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Some Results on Uniqueness of Meromorphic Functions Regarding Shifts and Derivatives

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Abstract

In this paper, we study the uniqueness problems of difference - differential polynomials which are the derivatives of difference products of meromorphic functions. The result of the paper extend some recent results of Meng and Li [7] and supplement to the previous result given by An and Khoai [1].

Keywords and prases: Uniqueness, Entire and Meromorphic functions, Weighted sharing, Difference-differential polynomial.

Subject classification: 30D35.

1. INTRODUCTION AND MAIN RESULTS

A meromorphic function f means meromorphic in the complex plane $\mathbb{C}$. If no poles occur, then f reduces to an entire function. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [3, 10]. For a non - constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}(r \to \infty, r \notin E)$.

Let f and g be two non-constant meromorphic functions on $\mathbb{C}$. Set $E(a,f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity m is counted m times in the set. If these zeros are only counted once, then we denote the set by $\overline{E}(a,f)$. If $E(a,f) = E(a,g)$, then we say that f and g share the value a CM; if $\overline{E}(a,f) = \overline{E}(a,g)$, then we say that f and g share the value a IM. Let m be a positive integer (or) infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a,f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a, f)$ the set of distinct a-points of f with multiplicities not greater than m.

For any constant a, we denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function for zeros of $f - a$ with multiplicity no more than k, and by $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{f-a}\right)$ be the counting function for zeros of $f - a$ with multiplicity at least k and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ be the corresponding one for which multiplicity is not counted. We set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

In 2001, I. Lahiri introduced the notion of weighted sharing, which measures how close a shared value is to being shared CM (or) to being shared IM. The definition is as follows.

Definition 1.1 [5, 6]. For a complex number $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. For a complex number $a \in \mathbb{C} \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with wieght k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a, k) to mean that f, g share the value a with wieght k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers $p, 0 \leq p < k$. Also we note that f, g share a value a IM (or) CM if and only if f, g share $(a, 0)$ (or) (a, ∞) respectively.

Definition 1.2 [5]. Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k a non-negative integer (or) ∞. We denote by $E_f(S, K)$ the set $\cup_{a \in S} E_k(a, f)$. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

In 1997, corresponding to the famous conjecture of Haymann [4], Yang and Hua[8] studied the unicity of differential monomials and obtain the following theorem.

Theorem A. Let f and g be two non - constant entire functions, $n \geq 6$ a positive integer. If $f^n f'$ and $g^n g'$ share 1CM, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2, c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.

In 2018, An and Khoai [1] considered the set of roots of unity of degree d and studied the relations of f and g when $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$ and they proved the following result.

Theorem B [1]. Let $f(z)$ and $g(z)$ be two non - constant meromorphic functions, and let n, d, k be positive integers with $n > 2k + \frac{2k+8}{d}, d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S) = E_{(g^n)^{(k)}}(S)$, then one of the following two cases holds:

1. $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$ for three non - zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$;

2. $f = tg$ with $t^{nd} = 1, t \in \mathbb{C}$.

In 2019, Chao Meng and Xu Li [7] proved the following results.

Theorem C. Let $f(z)$ and $g(z)$ be two non - constant meromorphic functions, and let n, d, k be positive integers with $n > 2k + \frac{3k+9}{d}, d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S, 1) = E_{(g^n)^{(k)}}(S, 1)$, then one of the following two cases holds:

1. $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$ for three non - zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$;

2. $f = tg$ with $t^{nd} = 1, t \in \mathbb{C}$.

Theorem D. Let $f(z)$ and $g(z)$ be two non - constant meromorphic functions, and let n, d, k be positive integers with $n > 2k + \frac{8k+14}{d}, d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n)^{(k)}}(S, 0) = E_{(g^n)^{(k)}}(S, 0)$, then one of the following two cases holds:

1. $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$ for three non - zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{nd}(nc)^{2kd} = 1$;

2. $f = tg$ with $t^{nd} = 1, t \in \mathbb{C}$.

In this paper, we consider Theorem B, C and D for difference implies that $f^n(z)$ will be replaced with $f^n(z)f(z+c)$, herein, c is a non - zero complex constant, n, k are positive integers. We present following four theorems which investigate the effect of weighted sharing on Theorem B, C and D.

