected [11]. In [14] the author discusses a method for dealing with fractional derivatives including singularities that is based on Colombeau's theory of algebras of generalized functions. Travis and Webb can consolidate and simplify some knowledge pertaining to the theory of strongly continuous cosine families of linear operators in Banach spaces based on the work of H. Fattorini [5]. They examined the outcome of an abstract semi-linear second order initial value problem in [16]:

$$\frac{d^2}{dt^2}W(t) = AW(t) + f\left(t, W(t), \frac{d}{dt}W(t)\right), W(t_0) = x, \quad \frac{d}{dt}W(t_0) = y.$$

$y, x \in X$.
Where W is a mapping from $\mathbb{R} \longrightarrow X$, X is a Banach space, f is a nonlinear mapping from $\mathbb{R} \times \mathbb{R} \times X \longrightarrow X$ and A is the infinitesimal generator of a strongly continuous cosine family of linear operators in X. In the first time A. Benmerrous and al [1] were able to give themselves the notion of the cosine family in the algebra of Colombeau, and from this they open a great way to study the semi-linear equations of the second order, In their paper they deal with the following abstract problem, taking the initial values as generalized functions.

$$\begin{cases} \frac{d^2}{dt^2}u(t,x) + Au(t,x) = F\left(t, u(t,x)\right), \\ u(0,x) = a_0(x) \\ \frac{\partial}{\partial t}u(0,x) = b_0(x) \end{cases}$$

Where A is an operator defined $\mathcal{G} \longrightarrow \mathcal{G}$, a_0 and b_0 are singular generalized functions and F satisfies L^∞ logarithmic type . Then they studied the associations for this abstract problem.

Our objective is to enlarge this last work in the situation of the Caputo's fractional derivative with the exponent between 1 and 2. But before discussing the notion of generalized α-cosine family, we will explain this notion in the ordinary case. We will explain the α-cosine family in case of a given linear operator and in the ordinary case, we will demonstrate the existence and uniqueness of the following problem

$$\begin{cases} D^\alpha x(t) - Ax(t) = f\left(t, x(t)\right), \quad t \in [0,T] \\ x(0) = x_0 \\ x'(0) = x'_0 \end{cases}$$

Where D^α is Caputo derivative of order α, $1 < \alpha < 2$, and A is a linear operator. Our second main goal is to study the same problem but this time in the general framework:

$$\begin{cases} \widetilde{D}^\alpha u(x,t) - Au(x,t) = F\left(t, u(x,t)\right), \quad t \in [0,T] \\ x(0,x) = a_0(x) \\ x'(0) = b_0(x) \end{cases}$$

Where A is an operator defined from $\mathcal{G} \longrightarrow \mathcal{G}$, a_0 and b_0 are singular generalized functions and F satisfies L^∞ logarithmic type.

The paper is structured as follows, in Section 2, we mention some notions of Colombeau's algebra, in section 3, we clarify the expression of generalized α-cosine family, and some results of this new concept, and we prove the existence and uniqueness of the abstract Cauchy problem, in section 4, we presented the definition of the generalized α-cosine family, we demonstrated the existence and the uniqueness of the solution of a cauchy problem.

2. PRELIMINARIES

In the following section, we will discuss the Colombeau generalized function (see also [1–3]).

Definition 1. *$\mathcal{A}_0(\mathbb{R}^n)$ is a set of functions ϕ in $C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\phi(t)dt = 1$. For $q \in \mathbb{N}, \mathcal{A}_q(\mathbb{R}^n) = \left\{\phi \in \mathcal{A}_0 : \int_{\mathbb{R}^n} t^i\phi(t)dt = 0, 0 < |i| \leq q\right\}$, where $t^i = t_1^{i_1}\cdots t_n^{i_n}$.*

In [3] sets

$$\overline{\mathcal{A}}_q(\mathbb{R}^n) = \{\phi(x_1, \dots x_n) = \phi(x_1)\cdots\phi(x_n) : \phi(x_i) \in \mathcal{A}_q(\mathbb{R})\},$$

are used because of applications to initial value problems. We shall follow the Colombeau original definition.

Obviously, if $\phi \in \mathcal{A}_q, q \in \mathbb{N}_0$, then for every $\varepsilon > 0, \phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n$, belongs to $\mathcal{A}_q$. If $\phi \in \mathcal{A}_0$, then its support number $d(\phi)$ is defined by

$$d(\phi) = \sup\{|x| : \phi(x) \neq 0\}.$$

Denote by $\mathcal{E}(\Omega)$ the set of functions

$$R : \mathcal{A}_0 \times \Omega \to \mathbb{C}, (\phi, x) \mapsto R(\phi, x),$$

which are in $C^\infty(\Omega)$ for every fixed ϕ. In the other words elements of $\mathcal{E}(\Omega)$ are functions $R : \mathcal{A}_0 \to C^\infty(\Omega)$. Note that for any $f \in C^\infty(\Omega)$, the mapping

$$(\phi, x) \mapsto f(x), (\phi, x) \in \mathcal{A}_0 \times \Omega,$$

defines an element in $\mathcal{E}(\Omega)$ which does not depend on ϕ. Conversely, if an element F in $\mathcal{E}(\Omega)$ does not depend on $\phi \in \mathcal{A}_0$ i.e.

$$F(\phi, x) = F(\psi, x), x \in \Omega, \text{ for every } \phi, \psi \in \mathcal{A}_0,$$

then it defines a function $f \in C^\infty(\Omega)$,

$$f(x) = F(\phi, x), x \in \Omega, \text{ for every } \phi \in \mathcal{A}_0.$$

In this sense, we identify $C^{\infty}(\Omega)$ with the corresponding subspace of $\mathcal{E}(\Omega)$.

Definition 2. *An element $R \in \mathcal{E}(\Omega)$ is moderate if for every $K \subset\subset \Omega$ and $\alpha \in \mathbb{N}_0^n$ there exists $N \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there exist $\eta > 0$ and $C > 0$ such that*

$$\|\partial^{\alpha} R\left(\phi_{\varepsilon}, x\right)\| \leq C\varepsilon^{-N}, x \in K, 0 < \varepsilon < \eta.$$

The set of all moderate elements is denoted by $\mathcal{E}_M(\Omega)$.

Definition 3. *An element $R \in \mathcal{E}_0(\mathbb{C})$ is moderate if there exists $N \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there exist $\eta > 0$ and $C > 0$ such that*

$$\|R\left(\phi_{\varepsilon}\right)\| < C\varepsilon^{-N}, 0 < \varepsilon < \eta.$$

The space of moderate elements is denoted by $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$).

Remark 1. *We shall also use the following equivalent definition. $R \in \mathcal{E}(\Omega)$ is moderate if for every $m \in \mathbb{N}$ there exist $N \in \mathbb{N}_0$ and $b \in \mathbb{R}$ such that for every $\phi \in \mathcal{A}_N$ where, for instance,*

$$\mu_m\left(R\left(\phi_{\varepsilon}, x\right)\right) = \mathcal{O}\left(\varepsilon^{b}\right), \varepsilon \to 0$$

$$\mu_m(f) = \sup_{\Omega_m} \sup_{\|\alpha\| \leq m} \|\partial f(x)\|.$$

The space $\mathcal{E}_0$ is similarly defined by using the absolute value instead of μ_m. Clearly $\mathcal{E}_M(\Omega)$ and $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$) are associative subalgebras of $\mathcal{E}(\Omega)$ and $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$).

Denote by Γ the set of sequences $\{a_q\}$ of positive numbers which strictly increase to infinity.

Definition 4. *An element $R \in \mathcal{E}_M(\Omega)$ is called null if for every $K \subset\subset \Omega$ and every $\alpha \in \mathbb{N}_0^n$ there exist $N \in \mathbb{N}_0$ and $\{a_q\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$ there exist $\eta > 0$ and $C > 0$ such that*

$$\|\partial^{\alpha} R\left(\phi_{\varepsilon}, x\right)\| \leq C\varepsilon^{a_q - N}, x \in K, 0 < \varepsilon < \eta.$$

The space of null elements is denoted by $\mathcal{N}(\Omega)$.

