Results Related to Value Distribution and Uniqueness of Entire Functions Concerning with Difference Polynomials

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Abstract

In this paper, we prove the uniqueness theorems concerning to the difference polynomials. With the concepts of weakly weighted sharing and relaxed weighted sharing we obtain some results which extend and generalizes the earlier results of B. Saha, S. Pal and T. Biswas [19].

Keywords: Nevanlinna Theory, Differentail Polynomial, Meromorphic Function, Uniqueness etc.,

1. INTRODUCTION AND MAIN RESULTS

In this research paper, we consider a meromorphic function which always meant a meromorphic function in the complex plane $\mathbb{C}$. Here authors are assumed that readers are known about basic notations of Nevanlinna theory and uses some of the notations like $m(r, f)$, $N(r, 0; f)$, $\overline{N}(r, \infty; f)$, and $T(r, f)$ etc., (see [1], [2], [3], [4]). Denote $S(r, f)$ any quantity which satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside of an exceptional set of finite linear measure and we are also denoting $S(r, f) = T(r, \alpha(z))$, where $\alpha(z)$ is a small function of $f$. Let $k$ be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a, f) = \{z \mid f - a = 0\}$, where a zero with multiplicity $k$ is counted $k$ times. If the zeros are counted only once, then we denote the set by $\overline{E}(a, f)$. If $E(a, f) = E(a, g)$ where $f$ and $g$ are two non-constant meromorphic functions, then we say that $f$ and $g$ share a CM (counting multiplicity). If $\overline{E}(a, f) = \overline{E}(a, g)$ then we say that $f$ and $g$ share a IM (ignoring multiplicity). Denoting $E_k(a, f)$ by the set of all $a$ points of $f$ with multiplicities not exceeding $k$, where an $a$ points is counted according to its multiplicity. Also we denote $\overline{E}_k(a, f)$ the set of distinct $a$ points of $f$ with multiplicities not exceeding $k$. We denote by $N_k(r, a; f)$ the counting function of zeros of $f - a$ with multiplicity less than or equal to $k$, and by $\overline{N}_k(r, a; f)$ the corresponding one for which multiplicity is not counted. Let $N_k(r, a; f)$ the counting function of zeros of $f - a$ with multiplicity greater than or equal to $k$, and by $\overline{N}_k(r, a; f)$ the corresponding one for which multiplicity is not counted. Set

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}_2(r, a; f) + \ldots + \overline{N}_{k}(r, a; f).$$
Let us define $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_0$ be a non-zero polynomial of degree $m$, where $a_m(\neq 0)$, $a_{m-1}, \ldots, a_0 (\neq 0)$ are complex constants and $m$ is a positive integer. Let $m_1$ be the number of distinct simple zeros and $m_2$ be the number of distinct multiple zeros of $P(z)$. Let $\Gamma_0 = m_1 + 2m_2$ and $\Gamma_1 = m_1 + m_2$. Let $m_1$ be the number of distinct simple zeros and $m_2$ be the number of distinct multiple zeros of $P(z)$. Let $\Gamma_0 = m_1 + 2m_2$ and $\Gamma_1 = m_1 + m_2$. Throughout the paper, we denote by $d = \gcd(\sigma_0, \ldots, \sigma_m)$ where $\sigma_j = n + \sigma + j$, if $a_j \neq 0$, $\sigma_j = n + m + \sigma$, if $a_j = 0$. Set

$$\Phi(z, f) = \prod_{j=1}^{d} (z + c_j)^{\mu_j},$$

where $d \in \mathbb{N}$ and $\mu_j \in \mathbb{N}$ for $j = 1, \ldots, d$. Let $\sigma = \sum_{j=1}^{d} \mu_j$.

2. SOME DEFINITIONS

The following definitions are necessary to prove our main results.