Theorem 1.1. Let $f(z)$ and $g(z)$ be two non - constant meromorphic functions, and let n, d, k be positive integers with $n > 2k - 1 + \frac{\frac{11}{2}k+10}{d}, d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n f(z+c))^{(k)}}(S, 1) = E_{(g^n g(z+c))^{(k)}}(S, 1)$, then one of the following two cases holds:

1. $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$ for three non - zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{(n+1)d}((n+1)c)^{2kd} = 1$;

2. $f = tg$ with $t^{(n+1)d} = 1, t \in \mathbb{C}$.

Theorem 1.2. Let $f(z)$ and $g(z)$ be two non - constant meromorphic functions, and let n, d, k be positive integers with $n > 2k-1+\frac{13k+22}{d}$, $d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n f(z+c))^{(k)}}(S, 0) = E_{(g^n g(z+c))^{(k)}}(S, 0)$, then conclusion of Theorem 1.1 holds.
Theorem 1.3. Let $f(z)$ and $g(z)$ be two non - constant entire functions, and let n, d, k be positive integers with $n > 2k-1+\frac{3k+7}{d}$, $d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n f(z+c))^{(k)}}(S, 1) = E_{(g^n g(z+c))^{(k)}}(S, 1)$, then conclusion of Theorem 1.1 holds.
Theorem 1.4. Let $f(z)$ and $g(z)$ be two non - constant entire functions, and let n, d, k be positive integers with $n > 2k-1+\frac{5k+11}{d}$, $d \geq 2$, and $S = \{a \in \mathbb{C} : a^d = 1\}$. If $E_{(f^n f(z+c))^{(k)}}(S, 0) = E_{(g^n g(z+c))^{(k)}}(S, 0)$, then conclusion of Theorem 1.1 holds.

2. SOME LEMMAS

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

Where F and G are non - constant meromorphic functions defined in the complex plane $\mathbb{C}$.
Lemma 2.1 [2]. Let H be defined as above. If F and G share (1, 1) and $H \neq 0$, then

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G),$$

the same inequality holds for $T(r,G)$.
Lemma 2.2 [9]. Let f be a non - constant meromorphic function, k be a positive integer, then

$$N_p\left(r,\frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

where $N_p\left(r,\frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Clearly $\overline{N}\left(r,\frac{1}{f^{(k)}}\right) = N_1\left(r,\frac{1}{f^{(k)}}\right)$.
Lemma 2.3 [2]. Let H be defined as above. If F and G share (1, 0) and $H \not\equiv 0$, then

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) + 2\overline{N}\left(r,\frac{1}{F}\right) + 2\overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,F) + S(r,G), \tag{2.1}$$

the same inequality holds for $T(r, G)$.

Lemma 2.4. Let f be a non - constant meromorphic functions and n, k be positive integers, $n > 2k$. Then

$$(n-2k+1)T(r,f) + kN(r,f) + N\left(r, \frac{f^{n-k+1}}{(f^n f(z+c))^{(k)}}\right) \le T\left(r, (f^n f(z+c))^{(k)}\right) + S(r,f).$$

Proof. Using the same as in [[1], Lemma 2.6], we can easily obtain Lemma 2.4. □

Lemma 2.5. Let $f(z)$ and $g(z)$ be two non - constant entire functions and n, s, k be positive integers, $n > k+1$. If $\big(f^n f(z+c)\big)^{(k)}\big(g^n g(z+c)\big)^{(k)} = h$, $h \in \mathbb{C}$, $h \neq 0$, then $f = \lambda_1 e^{\lambda z}$ and $g = \lambda_2 e^{-\lambda z}$ for three non - zero constants λ_1, λ_2 and λ such that $(-1)^k(\lambda_1\lambda_2)^{(n+1)}((n+1)\lambda)^{2k} = h$.