Definition 5. *The spaces of generalized functions $\mathcal{G}(\Omega)$ defined by*

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$$

The following description explains what the term "association" means in $\mathcal{G}(\mathbb{R})$.

Definition 6. *[2] Let $f, g \in \mathcal{G}(\mathbb{R})$.*
We said that f,g are associated, denoted $f \approx g$, if for all representative $f(\varphi_{\varepsilon}, x)$ and $g(\varphi_{\varepsilon}, x)$ and arbitrary $\xi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have:

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \|f(\varphi_{\varepsilon}, x) - g(\varphi_{\varepsilon}, x)\| \xi(x) dx = 0$$

Fractional Derivatives in Colombeau Algebra

In this section we will present the various definitions and features that we will need in the following.

A fractional integral is defined by: [12]

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad \alpha > 0$$

The fractional derivative of order $\alpha > 0$ in the Caputo sense is defined by: [12]

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau) d\tau}{(t-\tau)^{\alpha+1-m}} \quad , \quad m-1 < \alpha < m$$

Let (f_ε) be a representative of $F \in \mathcal{G}([0,+\infty[)$, then:

$$\begin{aligned}
D^{\alpha} f_{\varepsilon}(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau \quad 0 < \alpha < 1 \\
\sup_{t \in [0,T]} \|D^{\alpha} f_{\varepsilon}(t)\| &\leq \frac{1}{\Gamma(1-\alpha)} \sup_{t \in [0,T]} \| \int_0^t \frac{f'(\tau) d\tau}{(t-\tau)^{\alpha}} \| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \|f'\|_{L^{\infty}([0,T])} \sup_{t \in [0,T]} \int_0^t \frac{d\tau}{(t-\tau)^{\alpha}} d\tau \\
&\leq \frac{1}{\Gamma(1-\alpha)} \varepsilon^{-N} \frac{T^{1-\alpha}}{1-\alpha} \\
&\leq C_{\alpha,T} \varepsilon^{-N}
\end{aligned}$$

In general [14], for $m-1 < \alpha < m$

$$\begin{aligned}
\sup_{t \in [0,T]} \|D^{\alpha} f_{\varepsilon}(t)\| &\leq \frac{1}{\Gamma(m-\alpha)} \sup_{t \in [0,T]} \int_0^t \frac{\|f^{(m)}(\tau)\|}{(t-\tau)^{\alpha+1-m}} d\tau \\
&\leq \frac{1}{\Gamma(m-\alpha)} \left\|f^{(m)}\right\|_{L^{\infty}([0,T])} \sup_{t \in [0,T]} \int_0^t \frac{1}{(t-\tau)^{\alpha+1-m}} d\tau \\
&\leq \frac{1}{\Gamma(m-\alpha)} \varepsilon^{-N} \frac{T^{m-\alpha}}{m-\alpha} \\
&\leq C_{\alpha,T} \varepsilon^{-N}
\end{aligned}$$

The constant $C_{\alpha,T}$ depends on two parameters α and T.

Proposition 1. *[14] Let $(\omega_\epsilon(t))_\epsilon$ be a representative of ω(t) $\in \mathcal{G}([0,+\infty))$. The* *regularized Caputo αth fractional derivative of $(\omega_\epsilon(t))_\epsilon$, $\alpha > 0$, is defined by*

$$i_{frac} : \begin{cases} \mathcal{G}([0,+\infty)) \to \mathcal{G}([0,+\infty)) \\ \omega \to \left[\, \left(\widetilde{D^\alpha}\omega_\epsilon\right)_{\epsilon>0} \right] = \left[\, (D^\alpha \omega * \varphi_\epsilon)_{\epsilon>0} \right] \end{cases}$$

Proposition 2.

$$\left(\left(\widetilde{D^\alpha}\omega_\epsilon\right)_{\epsilon>0} \right) \approx \left((D^\alpha \omega_\epsilon)_{\epsilon>0} \right)$$

Proof. Let: $u_\varepsilon \in G([0,+\infty))$.

We have,

$$\begin{aligned} \|\tilde{D}^\alpha u_\varepsilon(t)\| &= \|D^\alpha u_\varepsilon * \varphi_\varepsilon(t)\| \\ &= \|\frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u_\varepsilon^{(2)}(\tau)d\tau}{(t-\tau)^{\alpha-1}} * \varphi_\varepsilon(t)\| \\ &\leqslant \|\frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{U_\varepsilon^{(2)}(\tau)}{(t-\tau)^\alpha - 1} d\tau\| \times \|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \\ &\leqslant \|D^\alpha U_\varepsilon(t)\| \times \|\varphi_\varepsilon\| L^\infty(\mathbb{R}^n) \end{aligned}$$

Then:

$$\begin{aligned} \|\widetilde{D^\alpha}U_\varepsilon(t) - D^\alpha U_\varepsilon(t)\| \leqslant & \|D^\alpha U_\varepsilon(t)\| \left(\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} - 1\right) \\ \leqslant & \frac{1}{\Gamma(2-\alpha)} \sup_{t\in[0,T]} \| \int_0^t \frac{u_\varepsilon^{(2)}(\tau)}{(t-\tau)^{\alpha-1}} d\tau\| \times \left(\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} - 1\right) \end{aligned}$$

$\leqslant \frac{1}{\Gamma(2-\alpha)} \sup_{t\in[0,T]} \|u_\varepsilon^{(2)}(t)\| \times \frac{T^{2-\alpha}}{2-\alpha} \times \left(\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} - 1\right)$
$\leqslant C_{T,\alpha}\varepsilon^{2-\alpha} \overset{\varepsilon\longrightarrow 0}{\longrightarrow} 0.$
So,

$$\widetilde{D}^\alpha U_\varepsilon \approx D^\alpha U_\varepsilon$$

□

Remark 2. *By this last property and in the rest of the paper we can write $D^\alpha u$ as a generalized function where u is a generalized function.*

3. FRACTIONAL COSINE FAMILY

Let (X,$\|.\|$) be a Banach space, and $T \geq 0$, $J = [0,T]$. Denote $\mathcal{C}(J,X)$ be the space of continuous functions from $J \longrightarrow X$ with the norm $\|x\|_\infty = \sup_{t\in J} |x(t)|$.

Consider the following problem :

$$\begin{cases} D^{\alpha}x(t) - Ax(t) = f\left(t, x(t)\right), \quad t \in [0, T] \\ x(0) = x_0 \\ x'(0) = x_0' \end{cases} \tag{1}$$

Where D^{α} is Caputo derivative of order α, $1 < \alpha \leq 2$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\left(T(t)\right)_{t\geq 0}$ in Banach space X, $x_0, x_0' \in X$ and $f : \mathcal{C}(J, X) \to \mathcal{C}(J, X)$ is a given (non-linear) operator.

3.1. Definition of integral solution

Definition 7. *[6] Define the Mittag-Leffler function by:*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}.$$

The Laplace transform plays a crucial role in solving the differential equations.

Definition 8. *[3] Define the Laplace transform of a function f by*

$$\mathcal{L}(f(t))(s) = \int_0^{+\infty} e^{-st} f(t) dt.$$

Proposition 3. *[3] Let f and g two functions, we have*

$$\mathcal{L}\left((f * g)(t)\right)(s) = \mathcal{L}\left(f(t)\right)(s)\mathcal{L}\left(g(t)\right)(s).$$

Definition 9. *[6]*

1. *The Gamma function is given by*

$$\forall x > 0, \quad \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

2. *The $\mathbb{B}$ function is defined by*

$$\forall x, y > 0, \quad \mathbb{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}.$$

Proposition 4. *[6] We have*

1. *For all $x, y \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\mathbb{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.*
2. *For all $x > 0$, $\Gamma(x + 1) = x\Gamma(x)$.*

It is easy to show the following proposition.