**Definition 1.** Let $a \in \mathbb{C} \cup \{\infty\}$. Denote by $N_E(r, a; f, g)\left(N_E(r, a; f, g)\right)$ by the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ with same multiplicities by $N_0(r, a; f, g)\left(N_0(r, a; f, g)\right)$ the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ IM. If

$$N(r, a; f) + N(r, a; g) - 2N_E(r, a; f, g) = S(r, f) + S(r, g)$$

then we say that $f$ and $g$ share the value a CM. If

$$N(r, a; f) + N(r, a; g) - 2N_0(r, a; f, g) = S(r, f) + S(r, g)$$

then we say that $f$ and $g$ share the value a IM.

**Definition 2.** [7] Let $f$ and $g$ share the value a IM and $k$ be a positive integer or infinity. Then $N_k^E(r, a; f, g)$ denotes the reduced counting function of those $a$ points of $f$ whose multiplicities are equal to the corresponding $a$ points of $g$, and both of their multiplicities are not greater than $k$. $N_k^0(r, a; f, g)$ denotes the reduced counting function of those $a$ points of $f$ which are points of $g$, and both of their multiplicities are not less than $k$.

In 2006, authors S. H Lin and W. C Lin [7] introduced the following definitions of weakly weighted sharing which is a scaling between sharing IM and CM.

**Definition 3.** [7] Let $a \in \mathbb{C} \cup \{\infty\}$ and $k$ be a positive integer or infinity. If

$$N(r, a; f) \leq k - N_k^E(r, a; f, g) = S(r, f).$$

$$N(r, a; g) \leq k - N_k^E(r, a; f, g) = S(r, g).$$

$$N(r, a; f) \geq k + 1 - N_{k+1}^0(r, a; f, g) = S(r, f).$$

$$N(r, a; g) \geq k + 1 - N_{k+1}^0(r, a; f, g) = S(r, g).$$
\[ \overline{N}(r, a; f) - \overline{N}_0(r, a; f, g) = S(r, f). \]
\[ \overline{N}(r, a; g) - \overline{N}_0(r, a; f, g) = S(r, g). \]
then we say that \( f \) and \( g \) share the value \( a \) weakly with weight \( k \) and we write \( f \) and \( g \) share "\((a, k)\)".

In 2007, A. Banerjee and S. Mukherjee [5] introduced a new type of sharing known as relaxed weighted sharing, weaker than weakly weighted sharing and is defined as follows.

**Definition 4.** [5] We denote by \( \overline{N}(r, a; |f| = p; |g| = q) \) the reduced counting function of common \( a \) points of \( f \) and \( g \) with multiplicities \( p \) and \( q \) respectively.

**Definition 5.** [5] Let \( a \in \mathbb{C} \cup \{\infty\} \) and \( k \) be a positive integer or infinity. Suppose that \( f \) and \( g \) share the value \( a \) IM. If for \( p \neq q \),
\[ \sum_{p,q \leq k} \overline{N}(r, a; |f| = p; |g| = q) = S(r), \]
then we say that \( f \) and \( g \) share the value \( a \) with weight \( k \) in a relaxed manner and in that case we write \( f \) and \( g \) share \((a, k)^*\).

In 2015, Pulak Sahoo [11] proved the following results.

**Theorem 1.** [11] Let \( f \) and \( g \) be two transcendental entire functions of finite order, and \( \alpha \) \((\neq 0, \infty)\) be a small function of both \( f \) and \( g \). Suppose that \( \eta \) is non-zero complex constant, \( n \) and \( m(\geq 1) \) are integers such that \( n \geq m + 6 \). If \( f^n f - 1 f(z + \eta) \) and \( g^n(g^m - 1)g(z + \eta) \) share "\((\alpha(z), 2)\)"., then \( f \equiv tg \) where \( t^m = 1 \).

**Theorem 2.** [11] Let \( f \) and \( g \) be two transcendental entire functions of finite order, and \( \alpha \) \((\neq 0, \infty)\) be a small function of both \( f \) and \( g \). Suppose that \( \eta \) is non-zero complex constant, \( n \) and \( m(\geq 1) \) are integers such that \( n \geq 2m + 8 \). If \( f^n f - 1 f(z + \eta) \) and \( g^n(g^m - 1)g(z + \eta) \) share \((\alpha(z), 2)^*\), then \( f \equiv tg \) where \( t^m = 1 \).