Proof. Using the same as in [[11], Lemma 1], we can easily obtain Lemma 2.5. □

3. PROOF OF THE MAIN RESULTS

3.1. Proof of Theorem 1.1. Let

$$F = \left(\big(f^n f(z+c)\big)^{(k)}\right)^d, G = \left(\big(g^n g(z+c)\big)^{(k)}\right)^d \tag{3.1}$$

Since $E_{(f^n f(z+c))^{(k)}}(S,1) = E_{(g^n g(z+c))^{(k)}}(S,1)$, we see that F and G share (1, 1).
If $H \not\equiv 0$, then by Lemma 2.1

$$T(r,F) \le N_2\left(r, \frac{1}{F}\right) + N_2\big(r,F\big) + N_2\left(r, \frac{1}{G}\right) + N_2\big(r,G\big) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + \frac{1}{2}\overline{N}\big(r,F\big) + S(r,F) + S(r,G). \tag{3.2}$$

By Lemma 2.4, we obtain

$$(n-2k+1)T(r,f) \le T(r, (f^n f(z+c))^{(k)}) + S(r,f) \le (k+1)(n+1)T(r,f) + S(r,f) \tag{3.3}$$

$$(n-2k+1)T(r,g) \le T(r, (g^n g(z+c))^{(k)}) + S(r,g) \le (k+1)(n+1)T(r,g) + S(r,g). \tag{3.4}$$

Since

$$T\left(r, \big(\big(f^n f(z+c)\big)^{(k)}\big)^d\right) = dT\left(r, \big(f^n f(z+c)\big)^{(k)}\right) + S(r,f) \tag{3.5}$$

$$T\bigg(r, \big(\big(g^n g(z+c)\big)^{(k)}\big)^d\bigg) = dT\bigg(r, \big(g^n g(z+c)\big)^{(k)}\bigg) + S(r,g). \tag{3.6}$$

On the other hand

$$N_2\big(r,F\big) = 4\overline{N}(r,f) \tag{3.7}$$

$$N_2\big(r,G\big) = 4\overline{N}(r,g) \tag{3.8}$$

$$\frac{1}{2}\overline{N}\big(r,F\big) = \overline{N}\big(r,f\big). \tag{3.9}$$

By Lemma 2.2, we have

$$\begin{aligned} N_2\bigg(r,\frac{1}{F}\bigg) &= N_2\bigg(r, \frac{1}{\big(\big(f^n f(z+c)\big)^{(k)}\big)^d}\bigg) \\ &= 2\overline{N}\bigg(r, \frac{1}{\big(f^n f(z+c)\big)^{(k)}}\bigg) \\ &\le 2(k+1)\overline{N}\bigg(r,\frac{1}{f}\bigg) + 2N\bigg(r,\frac{1}{f}\bigg) + 4k\overline{N}(r,f) + S(r,f) \qquad (3.10) \\ \frac{1}{2}\overline{N}\bigg(r,\frac{1}{F}\bigg) &= \frac{1}{2}\overline{N}\bigg(r, \frac{1}{\big(\big(f^n f(z+c)\big)^{(k)}\big)^d}\bigg) \\ &= \frac{1}{2}\overline{N}\bigg(r, \frac{1}{\big(f^n f(z+c)\big)^{(k)}}\bigg) \\ &\le \frac{k+1}{2}\overline{N}\bigg(r,\frac{1}{f}\bigg) + \frac{1}{2}N\bigg(r,\frac{1}{f}\bigg) + k\overline{N}(r,f) + S(r,f). \qquad (3.11) \end{aligned}$$

And

$$\begin{aligned} N_2\bigg(r,\frac{1}{G}\bigg) &= 2\overline{N}\bigg(r, \frac{1}{\big(g^n g(z+c)\big)^{(k)}}\bigg) \\ &\le 2\bigg(\overline{N}\bigg(r,\frac{1}{g^{n-k+1}}\bigg) + N\bigg(r, \frac{g^{n-k+1}}{\big(g^n g(z+c)\big)^{(k)}}\bigg)\bigg) + S(r,g) \\ &\le 2\bigg(\overline{N}\bigg(r,\frac{1}{g}\bigg) + N\bigg(r, \frac{g^{n-k+1}}{\big(g^n g(z+c)\big)^{(k)}}\bigg)\bigg) + S(r,g) \qquad (3.12) \\ N_2\bigg(r,\frac{1}{F}\bigg) &\le 2\bigg(\overline{N}\bigg(r,\frac{1}{f}\bigg) + N\bigg(r, \frac{f^{n-k+1}}{\big(f^n f(z+c)\big)^{(k)}}\bigg)\bigg) + S(r,f). \qquad (3.13) \end{aligned}$$