Proposition 5. *For all $\alpha > 0$, we get the following result*

$$\int_0^t E_{\alpha,1}(s^\alpha A)ds = tE_{\alpha,2}(t^\alpha A).$$

Proof.

$$\begin{aligned}\int_0^t E_{\alpha,1}(s^\alpha A)ds &= \int_0^t \sum_{n=0}^{\infty} \frac{s^{n\alpha}}{\Gamma(n\alpha+1)} A^n ds \\ &= \sum_{n=0}^{\infty} \frac{\int_0^t s^{n\alpha} ds}{\Gamma(n\alpha+1)} A^n \\ &= \sum_{n=0}^{\infty} \frac{t^{n\alpha+1}}{(n\alpha+1)\Gamma(n\alpha+1)} A^n \\ &= \sum_{n=0}^{\infty} \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} A^n \\ &= tE_{\alpha,2}\left(t^\alpha A\right).\end{aligned}$$

□

Lemma 1. *Let $f : \mathcal{C}(J,X) \to \mathcal{C}(J,X)$ be continuous.*

The problem 3.2 is equal to the integral equation

$$x(t) = x_0 + tx_0' + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[Ax(s) + f\left(s, x(s)\right)\right] ds, \quad t \in J, \tag{2}$$

$x : D(A) \longrightarrow D(A)$ provided that the integral in 2 exists.

We will need the following lemma.

Lemma 2. *For all $\alpha \in [1,2]$ and $s > 0$,*

1. $s^{\alpha-1}\left(s^\alpha - A\right)^{-1} = \mathcal{L}\big(E_{\alpha,1}\left(At^\alpha\right)\big)(s)$,

2. $s^{\alpha-2}\left(s^\alpha - A\right)^{-1} = \mathcal{L}\big(tE_{\alpha,2}\left(At^\alpha\right)\big)(s)$,

3. $\left(s^\alpha - A\right)^{-1} = \frac{1}{\Gamma(\alpha-1)}\mathcal{L}\left(\int_0^t (t-s)^{\alpha-2} E_{\alpha,1}(As^\alpha)ds\right)$.

Proof. 1. For $s > 0$,

$$\begin{aligned}\mathcal{L}\big(E_{\alpha,1}\left(At^{\alpha}\right)\big)(s) &= \mathcal{L}\Big(\sum_{n=0}^{+\infty}\frac{t^{\alpha n}A^{n}}{\Gamma(\alpha n+1)}\Big)\\ &= \sum_{n=0}^{+\infty}\mathcal{L}(t^{\alpha n})\frac{A^{n}}{\Gamma(\alpha n+1)}\\ &= \sum_{n=0}^{+\infty}\frac{1}{s^{n\alpha+1}}A^{n}\\ &= s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1}.\end{aligned}$$

2. For $s > 0$, $s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1} = \mathcal{L}\big(E_{\alpha,1}\left(At^{\alpha}\right)\big)(s)$, then

$$\begin{aligned}s^{\alpha-2}\left(s^{\alpha}-A\right)^{-1} &= s^{-1}s^{\alpha-1}\left(s^{\alpha}-A\right)^{-1}\\ &= \mathcal{L}(1)(s)\mathcal{L}\big(E_{\alpha,1}\left(At^{\alpha}\right)\big)(s)\\ &= \mathcal{L}\big(1 * E_{\alpha,1}\left(At^{\alpha}\right)\big)(s)\\ &= \mathcal{L}\Big(\int_0^t E_{\alpha,1}\left(At^{\alpha}\right)\Big)(s)\\ &= \mathcal{L}\big(tE_{\alpha,2}\left(t^{\alpha}A\right)\big)(s)\end{aligned}$$

3. From (1), we get

$$\begin{aligned}\left(s^{\alpha}-A\right)^{-1} &= s^{1-\alpha}\mathcal{L}\big(E_{\alpha,1}\left(At^{\alpha}\right)\big)(s)\\ &= \mathcal{L}\Big(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\mathcal{L}\big(E_{\alpha,1}\left(At^{\alpha}\right)\big)(s)\\ &= \mathcal{L}\Big(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * E_{\alpha,1}\left(At^{\alpha}\right)\Big)(s)\\ &= \mathcal{L}\Big(\int_0^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)}E_{\alpha,1}\left(A\delta^{\alpha}\right)d\delta\Big)(s),\end{aligned}$$

hence the desired result.

□

Proposition 6. *If*

$$x(t) = x_0 + tx_0' + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[Ax(s) + f\left(s, x(s)\right)\right] ds, \quad t \in J,$$

holds, then we have

$$x(t) = E_{\alpha,1}(At^\alpha)x_0 + tE_{\alpha,2}(At^\alpha)x_0' + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f\left(s, x(s)\right) ds.$$

Proof. Since $x(t) = x_0 + tx_0' + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[Ax(s) + f\left(s, x(s)\right)\right] ds$, using the Laplace transform, we obtain

$$\begin{aligned}
\mathcal{L}\big(x(t)\big)(s) &= \mathcal{L}\big(x_0 + tx_0' + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(Ax(\tau) + f\left(\tau, x(\tau)\right)\right) d\tau\big)(s) \\
&= \mathcal{L}(x_0)(s) + \mathcal{L}(tx_0')(s) + \mathcal{L}\big(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(Ax(\tau) + f\left(\tau, x(\tau)\right)\right) d\tau\big)(s) \\
&= \frac{x_0}{s} + \frac{x_0'}{s^2} + \mathcal{L}\big(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(Ax(\tau) + f\left(\tau, x(\tau)\right)\right) d\tau\big)(s) \\
&= \frac{x_0}{s} + \frac{x_0'}{s^2} + \frac{1}{s^\alpha}\big(A\mathcal{L}\left(x(t)\right)(s) + \mathcal{L}\left(f(t, x(t))\right)(s)\big).
\end{aligned}$$

We can deduce
$\mathcal{L}\big(x(t)\big)(s) =$
$s^{\alpha-1}\left(s^\alpha - A\right)^{-1} x_0 + s^{\alpha-2}\left(s^\alpha - A\right)^{-1} x_0' + \left(s^\alpha - A\right)^{-1} \mathcal{L}\left(f(t, x(t))\right)(s) x_0.$
Now, use the lemma 2 to conclude.

□

Definition 10. *By the integral solution of the Cauchy problem 3.2, we mean that the function $x \in \mathcal{C}(J, X)$ which satisfies*

$$x(t) = E_{\alpha,1}(At^\alpha)x_0 + tE_{\alpha,2}(At^\alpha)x_0' + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} E_{\alpha,1}(As^\alpha) f\left(s, x(s)\right) ds.$$

Now, we put $G_\alpha(t) = E_{\alpha,1}(At^\alpha)$ and $H_\alpha(t) = tE_{\alpha,2}(At^\alpha)$.

Proposition 7. $\forall t \geq 0$, *we have*

1. $G_\alpha(0) = I$,
2. $I^\alpha G_\alpha(t) = AG_\alpha(t)$,
3. $D^\alpha G_\alpha(t) = AG_\alpha(t)$.