**Theorem 3.** [11] Let \( f \) and \( g \) be two transcendental entire functions of finite order, and \( \alpha \) \((\neq 0, \infty)\) be a small function of both \( f \) and \( g \). Suppose that \( \eta \) is non-zero complex constant, \( n \) and \( m(\geq 1) \) are integers such that \( n \geq 4m + 12 \). If \( E_2(\alpha(z), f^n f - 1 f(z + \eta)) = E_2(\alpha(z), g^n(g^m - 1)g(z + \eta)) \), then \( f \equiv tg \) where \( t^m = 1 \).

In 2018, Pulak Sahoo and Gurudas Biswas [13] proved the following result.

**Theorem 4.** [13] Let \( f \) and \( g \) be two transcendental entire functions of finite order, and \( \alpha \) \((\neq 0, \infty)\) be a small function of both \( f \) and \( g \). Suppose that \( \eta \) is non-zero complex constant, \( n \) and \( m(\geq 1) \), \( k(\geq 0) \) are integers such that \( n \geq 2k + m + 6 \). If \( f^n f - 1 f(z + \eta))^{(k)} \) and \( g^n(g^m - 1)g(z + \eta))^{(k)} \) share "\((\alpha(z), 2)\)"., then \( f \equiv tg \) where \( t^m = 1 \).
Theorem 5. [13] Let \( f \) and \( g \) be two transcendental entire functions of finite order, and \( \alpha \) (\( \neq 0, \infty \)) be a small function of both \( f \) and \( g \). Suppose that \( \eta \) is non-zero complex constant, \( n \) and \( m(\geq 1), k(\geq 0) \) are integers such that \( n \geq 3k + 2m + 8 \). If 
\[
(f^n(f^m - 1)f(z + \eta)^{(k)}) \text{ and } (g^n(g^m - 1)g(z + \eta)^{(k)}) \text{ share } (\alpha(z), 2)^*,
\]
then \( f \equiv tg \) where \( t^m = 1 \).

Theorem 6. [13] Let \( f \) and \( g \) be two transcendental entire functions of finite order, and \( \alpha \) (\( \neq 0, \infty \)) be a small function of both \( f \) and \( g \). Suppose that \( \eta \) is non-zero complex constant, \( n \) and \( m(\geq 1), k(\geq 0) \) are integers such that \( n \geq 5k + 4m + 12 \). If 
\[
E_2) \{\alpha(z), (f^n(f^m - 1)f(z + \eta)^{(k)}) = E_2) \{\alpha(z), (g^n(g^m - 1)g(z + \eta)^{(k)}) \text{, then } f \equiv tg \text{ where } t^m = 1 \}
\]

In 2022, B. Saha, S. Pal and T. Biswas [19] proved the following results.

Theorem 7. [19] Let \( f \) and \( g \) be two transcendental entire functions of finite order, \( c_j \) \( (j = 1, \ldots, d) \) be finite complex constants and and \( \alpha \) (\( \neq 0 \)) be a small function of both \( f \) and \( g \) with finitely many zeros. Suppose that \( n(\geq 1), m(\geq 1) \) and \( k(\geq 0) \) are positive integers satisfying \( n \geq \max\{2k + m + \sigma + 5, \sigma + 2d + 3\} \). If 
\[
(f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j})^{(k)} \text{ and } (g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j})^{(k)} \text{ share } \text{”}(\alpha, 2)\text{” then } f \equiv tg
\]
for some constant \( t \) such that \( t^{n+\sigma} = t^m = 1 \).

Theorem 8. [19] Let \( f \) and \( g \) be two transcendental entire functions of finite order, \( c_j \) \( (j = 1, \ldots, d) \) be finite complex constants and and \( \alpha \) (\( \neq 0 \)) be a small function of both \( f \) and \( g \) with finitely many zeros. Suppose that \( n(\geq 1), m(\geq 1) \) and \( k(\geq 0) \) are integers satisfying \( n \geq \max\{3k + 2m + 2\sigma + 6, \sigma + 2d + 3\} \). If 
\[
(f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j})^{(k)} \text{ and } (g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j})^{(k)} \text{ share } (\alpha, 2)^* \text{ then conclusion of Theorem 7 holds.}
\]

Theorem 9. [19] Let \( f \) and \( g \) be two transcendental entire functions of finite order, \( c_j \) \( (j = 1, \ldots, d) \) be finite complex constants and and \( \alpha \) (\( \neq 0 \)) be a small function of both \( f \) and \( g \) with finitely many zeros. Suppose that \( n(\geq 1), m(\geq 1) \) and \( k(\geq 0) \) are integers satisfying \( n \geq \max\{5k + 4m + 4\sigma + 8, \sigma + 2d + 3\} \). If 
\[
E_2) \{\alpha(z), (f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j})^{(k)} \text{ and } E_2) \{\alpha(z), (g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j})^{(k)} \text{ then conclusion of Theorem 7 holds.}
\]

Now we prove the following results.
Theorem 10. Let \( f \) and \( g \) be two transcendental entire functions of finite order, \( c_j \) \((j = 1, \ldots, d)\) be finite complex constants and \( \alpha (\neq 0) \) be a small function of both \( f \) and \( g \) with finitely many zeros. Suppose that \( n(\geq 1), m(\geq 1) \) and \( k(\geq 0) \) are positive integers satisfying \( n \geq \max \{2k + \Gamma_1 + \sigma + 5, \sigma + 2d + 3\} \). If 
\[
\left( f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j} \right)^{(k)} \quad \text{and} \quad \left( g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j} \right)^{(k)} \quad \text{share "} \alpha, 2^{\ast} \text{" then } f \equiv tg \text{ for some constant } t \text{ such that } t^{n+\sigma} = t^{m} = 1.
\]

Theorem 11. Let \( f \) and \( g \) be two transcendental entire functions of finite order, \( c_j \) \((j = 1, \ldots, d)\) be finite complex constants and \( \alpha (\neq 0) \) be a small function of both \( f \) and \( g \) with finitely many zeros. Suppose that \( n(\geq 1), m(\geq 1) \) and \( k(\geq 0) \) are integers satisfying \( n \geq \max \{3k + 2\Gamma_1 + 2\sigma + 6, \sigma + 2d + 3\} \). If 
\[
\left( f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j} \right)^{(k)} \quad \text{and} \quad \left( g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j} \right)^{(k)} \quad \text{share} \ (\alpha, 2) \ast \text{ then conclusion of Theorem 10 holds.}
\]

Theorem 12. Let \( f \) and \( g \) be two transcendental entire functions of finite order, \( c_j \) \((j = 1, \ldots, d)\) be finite complex constants and \( \alpha (\neq 0) \) be a small function of both \( f \) and \( g \) with finitely many zeros. Suppose that \( n(\geq 1), m(\geq 1) \) and \( k(\geq 0) \) are integers satisfying \( n \geq \max \{5k + 5\Gamma_1 + 5\sigma + 6, \sigma + 2d + 3\} \). If 
\[
\left( \alpha(z), \left( f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j} \right)^{(k)} \right) \quad \text{and} \quad \left( \alpha(z), \left( g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j} \right)^{(k)} \right) \quad \text{then conclusion of Theorem 10 holds.}
\]

3. SOME LEMMAS

The following sequence of Lemmas will be helpful to prove our main results.