Proof. 1. It is clear that $G_\alpha(0) = I$,
2. Setting $\varphi(t) = t^{n\alpha}$,

$$\begin{aligned} I^\alpha(\varphi(t)) &= \frac{1}{\Gamma(\alpha)} \int_0^t s^{n\alpha}(t-s)^{\alpha-1} ds \\ &= \frac{t^{(n+1)\alpha}}{\Gamma(\alpha)} \mathbb{B}(n\alpha+1, \alpha) \\ &= \frac{t^{(n+1)\alpha}\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)}, \end{aligned}$$

which give

$$\begin{aligned} I^\alpha E_{\alpha,1}\left(At^\alpha\right) &= \sum_{n=0}^{+\infty} \frac{I^\alpha(\varphi(t))}{\Gamma(n\alpha+1)} A^n \\ &= t^\alpha \sum_{n=0}^{+\infty} \frac{t^{(n+1)\alpha}}{\Gamma\left((n+1)\alpha+1\right)} A^n \\ &= AE_{\alpha,1}\left(At^\alpha\right). \end{aligned}$$

3.

$$\begin{aligned} D^\alpha\varphi(t) &= \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha}\varphi''(s) ds \\ &= \frac{n\alpha\left(n\alpha-1\right)}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} s^{n\alpha-2} \\ &= \frac{n\alpha\left(n\alpha-1\right)t^{n\alpha-\alpha}}{\Gamma(2-\alpha)} \int_0^1 x^{(n\alpha-1)-1}(1-x)^{(2-\alpha)-1} \\ &= \frac{n\alpha\left(n\alpha-1\right)t^{n\alpha-\alpha}}{\Gamma(2-\alpha)} \mathbb{B}\left(2-\alpha, n\alpha-1\right) \\ &= \frac{\Gamma(n\alpha+1)t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}. \end{aligned}$$

Which implies that

$$\begin{aligned} D^\alpha E_{\alpha,1}\left(t^\alpha A\right) &= \sum_{n=0}^{+\infty} \frac{D^\alpha\varphi(t)}{\Gamma(n\alpha+1)} A^n \\ &= \sum_{n=1}^{+\infty} \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} A^n \\ &= AE_{\alpha,1}\left(t^\alpha A\right). \end{aligned}$$

□

Proposition 8. *1. $G_\alpha(t)$ and $H_\alpha(t)$ commute for all $t \in \mathbb{R}+$.*

2. $H_\alpha(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$.

3. There existe a constant $M > 0$ and $\omega \geq 0$ such that

$$\sup_{t\geq 0} \|G_\alpha(t)\| \leq Me^{\omega t}.$$

4. For all $t, t' \in \mathbb{R}+$,

$$\sup_{t\geq 0} \|H_\alpha(t') - H_\alpha(t)\| \leq \frac{M}{\omega} \| \left(e^{\omega t'} - e^{\omega t} \right) \|.$$

Proof. 1. obvious

2. obvious

3. The Γ function is decreasing and the function $t \to t^{n\alpha}$ is increasing, then $G_\alpha(t) \leq \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!}|A|^n$, where $|A| = \sup_{\|x\|\geq 1} \|Ax\|$, for the rest of the proof see [15].

4. Only apply (1) and inequality of finite increments.

□

3.2. Existence and uniqueness

In this subsection we recall the problem 3.2

$$\begin{cases} D^\alpha x(t) - Ax(t) = f(t, x(t)), & t \in [0, T] \\ x(0) = x_0 & \\ x'(0) = x'_0 & \end{cases}$$

Theorem 1. *Suppose that $f : J \times X \to X$ is continuous and Lipschitzian with respect to the second argument.T*
hen for any $x_0, x'_0 \in X$ such that $C_\alpha(t)x_0, S_\alpha x'_0 \in D(A)$ for all $t \in J$, the problem 3.2 has a unique integral solution.

Proof. We denote $\mathcal{C} = \mathcal{C}(J, X)$.
Define the operator:

$$P : \begin{cases} \mathcal{C} \to \mathcal{C} \\ x \to Px, \end{cases}$$

with, $(Px)(t) = G_\alpha(t)x_0 + H_\alpha(t)x'_0 + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} G_\alpha(s) f(s, x(s))\, ds.$
We give the proof in sevral step:
First let's check that $Px \in \mathcal{C}$ for all $x \in \mathcal{C}$.
Step 1: Let $x \in \mathcal{C}$

$\|(Px)(t+h)-(Px)(t)\| \leq \|G_\alpha(t+h)x_0 - G_\alpha(t)x_0\| + \|H_\alpha(t+h)x_0' - H_\alpha(t)x_0'\| + \frac{1}{\Gamma(\alpha-1)}(\| \int_0^{t+h} (t+h-s)^{\alpha-2}G_\alpha(s)f(s,x(s))ds \; - \; \int_0^t (t-s)^{\alpha-2}G_\alpha(s)f(s,x(s))ds\|)$.
The continuity of the functions $t \to G_\alpha(t)x_0$ and $t \to H_\alpha(t)x_0'$ ensures,

$$\lim_{h\to 0} G_\alpha(t+h)x_0 - G_\alpha(t)x_0 = 0,$$

and

$$\lim_{h\to 0} H_\alpha(t+h)x_0' - H_\alpha(t)x_0' = 0,$$

Now,

$$\begin{aligned}
&\int_0^{t+h} (t+h-s)^{\alpha-2}G_\alpha(s)f(s,x(s))ds - \int_0^t (t-s)^{\alpha-2}G_\alpha(s)f(s,x(s))ds \\
&= \int_0^t \big((t+h-s)^{\alpha-2} - (t-s)^{\alpha-2}\big)G_\alpha(s)f(s,x(s))ds \\
&+ \int_t^{t+h} (t+h-s)^{\alpha-2}G_\alpha(s)f(s,x(s))ds.
\end{aligned}$$

But,

$$\begin{aligned}
&\| \int_0^t ((t+h-s)^{\alpha-2} - (t-s)^{\alpha-2})G_\alpha(s)f(s,x(s))ds\| \\
&\leq \sup_{t\in[0,T]} |||G_\alpha(t)|||.\|f\|_\infty \int_0^t (t+h-s)^{\alpha-2} - (t-s)^{\alpha-2}ds \\
&\leq \sup_{t\in[0,T]} |||G_\alpha(t)|||.\|f\|_\infty \frac{(t+h)^{\alpha-1} + t^{\alpha-1} - h^{\alpha-1}}{\alpha-1},
\end{aligned}$$

so,

$$\lim_{h\to 0} \int_0^t \big((t+h-s)^{\alpha-2} - (t-s)^{\alpha-2}\big)G_\alpha(s)f(s,x(s))ds = 0.$$

In other hand

$$\| \int_t^{t+h} (t+h-s)^{\alpha-2}G_\alpha(s)f(s,x(s))ds\| \leq \sup_{t\in[0,T]} G_\alpha(t)\|f\|_\infty \frac{h^{\alpha-1}}{\alpha-1},$$

so,

$$\lim_{h\to 0} \int_t^{t+h} (t+h-s)^{\alpha-2}G_\alpha(s)f(s,x(s))ds = 0.$$

Finally, Px belong to $\mathcal{C}$.
In second step we will exploit Banach's fixed point theorem, since the space $\mathcal{C}$ is a Banach space.

Step 2: Let $x_1, x_2 \in \mathcal{C}$, $t \in J$, with $x_1(0) = x_2(0)$ and $x_1'(0) = x_2'(0)$. We have,

$$\begin{aligned}|(Px_1)(t) - (Px_2)(t)| &= \frac{1}{\Gamma(\alpha-1)}|\int_0^t (t-s)^{\alpha-2}(f(s,x_1(s)) - f(s,x_2(s)))ds| \\ &\leq \frac{Lt^{\alpha-1}}{\Gamma(\alpha)}\|x_1 - x_2\|_\infty,\end{aligned}$$

which implies that

$$\|Px_1 - Px_2\|_\infty \leq \frac{LT^{\alpha-2}t}{\Gamma(\alpha)}\|x_1 - x_2\|$$

We set $\eta = \frac{LT^{\alpha-2}t}{\Gamma(\alpha)}$, by induction, we have for all $t \in [0, T]$

$$\|(P^n x_1 - P^n x_2\| \leq \frac{\eta^n}{n!}\|x_1 - x_2\|, \quad \forall n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} \frac{\eta^n}{n!} = 0$, then there exists $n \in \mathbb{N}$ such that follows that P^n is a contraction and there exists a unique $x \in \mathcal{C}$ such that $P^n x = x$. Furthermore, we have $P^n(Px) = P(P^n x) = Px$. Hence, Px is a unique fixed point of P^n.

So we conclude that x is the unique integral solution of 3.2. □

3.3. Case $\alpha = 2$

In this case we have, $G_2(t) = E_{\alpha,1}(At^2) = \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!}A^n$ and $H_2(t) = tE_{\alpha,2}(At^2) = \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!}A^n$.