We denote \( H \) by the following function.
\[
H = \frac{F''}{F'} - \frac{2F'}{F - 1} - \frac{G''}{G'} + \frac{2G'}{G - 1}
\]

Lemma 1. [1] Suppose \( f \) is a meromorphic function in the complex plane \( \mathbb{C} \) and the polynomial is defined by \( P(z) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0 \), where \( a_n(\neq 0) \), \( a_0, a_1, \ldots, a_{n-1} \) are small functions of \( f \). Then 
\[
T(r, P(f)) = nT(r, f) + S(r, f).
\]
Lemma 2. [5] Let $H$ be defined as above. If $F$ and $G$ share \((1, 2)\) and $H \not= 0$, then\[ T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) \]
\[- \sum_{p=3}^{\infty} \overline{N}_p \left( r, \frac{G}{F^2} \right) + S(r, F) + S(r, G).\]

Lemma 3. [5] Let $H$ be defined as above. If $F$ and $G$ share \((1, 2)^*\) and $H \not= 0$, then\[ T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; G) \]
\[- m(r, 1; G) + S(r, F) + S(r, G).\]

Lemma 4. [6] Let $F$ and $G$ be two non-constant entire functions and $p \geq 2$ be an integer. If $\overline{E}_p(1, F) = \overline{E}_p(1, G)$ and $H \not= 0$, then\[ T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) + S(r, G).\]

Lemma 5. [14] Let $H$ be defined as above. If $F$ and $G$ share \((1, 2)^*\) and $H \equiv 0$ and\[ \lim_{r \to \infty} \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) \leq 1, r \in I.\]
where $T(r) = \max\{T(r, F), T(r, G)\}$ and $I$ is a set with linear measure, Then $F \equiv G$ or $FG \equiv 1$.

Lemma 6. [20] Let $f$ be a non-constant meromorphic function, and let $s, k$ be two positive integers. Then\[ N_s \left( r, \frac{1}{F} \right) \leq T(r, f^{(k)}) - T(r, f) + N_{s+k} \left( r, \frac{1}{f} \right) + S(r, f).\]
\[ N_s \left( r, \frac{1}{F} \right) \leq k\overline{N}(r, f) + N_{s+k} \left( r, \frac{1}{f} \right) + S(r, f).\]
Clearly, $\overline{N}(r, \frac{1}{f^{(s)}}) = N_1 \left( r, \frac{1}{f^{(s)}} \right)$.

Lemma 7. [13] Let $f$ and $g$ be two transcendental entire function of finite order and $c_j (j = 1, 2, \ldots, s)$ be finite complex constants. Let $m(\geq 1)$ and $n(\geq 1)$ be integers such that $n \geq \sigma + 2s + 3$. If\[ f^n(f^m - 1) \prod_{j=1}^{s} f(z + c_j)^{\mu_j} \equiv g^n(g^m - 1) \prod_{j=1}^{s} g(z + c_j)^{\nu_j}\]
then $f \equiv tg$ for some constant $t$ such that $t^m = t^{n+\sigma} = 1$

Lemma 8. [19] Let $f$ and $g$ be two entire functions, $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$ be integers and let us define $F = \left( f^n(f^m - 1) \prod_{j=1}^{s} f(z + c_j)^{\mu_j} \right)^{(k)}$ and $G = \left( g^n(g^m - 1) \prod_{j=1}^{s} g(z + c_j)^{\nu_j} \right)^{(k)}$. If there exists non-zero constants $c_1$ and $c_2$ such that $\overline{N}(r, c_1; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, c_2; G) = \overline{N}(r, 0; F)$ then $n \leq 2k + m + \sigma + 2$. 
4. PROOF OF MAIN RESULTS

Proof of Theorem 10.

Proof. Let \( F = \frac{F^{(k)}}{\alpha(x)} \) and \( G = \frac{G^{(k)}}{\alpha(x)} \) where \( F_1 = f^n(f^n - 1) \prod_{j=1}^d f(z + c_j)^{\mu_j} \) and \( G_1 = g^n(g^n - 1) \prod_{j=1}^d g(z + c_j)^{\mu_j} \). Then \( F \) and \( G \) are transcendental meromorphic functions that share “\((1, 2)\)” except the zeros and poles of \( \alpha(z) \). From Lemma 1, Lemma 2 and Lemma 6 we see that

\[
N_2(r, 0; F) \leq N_2(r, 0; F_1^{(k)}) \\
\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+2}(r, 0; F_1) + S(r, f) \\
\leq T(r, F) - (n + \Gamma_1 + \sigma)T(r, f) + N_{k+2}(r, 0; F_1) + S(r, f).
\]

which gives

\[
(n + \Gamma_1 + \sigma)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}(r, 0; F_1) + S(r, f). \tag{1}
\]

Also, by Lemma 6, we obtain,

\[
N_2(r, 0; F) \leq N_2(r, 0; F_1^{(k)}) + S(r, f) \\
\leq N_{k+2}(r, 0; F_1) + S(r, f). \tag{2}
\]

Similarly,

\[
N_2(r, 0; G) \leq N_{k+2}(r, 0; G_1) + S(r, g). \tag{3}
\]

By using the inequalities (2) and (3) and Lemma 2 we get from (1)

\[
(n + \Gamma_1 + \sigma)T(r, f) \leq N_2(r, 0; G) + N_2(r, \infty; F) \\
+ N_2(r, \infty; G) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, g) \\
\leq N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + S(r, f) + S(r, g) \\
\leq (k + 2) \{ N(r, 0; F_1) + N(r, 0; G_1) \} \\
+ N(r, 1; f^n) + N(r, 1; g^n) + N \left[ r, 0; \prod_{j=1}^d f(z + c_j)^{\mu_j} \right] \\
+ N \left[ r, 0; \prod_{j=1}^d g(z + c_j)^{\mu_j} \right] + S(r, f) + S(r, g) \\
\leq (k + \Gamma_1 + \sigma + 2) \{ T(r, f) + T(r, g) \} + S(r, f) + S(r, g).
\]

Therefore,

\[
(n + \Gamma_1 + \sigma)T(r, f) \leq (k + \Gamma_1 + \sigma + 2) \{ T(r, f) + T(r, g) \} + S(r, f) + S(r, g). \tag{4}
\]
Similarly,

\[(n + \Gamma_1 + \sigma)T(r, g) \leq (k + \Gamma_1 + \sigma + 2)\{T(r, f) + T(r, g)\} + + S(r, f) + S(r, g). \tag{5}\]

Adding the inequalities (4) and (5), we get,

\[(n - 2k - \Gamma_1 - \sigma - 4)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g).\]

which is obviously a contradiction as \(n \geq 2k + \Gamma_1 + \sigma + 5\).

Consider the case when \(H \equiv 0\). i.e.,

\[H = F'' F' - 2F' F - G'' G' + 2G' G - 1 \equiv 0\]

Integrating the above equation, we get,

\[\frac{1}{F - 1} = \frac{P}{G - 1} + Q \tag{6}\]

where \(P \neq 0\) and \(Q\) are integrating constants. From the equation (6) it is clear that \(F\) and \(G\) share 1 CM and hence they share \("(1, 2)\)\). Therefore \(n \geq 2k + \Gamma_1 + \sigma + 5\). Upon considering the some of the cases separately, we obtain as follows.

**Case 1.** Suppose \(Q \neq 0\) and \(P = Q\) then from equation (6), we get,

\[\frac{1}{F - 1} = \frac{QG}{G - 1}. \tag{7}\]

If \(B = -1\) then from equation (7), we get, \(FG \equiv 1\).

i.e., \[f^n(f - 1)(f^{m-1} + \ldots + 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j} \equiv \alpha^2. \]

It can be easily verified from above that, \(N(r, 0; f) = S(r, f)\) and \(N(r, 1; f) = S(r, f)\). Thus

\[\delta(0, f) + \delta(1, f) + \delta(\infty, f) = 3.\]

which is not possible.