Proposition 9. *For all $t, s \in \mathbb{R}$, We have*

$$G_2(t+S) + G_2(t-s) = 2G_2(t)G_2(s).$$

Proof. Let $t, s \in \mathbb{R}$. Now, calculate $G_2(t+S) + G_2(t-s)$,

$$\begin{aligned}G_2(t+S) + G_2(t-s) &= \sum_{n=0}^{+\infty} \frac{(t+s)^{2n}}{(2n)!}A^n + \sum_{n=0}^{+\infty} \frac{(t-s)^{2n}}{(2n)!}A^n \\ &= \sum_{n=0}^{+\infty} \left((s+t)^{2n} + (s-t)^{2n}\right)\frac{A^n}{(2n)!} \\ &= \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{2n} \binom{2n}{k} t^{2n-k}\left(s^k + (-s)^k\right)\right)\frac{A^n}{(2n)!} \\ &= 2\sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \binom{2n}{2k} t^{2n-2k}s^{2k}\right)\frac{A^n}{(2n)!}.\end{aligned}$$

On the other hand, use Cauchy's product

$$\begin{aligned}G_2(t)G_2(s) &= \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!}A^n \sum_{n=0}^{+\infty} \frac{s^{2n}}{(2n)!}A^n \\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{n} \frac{s^{2k}}{(2K)!}\frac{t^{2n-2k}}{(2n-2k)!}A^n \\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{n} \binom{2n}{2k} t^{2n-2k}s^{2k}A^n,\end{aligned}$$

Hence the result. □

Remark 3. *1.* $G_2(0) = I$*,*

2. $H_2(t+S) + H_2(t-s) = 2H_2(t)G_2(s)$*,*

3. $H_2(t+S) - H_2(t-s) = 2G_2(t)H_2(s)$*.*

4. IN COLOMBEAU'S ALGEBRA

let's start this section with the notion of generalized α-cosine family.

4.1. Generalized α-cosine family

Let $(X, \|.\|)$ denote a Banach space, and $\mathcal{L}(X)$ denote the space of all linear continuous mappings $X \to X$.
Before we define the generalized cosine family, we will state that an application from $\mathcal{G} \longrightarrow \mathcal{G}$ must be linear.

Definition 11. *Let X be a locally convex space with a semi-norm familly $(p_i)_{i\in I}$.*

We define $\mathcal{E}_M(X)$ by the set of $(x_\varepsilon)_\varepsilon \in (X)^{[0,1]}/\exists m \in \mathbb{N}, \forall i \in I, p_i(x_\varepsilon) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^{-m})$
And
$\mathcal{N}(X)$ by $(x_\varepsilon)_\varepsilon \in (X)^{[0,1]}/\forall m \in \mathbb{N}, \forall i \in I, \quad p_i(x_\varepsilon) = \mathcal{O}_{\epsilon\to 0}(\varepsilon^{m})$

And the Colombeau algebra type by:

$$\tilde{X} = \mathcal{E}_M(X)/\mathcal{N}(X)$$

Initially, we want to see if we can define a map $A : \tilde{X} \longrightarrow \tilde{X}$ using a provided family $(A_\varepsilon)_{\varepsilon\in[0,1]}$ of maps $Avarepsilon : XlongrightarrowX$, in which A_ε is a linear and continuous operator.
The next lemma expresses the basic requirement:

Lemma 3. *Let* $(A_\varepsilon)_{\varepsilon\in[0,1]}$ *represen a given family of maps* $A_\varepsilon : X \longrightarrow X$.
For each $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$ *and* $(y_\varepsilon)_\varepsilon \in \mathcal{N}(X)$, *suppose that:*

(1) $(A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$
(2) $(A_\varepsilon (x_\varepsilon + y_\varepsilon))_\varepsilon - (A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{N}(X)$

Then

$$A : \begin{cases} \tilde{X} \longrightarrow \tilde{X} \\ x = [x_\varepsilon] \longmapsto Ax = [A_\varepsilon x_\varepsilon] \end{cases}$$

is well defined.

Proof. We can see from the first property that the class $[(A_\varepsilon x_\varepsilon)_\varepsilon] \in \tilde{X}$. Let $x_\varepsilon + y_\varepsilon$ be another representative of $x = [x_\varepsilon]$ From the second property we have

$$(A_\varepsilon (x_\varepsilon + y_\varepsilon))_\varepsilon - (A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{N}(X)$$

and

$$[(A_\varepsilon (x_\varepsilon + y_\varepsilon))_\varepsilon] = [(A_\varepsilon x_\varepsilon))_\varepsilon] \text{ in } \tilde{X}$$

Then A is well defined. □

Now we will presented a notion of generalized α-cosine family.

Definition 12. *Let* $E_M([0,+\infty[,\mathcal{L}(X))$ *be the space of nets* $(C_{\alpha,\varepsilon})_\varepsilon$ *of continuous mappings* $C_{\alpha,\varepsilon} : [0,+\infty[\to \mathcal{L}(X), \varepsilon \in]0,1[$ *with for every* $T > 0$ *there exists* $a \in \mathbb{R}$ *such that*

$$\sup_{t\in[0,T]} \|C_{\alpha,\varepsilon}(t)\| = \mathcal{O}(\varepsilon^a) \quad \textit{as} \quad \varepsilon \to 0 \tag{3}$$

Definition 13. *Let* $N([0,+\infty[,\mathcal{C}(X))$ *be the space of nets* $(N_\varepsilon)_z$ *of continuous mappings* $N_\varepsilon : [0,+\infty[\to \mathcal{L}(X), \varepsilon \in]\,0,1[$ *with for every* $T > 0$ *and* $b \in \mathbb{R}$, *we have:*

$$\sup_{t\in[0,T]} \|N_\varepsilon(t)\| = \mathcal{O}\left(\varepsilon^b\right) \quad \textit{as} \quad \varepsilon \to 0 \tag{4}$$

Proposition 10. $E_M([0,+\infty[,\mathcal{L}(X))$ *is an algebra with respect to composition and* $N([0,+\infty[,\mathcal{L}(X))$.. *is an ideal of* $E_M([0,+\infty[,\mathcal{L}(X))$.

Proof. Let $(S_{\alpha,\varepsilon}(t))_\varepsilon \in E_M([0,+\infty[,\mathcal{L}(X))$ and $(N_\varepsilon(t))_\varepsilon \in N([0,+\infty[,\mathcal{L}(X))$. We will prove only the second assertion, i.e., that:

$$(S_{\alpha,\varepsilon}(t)N_\varepsilon(t))_\varepsilon, (N_\varepsilon(t)S_{\alpha,\varepsilon}(t))_\varepsilon \in N([0,+\infty[,\mathcal{L}(X))$$

where $S_{\alpha,\varepsilon}(t)N_\varepsilon(t)$ denotes the composition.
Let $\epsilon \in]0,1[$. By (2) and (3), for some $a \in \mathbb{R}$ and every $b \in \mathbb{R}$,

$$\|S_{\alpha,\varepsilon}(t)N_\varepsilon(t)\| \le \|S_{\alpha,\varepsilon}(t)\| \, \|N_\varepsilon(t)\| = O\left(\epsilon^{a+b}\right) \quad \text{as} \quad \epsilon \to 0.$$

The same holds for:

$$\|N_\varepsilon(t)S_{\alpha,\varepsilon}(t)\| \leq \|N_\varepsilon(t)\|\,\|S_{\alpha,\varepsilon}(t)\| = \mathcal{O}\left(\epsilon^{a+b}\right) \quad \text{as} \quad \epsilon \to 0.$$

□

Definition 14. *We define the Colombenu type algebra as the factor algebra:*

$$G\left([0,+\infty[,\mathcal{L}(X)\right) = E_M([0,+\infty[,\mathcal{L}(X))/N([0,+\infty[,\mathcal{L}(X))$$

Remark 4. *Let $C_\alpha \in G([0,+\infty[,\mathcal{L}(X))$.*
We denoted by $C_\alpha = [(C_{\alpha,\varepsilon})]$ with $C_{\alpha,\varepsilon} \in E_M([0,+\infty[,\mathcal{L}(X))$.