If \(B = -1\) from equation (7), we have, \(\frac{1}{F} = \frac{QG}{(1+Q)G - 1}\) and so \(\overline{N}(r, \frac{1}{1+Q}; G) = \overline{N}(r, 0; F)\). Using Lemma 6 and Second main theorem of Nevanlinna, we get

\[T(r, G) \leq \overline{N}(r, 0; G) + \overline{N}(r, \frac{1}{1+Q}; G) + \overline{N}(r, \infty; G) + S(r, G)\]

\[\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G)\]

\[\leq N_{k+1}(r, 0; F_1) + T(r, G) + N_{k+1}(r, 0; G_1) - (n + \Gamma_1 + \sigma)T(r, g) + S(r, g).\]
Therefore,
\[(n + \Gamma_1 + \sigma)T(r, g) \leq (k + \Gamma_1 + \sigma + 1)\{T(r, f) + T(r, g)\} + S(r, g).\]  
(8)

Likewise, we also get,
\[(n + \Gamma_1 + \sigma)T(r, f) \leq (k + \Gamma_1 + \sigma + 1)\{T(r, f) + T(r, g)\} + S(r, f).\]  
(9)

From the inequalities (8) and (9) we obtain a contradiction as \(n \geq 2k + \Gamma_1 + \sigma + 2\).

**Case 2.** Let \(Q \neq 0 \text{ and } P \neq Q\), then from equation (6) we get,
\[F = \left(\frac{Q + 1}{QG + (P - Q)}\right)\]
and so \(\mathcal{N} \left( r, \frac{Q - P + 1}{Q}; G \right) = \mathcal{N}(r, 0; F)\). By providing the same argument as in case 1, we obviously get a contradiction.

**Case 3.** If \(Q = 0 \text{ and } P \neq 0\) then from equation (6) we get \(F = \frac{G + P - 1}{P}\) and \(G = PF - (P - 1)\). If \(P \neq 1\), it follows that \(\mathcal{N} \left( r, \frac{P - 1}{P}; F \right) = \mathcal{N}(r, 0; G)\) and \(N(r, 1 - A; G) = N(r, 0; F)\). Now by using Lemma 8, it can be shown that \(n \leq 2k + m + \sigma + 2\), a contradiction. Thus \(P = 1\) and then \(F \equiv G\) i.e.,
\[\left[f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j}\right]^{(k)} \equiv \left[g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j}\right]^{(k)}\]

Anti-Differentiate the above equation, we get,
\[\left[f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j}\right]^{(k-1)} \equiv \left[g^n(g^m - 1)g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j}\right]^{(k-1)} + E_{k-1}.\]

where \(E_{k-1}\) is a constant. If \(E_{k-1} \neq 0\), using Lemma 8 it follows that \(n \leq 2k + m + \sigma + 2\), which is a contradiction. Hence \(E_{k-1} = 0\). Repeating the above process \(k\) times we get
\[\left[f^n(f^m - 1) \prod_{j=1}^{d} f(z + c_j)^{\mu_j}\right] \equiv \left[g^n(g^m - 1) \prod_{j=1}^{d} g(z + c_j)^{\mu_j}\right]
which gives \(f = tg\), where \(t\) is a constant satisfying \(t^m = 1\). This completes the proof of Theorem 10.

\[\square\]

**Proof of Theorem 11.**
Proof. Let \( F, G, F_1 \) and \( G_1 \) be defined as in the proof of Theorem 10. Then \( F \) and \( G \) are transcendental meromorphic functions that share \((1, 2)^*\) except the zeros and poles of \( \alpha(z) \). Let \( H \neq 0 \). Then by using Lemma 3, Lemma 6 and Lemma 7, we get,

\[
(n + \Gamma_1 + \sigma)T(r, f) \leq N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, g) + (k \Gamma_1 + \sigma + 2)T(r, f) + S(r, f) + S(r, g).
\]

Therefore,

\[
(n + \Gamma_1 + \sigma)T(r, f) \leq (2k + 2 \Gamma_1 + 2 \sigma + 3)T(r, f) + (k + \Gamma_1 + \sigma + 2)T(r, g) + S(r, f) + S(r, g). \tag{10}
\]

Likewise,

\[
(n + \Gamma_1 + \sigma)T(r, g) \leq (2k + 2 \Gamma_1 + 2 \sigma + 3)T(r, g) + (k \Gamma_1 + \sigma + 2)T(r, f) + S(r, f) + S(r, g). \tag{11}
\]

Adding the inequalities (10) and (11), we get,

\[
(n + \Gamma_1 + \sigma)(T(r, f) + T(r, g)) \leq (3k + 3 \Gamma_1 + 3 \sigma + 5)T(r, f) + T(r, g) + S(r, f) + S(r, g).
\]

which is a contradiction as \( n \geq 3k + 2 \Gamma_1 + 2 \sigma + 5 \). Thus \( H \equiv 0 \). Proceeding similarly as done in Theorem 10 we get the proof of Theorem 11.

Proof of Theorem 12.

Proof. Let \( F, G, F_1 \) and \( G_1 \) be defined as in the proof of Theorem 10. Then \( F \) and \( G \) are transcendental meromorphic functions such that \( \overline{E}_2(1; F) = \overline{E}_2(1; G) \) except the zeros and poles of \( \alpha(z) \). Let \( H \neq 0 \). Then by using Lemma 4, Lemma 6, we get,

\[
(n + \Gamma_1 + \sigma)T(r, f) \leq N_2(r, 0; G) + 2 \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + N_{k+2}(r, 0; F_1) + S(r, f) + S(r, g) + N_{k+2}(r, 0; G_1) + 2N_{k+1}(r, 0; F_1) + N_{k+1}(r, 0; G_1) + S(r, f) + S(r, g) \leq (3k + 3 \Gamma_1 + 3 \sigma + 4)T(r, f) + (2k + 2 \Gamma_1 + 2 \sigma + 3)T(r, g) + S(r, f) + S(r, g).
\]

Therefore,

\[
(n + \Gamma_1 + \sigma)T(r, f) \leq (3k + 3 \Gamma_1 + 3 \sigma + 4)T(r, f) + (2k + 2 \Gamma_1 + 2 \sigma + 3)T(r, g) + S(r, f) + S(r, g). \tag{12}
\]

Likewise,

\[
(n + \Gamma_1 + \sigma)T(r, g) \leq (3k + 3 \Gamma_1 + 3 \sigma + 4)T(r, g) + (2k + 2 \Gamma_1 + 2 \sigma + 3)T(r, f) + S(r, f) + S(r, g). \tag{13}
\]
Adding the inequalities (12) and (13), we get,

\[(n+\Gamma_1+\sigma)\{T(r,f)+T(r,g)\} \leq (5k+5\Gamma_1+5\sigma+6)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\]

which is a contradiction as \(n \geq 5k+5\Gamma_1+5\sigma+6\). Thus \(H \equiv 0\). Proceeding similarly as done in Theorem 10 we get the proof of Theorem 12.

5. CONCLUSION
The main aim of this paper is to generalise and extend the some of the results of difference polynomial to Linear Difference Polynomial. Here, we considered the Linear Difference Polynomials sharing \(\sim (1,2)\), \(\sim (1,2)^*\) and without weighted sharing. The concepts of weakly weighted sharing, relaxed weighted sharing and without weighted sharing was introduced by A. Banerjee and S. Mukherjee [5] in 2007. By this way, we have obtained three results which extends the previous results of Pulak Sahoo - Gurudas Biswas [13].

6. OPEN QUESTIONS
Next, we pose an open question for further research.
1. What can be said about the transcendental meromorphic functions \(f\) and \(g\) if we consider the difference polynomial of the form \(\prod_{j=1}^{d} f(z + c_j)^{\mu_j}(k)\) where \(\prod_{j=1}^{d} f(z + c_j)^{\mu_j}\) is called Linear Difference Polynomial.

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AVAILABILITY OF DATA AND MATERIALS
Data sharing will not be applicable to this article as no data sets were generated or analysed during the current study.

COMPETING INTEREST
The authors declare that they have no competing interest.

REFERENCES