Definition 15. *We say $C_\alpha = [(C_{\alpha,\varepsilon})]$ with $C_\varepsilon \in E_M([0,+\infty[,\mathcal{L}(E))$ the generalized cosine-family if:*
(1) $C_\varepsilon(0) = I$.
(2) $D^\alpha C_\alpha = AC_\alpha$
(3) $C_\varepsilon(t)x$ is continuous in t on $\mathbb{R}^+$ for each fired $x \in X$.

Definition 16. *We say that $S_\alpha = [(S_{\alpha,\varepsilon})]$ is the generalized sine-family associated with $C_\alpha = [(C_{\alpha,\varepsilon})]$ generalized cosine-family if for all $\varepsilon \in]0,1[$, we have:*

$$S_{\alpha,\varepsilon}(t) = \int_0^t C_{\alpha,\varepsilon}(\tau)d\tau.$$

Proposition 11. *S=[($S_{\alpha,\varepsilon} \in G([0,+\infty,\mathcal{L}(X)))$]*

Proof. Let $C_\alpha = [(C_{\alpha,\varepsilon})]$ generalized cosine-family and $t \in [0,T]$ we have:

$$S_{\alpha,\varepsilon}(t) = \int_0^t C_{\alpha,\varepsilon}(\tau)d\tau$$

Then,

$$\sup_{t\in[0,T]} \|S_{\alpha,\varepsilon}(t)\| \leq T \sup_{t\in[0,T]} \|C_{\alpha,\varepsilon}(t)\|$$

As

$$C_{\alpha,\varepsilon} \in E_M\left([0,+\infty[,\mathcal{L}(X)\right) \text{ then } S_{\alpha,\varepsilon} \in E_M([0,+\infty[,\mathcal{L}(X))$$

finally:

$$S_\alpha = [(S_{\alpha,\varepsilon})] \in G([0,+\infty[,\mathcal{L}(X))$$

□

Proposition 12. *Let $C_\alpha = [(C_{\alpha,\varepsilon})]$, be a strongly continuous generalized cosine family with associated generalized sine family $S_\alpha = [(S_{\alpha,\varepsilon})]$ we have:*
(1) $C_{\alpha,\varepsilon}(s), S_{\alpha,\varepsilon}(s), C_{\alpha,\varepsilon}(t)$, and $S_{\alpha,\varepsilon}(t)$ commute for all $s,t \in \mathbb{R}^+$.
(3) $S_{\alpha,\varepsilon}(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$

(6) there exist constants M and $w \geq 0$ such that:

$$|C_{\alpha,\varepsilon}(t)| \leq Me^{w|t|}$$

(7)

$$|S_{\alpha,\varepsilon}(t) - S_{\alpha,\varepsilon}(t')| \leq M|\int_{t'}^{t} e^{w|s|}ds| \quad \forall t, t' \in \mathbb{R}^+.$$

Definition 17. *Let $X_{2,\varepsilon} = \{x \in X : t \longrightarrow D^\alpha C_{\alpha,\varepsilon}(t)x$ is continuous in $t \in \mathbb{R}^+\}$*

We use the lemma 1 the infinitesimal generator of a strongly continuous generalized cosine family $C_\alpha = [(C_{\alpha,\varepsilon})], t \in \mathbb{R}^+$, is the operator $A = [(A_\varepsilon)]$ with $A_\varepsilon : X \to X$

$$A_\varepsilon x = D^\alpha C_\varepsilon(0)x$$

with:
$$D(A_\varepsilon) = X_{2,\varepsilon}$$

4.2. Existence solution

In this subsection consider the following problem:

$$\begin{cases} D^\alpha u(x,t) + Au(x,t) = F(t, u(x,t)), & t \in [0,a] \\ x(0,x) = a_0(x) \\ x'(0) = b_0(x) \end{cases} \tag{5}$$

with $a_0(x), b_0(x) \in D'(\mathbb{R}^n)$.
Now we will transform the problem to the Colombeau algebra.

$$\begin{cases} D^\alpha u_\varepsilon(t,x) + A_\varepsilon u_\varepsilon(t,x) = F_\varepsilon(t, u_\varepsilon(t,x)) \quad x \in \mathbb{R}^n, \quad t \geq 0 \\ \quad u_\varepsilon(0,x) = a_{0,\varepsilon}(x) \\ \dfrac{d}{dt}u_\varepsilon(0,x) = b_{0,\varepsilon}(x) \end{cases}$$

with $a_{0,\varepsilon}(x), b_{0,\varepsilon}(x)$ are the regularizetion of $a_0(x)$ and $b_0(x)$ respectively, and, as stated definition (12) $A = [(A_\varepsilon)]$ is the infinitesimal generator of generalized cosine family $C = [(C_\varepsilon)]$.

Definition 18. *An element $f \in \mathcal{G}[\mathbb{R}^n]$ is L^∞ logarithmic type if it has a representative $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M[\mathbb{R}^n]$ such that*

$$\|f_\varepsilon\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\log(\varepsilon)) \quad as \quad \varepsilon \to 0$$

Theorem 2. *Let ∇F_ε is L^∞ log-type and the generalized sine family $S = [(S_\varepsilon)]$ a associated of the generalized cosine family $C = [(C_\varepsilon)]$ verify the property: there exist $M > 0$ and $\varepsilon_0 > 0$ for every $(f_\varepsilon)_\varepsilon \subset L^\infty(\mathbb{R}^n)$ and $T > 0$*

$$\sup_{t \in [0,T]} \|S_\varepsilon(t) f_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq M \|f_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \quad as \quad \varepsilon < \varepsilon_0$$

The problem (5) has a unique solution in $\mathcal{G}\left(\mathbb{R}^{+}\times\mathbb{R}^{n}\right)$.

Proof. The integral solution of the equation 2 is:

$$u_{\varepsilon}(t,x)=C_{\alpha,\varepsilon}(t)a_{0,\varepsilon}(x)+S_{\alpha,\varepsilon}(t)b_{0,\varepsilon}(x)+I^{\alpha-1}C_{\alpha,\varepsilon}(t)F_{\varepsilon}\left(t,u_{\varepsilon}(t,x)\right)$$

Which implies that:

$$\begin{aligned}\|u_{\varepsilon}(t,.)\|_{L^{\infty}(\mathbb{R}^{n})}&\leq\|C_{\alpha,\varepsilon}(t)\|\,\|a_{0,\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^{n})}+\|S_{\alpha,\varepsilon}(t)\|\,\|b_{0,\varepsilon}(x)\|_{L^{\infty}(\mathbb{R}^{n})}\\&+\frac{1}{\Gamma(\alpha-1)}\int_{0}^{t}(t-s)^{\alpha-2}\,\|C_{\alpha,\varepsilon}(t)\|\,\|F_{\varepsilon}\left(s,u_{\varepsilon}(s,.)\right)\|_{L^{\infty}(\mathbb{R}^{n})}\,ds,\end{aligned}$$

then:

$$\begin{aligned}\|u_{\varepsilon}(t,.)\|_{L^{\infty}(\mathbb{R}^{n})}&\leq\sup_{\tau\in[0,T]}\|C_{\varepsilon}(\tau)\|\,\|a_{0,\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^{n})}+\sup_{\tau\in[0,T]}\|S_{\varepsilon}(\tau)\|\,\|b_{0,\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^{n})}\\&+\sup_{\tau\in[0,T]}\|C_{\alpha,\varepsilon}(\tau)\|\int_{0}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,\|F_{\varepsilon}\left(s,u_{\varepsilon}(s,.)\right)\|_{L^{\infty}(\mathbb{R}^{n})}\,ds.\end{aligned}$$

The first approximation of F_{ε} :

$$F_{\varepsilon}\left(s,u_{\varepsilon}(s,.)\right)=F_{\varepsilon}(s,0)+\nabla F_{\varepsilon}u_{\varepsilon}(s,.)+N_{\varepsilon}(s)$$

with $N_{\varepsilon}\in\mathcal{N}\left(\mathbb{R}^{+}\right)$
Then

$$\begin{aligned}\|u_{\varepsilon}(t,.)\|_{L^{\infty}(\mathbb{R}^{n})}&\leq\sup_{\tau\in[0,T]}\|C_{\alpha,\varepsilon}(\tau)\|\,\|a_{0,\varepsilon}\|_{L^{\infty}(\mathbb{R}^{n})}+\sup_{\tau\in[0,T]}\|S_{\alpha,\varepsilon}(\tau)\|\,\|b_{0,\varepsilon}\|_{L^{\infty}(\mathbb{R}^{n})}\\&+\sup_{\tau\in[0,T]}\|S_{\alpha,\varepsilon}\|\int_{0}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}F_{\varepsilon}(s,0)ds\\&+\sup_{\tau\in[0,T]}\|C_{\alpha,\varepsilon}(\tau)\|\|\nabla F_{\varepsilon}\|\int_{0}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|u_{\varepsilon}(s,.)\|_{L^{\infty}(\mathbb{R}^{n})}ds\\&+\sup_{\tau\in[0,T]}\|C_{\alpha,\varepsilon}(\tau)\|\int_{0}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}N_{\varepsilon}(s)ds\end{aligned}$$

We get

$$\begin{aligned}&\|u_{\varepsilon}(t,.)\|_{L^{\infty}(\mathbb{R}^{n})}\leq\sup_{\tau\in[0,T]}\|C_{\alpha,\varepsilon}(\tau)\|\,\|a_{0,\varepsilon}\|_{L^{\infty}(\mathbb{R}^{n})}\\&+\sup_{\tau\in[0,T]}\|S_{\alpha,\varepsilon}\|\|b_{0,\varepsilon}\|_{L^{\infty}(\mathbb{R}^{n})}+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\sup_{\tau\in[0,T]}\|C_{\alpha,\varepsilon}(\tau)\|\sup_{\tau\in[0,T]}\|F_{\varepsilon}(\tau,0)\|\\&+\sup_{\tau\in[0,T]}\|C_{\varepsilon}(\tau)\|\|\nabla F_{\varepsilon}\|\int_{0}^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|u_{\varepsilon}(s,.)\|_{L^{\infty}(\mathbb{R}^{n})}ds\\&+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\sup_{\tau\in[0,T]}\|C_{\varepsilon}(\tau)\|\sup_{\tau\in[0,T]}\|N_{\varepsilon}(\tau)\|\end{aligned}$$

So,

$$
\begin{aligned}
\|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq & \sup_{\tau\in[0,T]} \|C_\varepsilon(\tau)\| \, \|a_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} \\
& + \sup_{\tau\in[0,T]} \|S_\varepsilon(\tau)\| \, \|b_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} \\
& + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\| \sup_{\tau\in[0,T]} \|F_\varepsilon(\tau,0)\| \\
& + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\| \sup_{\tau\in[0,T]} \|N_\varepsilon(\tau)\| \\
& + \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\| \, \|\nabla F_\varepsilon\| \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|u_\varepsilon(s,.)\|_{L^\infty(\mathbb{R}^n)} ds.
\end{aligned}
$$

By the Granwall's inequality:

$$
\begin{aligned}
\|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq & \left(\sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\| \, \|a_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} \right. \\
& + \sup_{\tau\in[0,T]} \|S_{\alpha,\varepsilon}(\tau)\| \, \|b_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^n)} \\
& + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\| \sup_{\tau\in[0,T]} \|F_\varepsilon(\tau,0)\| \\
& \left. + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sup_{\tau\in[0,T]} \|S_\varepsilon\| \sup_{\tau\in[0,T]} \|N_\varepsilon\| \right) \\
& \times \exp\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} \sup_{\tau\in[0,T]} \|S_\varepsilon(\tau)\| \|\nabla F_\varepsilon\| \right).
\end{aligned}
$$

Since $C \in G([0,+\infty[,\mathcal{L}(X)), S \in G([0,+\infty[,\mathcal{L}(X)), a_0 \in \mathcal{G}(\mathbb{R}^n), b_0 \in \mathcal{G}(\mathbb{R}^n) \quad (N_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$ and ∇F is $L^\infty-$ logtype there exist $M \in \mathbb{N}$ such that
$\sup_{t\in[0,T]} \|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}\left(\varepsilon^{-M}\right), \quad \varepsilon \to 0$ □

4.3. Uniqueness.

Proof. Suppose that there exist two solutions $u_{1,\varepsilon}(t,.)$, $u_{2,\varepsilon}(t,.)$ to problem (5), then:

$$
\begin{cases}
D^\alpha u_{1,\varepsilon}(t,x) + A_\varepsilon u_{1,\varepsilon}(t,x) - D^\alpha u_{2,\varepsilon} & (t,x) - A_\varepsilon u_{2,\varepsilon}(t,x) \\
& = F_\varepsilon(t, u_{1,\varepsilon}(t,x)) - F_\varepsilon(t, u_{2,\varepsilon}(t,x)) \\
& x \in \mathbb{R}^n, \quad t \geq 0 \\
u_{1,\varepsilon}(0,x) - u_{2,\varepsilon}(0,x) = N_{0,\varepsilon}(x) & \\
\frac{d}{dt} u_{1,\varepsilon}(0,x) - \frac{d}{dt} u_{2,\varepsilon}(0,x) = \bar{N}_{0,\varepsilon}(x) &
\end{cases}
\tag{6}
$$

Then:

$$\begin{cases} D^{\alpha}\left(u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\right)+A_{\varepsilon}\left(u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\right)=F_{\varepsilon}\left(t,u_{1,\varepsilon}(t,x)\right) \\ \qquad -F_{\varepsilon}\left(t,u_{2,\varepsilon}(t,x)\right) \\ \qquad x\in\mathbb{R}^n,\quad t\geq 0 \\ u_{1,\varepsilon}(0,x)-u_{2,\varepsilon}(0,x)=N_{0,\varepsilon}(x) \\ \frac{d}{dt}u_{1,\varepsilon}(0,x)-\frac{d}{dt}u_{2,\varepsilon}(0,x)=\bar{N}_{0,\varepsilon}(x) \end{cases} \tag{7}$$

With $\left(N_{0,\varepsilon}\right)_{\varepsilon},\left(\widetilde{N}_{0,\varepsilon}\right)_{\varepsilon}\in\mathcal{N}\left(\mathbb{R}^{+}\right)$.
The integral solution of the equation (7) is:

$$\begin{aligned} u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x) &= C_{\alpha,\varepsilon}(t)N_{0,\varepsilon}(x)+S_{\alpha,\varepsilon}(t)\widetilde{N}_{0,\varepsilon}(x) \\ &+\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}C_{\alpha,\varepsilon}(t)\left(F_{\varepsilon}\left(s,u_{1,\varepsilon}(s,x)\right)-F_{\varepsilon}\left(s,u_{2,\varepsilon}(s,x)\right)\right)ds \end{aligned}$$

Then:
$\left\|u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\right\|_{L^{\infty}(\mathbb{R}^n)}\leq\left\|C_{\alpha,\varepsilon}(t)\right\|\left\|N_{0,\varepsilon}(.)\right\|_{L^{\infty}(\mathbb{R}^n)}$
$+\left\|S_{\alpha,\varepsilon}(t)\right\|\left\|\tilde{N}_{0,\varepsilon}(\cdot)\right\|_{L^{\infty}(\mathbb{R}^n)}$
$+\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|C_{\alpha,\varepsilon}(t)\right\|\left\|F_{\varepsilon}\left(s,u_{1,\varepsilon}(s,.)\right)-F_{\varepsilon}\left(s,u_{2,\varepsilon}(s,.)\right)\right\|_{L^{\infty}(\mathbb{R}^n)}ds.$
Which implies that:

$$\begin{aligned} \left\|u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\right\|_{L^{\infty}(\mathbb{R}^n)} &\leq \sup_{\tau\in[0,T]}\left\|C_{\alpha,\varepsilon}(\tau)\right\|\left\|N_{0,\varepsilon}(.)\right\|_{L^{\infty}(\mathbb{R}^n)} \\ &+\sup_{\tau\in[0,T]}\left\|S_{\alpha,\varepsilon}(\tau)\right\|\left\|\tilde{N}_{0,\varepsilon}(.)\right\|_{L^{\infty}(\mathbb{R}^n)} \\ &+\sup_{\tau\in[0,T]}\left\|C_{\varepsilon}(\tau)\right\| \\ &\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|F_{\varepsilon}\left(s,u_{1,\varepsilon}(s,.)\right)-F_{\varepsilon}\left(s,u_{2,\varepsilon}(s,.)\right)\right\|_{L^{\infty}}ds. \end{aligned}$$

The first approximation of $F_{\varepsilon}\left(s,u_{1,\varepsilon}(s,.)\right)-F_{\varepsilon}\left(s,u_{2,\varepsilon}(s,.)\right)$ is given by:

$$F_{\varepsilon}\left(s,u_{1,\varepsilon}(s,.)\right)-F_{\varepsilon}\left(s,u_{2,\varepsilon}(s,.)\right)=\left\|\nabla F_{\varepsilon}\right\|\left(u_{1,\varepsilon}(s,.)\ -u_{2,\varepsilon}(s,.)\right)+N_{\varepsilon}(s),$$

with $(N_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$.
Then

$$\|u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \le \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\| \, \|N_{0,\varepsilon}(.)\|_{L^\infty(\mathbb{R}^n)}$$
$$+ \sup_{\tau\in[0,T]} \|S_{\alpha,\varepsilon}(\tau)\| \left\|\tilde{N}_{0,\varepsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)}$$
$$+ \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\|$$
$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha-1)} \|\nabla F_\varepsilon\| \|u_{1,\varepsilon}(s,.) - u_{2,\varepsilon}(s,.)\|_{L^\infty(\mathbb{R}^n)} ds$$
$$+ \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\| \int_0^t N_\varepsilon(s) ds$$

So,

$$\|u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \le \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\| \, \|N_{0,\varepsilon}(.)\|_{L^\infty(\mathbb{R}^n)}$$
$$+ \sup_{\tau\in[0,T]} \|S_{\alpha,\varepsilon}(\tau)\| \left\|\widetilde{N}_{0,\varepsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)}$$
$$+ \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sup_{\tau\in[0,T]} \|S_\varepsilon(\tau)\| \sup_{\tau\in[0,T]} \|N_\varepsilon(s)\|$$
$$+ \sup_{\tau\in[0,T]} \|C_{\alpha,\varepsilon}(\tau)\|$$
$$\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \|\nabla F_\varepsilon\| \left\|u_{1,\varepsilon}(s,.) - u_{2,\varepsilon}(s,.)\right\|_{L^\infty(\mathbb{R}^n)} ds$$

By the Granwall's inequality:

$$||u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.)||_{L^\infty(\mathbb{R}^n)} \le \Big(\sup_{\tau\in[0,T]} ||C_{\alpha,\varepsilon}(\tau)|| ||N_{0,\varepsilon}(.)||_{L^\infty} + \sup_{\tau\in[0,T]} ||S_{\alpha,\varepsilon}|| ||\widetilde{N}_{0,\varepsilon}(.)||_{L^\infty}$$
$$+ \frac{T^{\alpha-1}}{\Gamma(\alpha)} \sup_{\tau\in[0,T]} \|S_\varepsilon(\tau)\| \sup_{\tau\in[0,T]} |N_\varepsilon(s)| \Big)$$

$$\times \exp\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} \sup_{\tau\in[0,T]} \|S_\varepsilon(\tau)\| \|\nabla F_\varepsilon\| \right).$$

Since:
$C \in G([0,+\infty[, \mathcal{L}(X)), S \in G([0,+\infty[, \mathcal{L}(X)), (N_{0,\varepsilon})_\varepsilon, (\widetilde{N}_{0,\varepsilon})_\varepsilon \in \mathcal{N}(\mathbb{R}^+)(N_\varepsilon)_\varepsilon$

$\in \mathcal{N}(\mathbb{R}^+)$ and ∇F is L^∞ - logtype and for every $q \in \mathbb{N}$ such that:

$$\sup_{t\in[0,T]} \|u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\varepsilon^q) \quad \varepsilon \to 0$$

□

REFERENCES

[1] Benmerrous, A., Chadli, L.s., Moujahid, A. et al. Generalized Cosine Family. J Elliptic Parabol Equ (2022). https://doi.org/10.1007/s41808-022-00156-x.

[2] Benmerrous, A., Chadli, L.S., Moujahid, A., M'hamed, E., Melliani, S., Generalized solution of Schrödinger equation with singular potential and initial data. Int. J. Nonlinear Anal. Appl. 13(1), 3093-3101 (2022). http://dx.doi.org/10.22075/ijnaa.2021.23609.2565.

[3] J.F.Colombeau, Elementary Introduction to New Generalized Function, North Holland, Amsterdam, 1985.

[4] J.F.Colombeau, New Generalized Function and Multiplication of Distribution, North Holland, Amsterdam / New York / Oxford, 1984.

[5] Fattorini, H. O., Ordinary differential equations in linear topological spaces, II, J. Differential Equations, 6 (1969), 50–70.

[6] Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V., Mittag-Leffler Functions, Related Topics and Applications (Springer, New York, 2014).

[7] Gorenflo, R., Mainardi, F., Fractional calculus; Integral an differential equations of fractional order, in: Fractals and Fractional Calculus in Continuum Mechanics, in: A. Carpinteri, F. Mainardi (Eds.), CISM Lecture notes, Springer Verlag, Wien and New York, 1997, pp. 223–276.

[8] Jenkins, PF., Making sense of the chest x-ray: a hands-on guide. New York (NY): Oxford University Press; 2005.

[9] Kilbas, A.A., Srivastava, H.M., Trujillo,J.J., Theory and Applications of Fractional Differential Equations, in: Jan van Mill (Ed.), North-Holland Mathematics Studies, vol. 204, Amsterdam, Netherlands, 2006.

[10] Mainardi,F., Paradisi,P., Gorenflo,R., Probability distributions generated by fractional diffusion equations, in: J. Kertesz, I. Kondor (Eds.), Econophysics: An Emerging Science, Kluwer, Dordrecht, 2000.

[11] Oberguggenberger, M., Multiplication of Distributions and Applications to Partial Differential Equations, Pitman Research Notes in Mathematics, (1992).

[12] Podlubny, I., Fractional Differential Equations, Academic Press, San Diego, 1999.

[13] SEGAL, I., Non-linear semi-groups, Ann. Math., 78 (1963), 339–364.

[14] Stojanovic,M., Extension of Colombeau algebra to derivatives of arbitrary order D^{α}, $\alpha \geq 0$. Application to ODEs and PDEs with entire and fractional derivatives. 71 (2009) 5458–5475.

[15] Stojanovic, M., Foundation of the fractional calculus in generalized function algebra. Analysis and Applications, Vol. 10, No. 4 (2012) 439–467 World Scientific Publishing Company DOI: 10.1142/S0219530512500212.

[16] Travis,C.C., Webb, G. F., Cosine families and abstract nonlinear second order differential equations, Acta Mathematica Hungarica, 1588–2632, 1978.

[17] Trujillo, J.J., On best fractional derivative to be applied in fractional modeling. The fractional Fourier transform, in: Proc. FDA08, Ankara, 5–7 November, 2008.

[18] Weisstein, Eric, W. "Laplace Transform". mathworld.wolfram.com. Retrieved 2020-08-08.