Combining (3.2), (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12), we deduce

$$\begin{aligned} T\left(r, \left(\left(f^n f(z+c)\right)^{(k)}\right)^d\right) &\leq \frac{5k+5}{2}\overline{N}\left(r, \frac{1}{f}\right) + \frac{5}{2}N\left(r, \frac{1}{f}\right) + (5k+5)\overline{N}\big(r, f\big) \\ &\quad + 4\overline{N}(r, g) + 2\overline{N}\left(r, \frac{1}{g}\right) + 2N\left(r, \frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right) \\ &\quad + S(r, f) + S(r, g) \\ &\leq \left(\frac{11k+20}{2}\right)T(r, f) + 2k\overline{N}(r, f) + 6T(r, g) \\ &\quad + 2N\left(r, \frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right) + S(r, f) + S(r, g). \end{aligned} \tag{3.14}$$

Similarly, we have

$$\begin{aligned} T\left(r, \left(\left(g^n g(z+c)\right)^{(k)}\right)^d\right) &\leq \left(\frac{11k+20}{2}\right)T(r, g) + 2k\overline{N}(r, g) + 6T(r, f) \\ &\quad + 2N\left(r, \frac{f^{n-k+1}}{\left(f^n f(z+c)\right)^{(k)}}\right) + S(r, f) + S(r, g). \end{aligned} \tag{3.15}$$

By Lemma 2.4, we have

$$\begin{aligned} &(n-2k+1)dT(r, f) + kdN(r, f) \\ &\quad + dN\left(r, \frac{f^{n-k+1}}{\left(f^n f(z+c)\right)^{(k)}}\right) \\ &\quad \leq dT\big(r, \left(f^n f(z+c)\right)^{(k)}\big) + S(r, f). \end{aligned} \tag{3.16}$$

$$\begin{aligned} &(n-2k+1)dT(r, g) + kdN(r, g) \\ &\quad + dN\left(r, \frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right) \\ &\quad \leq dT\big(r, \left(g^n g(z+c)\right)^{(k)}\big) + S(r, g). \end{aligned} \tag{3.17}$$

From (3.14) to (3.17), we have

$$\begin{aligned}
&(n-2k+1)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k+1}}{\left(f^n f(z+c)\right)^{(k)}}\right)\\
&+(n-2k+1)dT(r,g)+kdN(r,g)+dN\left(r,\frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right)\\
&\le\left(\frac{11k+20}{2}\right)T(r,f)+2k\overline{N}(r,f)+6T(r,g)+2N\left(r,\frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right)\\
&+\left(\frac{11k+20}{2}\right)T(r,g)+2k\overline{N}(r,g)+6T(r,f)+2N\left(r,\frac{f^{n-k+1}}{\left(f^n f(z+c)\right)^{(k)}}\right)\\
&+S(r,f)+S(r,g). \qquad (3.18)
\end{aligned}$$

Since $d \geq 2$

$$dN\left(r,\frac{f^{n-k+1}}{\left(f^n f(z+c)\right)^{(k)}}\right)\ge 2N\left(r,\frac{f^{n-k+1}}{\left(f^n f(z+c)\right)^{(k)}}\right) \qquad (3.19)$$

$$dN\left(r,\frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right)\ge 2N\left(r,\frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right) \qquad (3.20)$$

$$kdN(r,f)\ge 2k\overline{N}(r,f) \qquad (3.21)$$

$$kdN(r,g)\ge 2k\overline{N}(r,g). \qquad (3.22)$$

Therefore

$$\begin{aligned}
&\left(nd-2kd+d-\frac{11}{2}k-10\right)T\big(r,f\big)+\left(nd-2kd+d-\frac{11}{2}k-10\right)T\big(r,g\big)\\
&\le S\big(r,f\big)+S\big(r,g\big), \qquad (3.23)
\end{aligned}$$

which contradicts with $n > 2k-1+\frac{\frac{11}{2}k+10}{d}$. Hence $H \equiv 0$. By integration, we get

$$\frac{1}{G-1}=\frac{A}{F-1}+B. \qquad (3.24)$$

Where $A \neq 0$ and B are constants. Thus

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)} \qquad (3.25)$$

and

$$F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}. \tag{3.26}$$

Next we consider the following three subcases:
Subcase 1. $B\neq 0,-1$. Then from (3.26) we get

$$\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)=\overline{N}(r,F). \tag{3.27}$$

By Nevanlinna Second Fundamental Theorem and (3.12), we get

$$\begin{aligned}T(r,G)&\leq \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-\frac{B+1}{B}}\right)+S(r,G)\\&\leq \overline{N}(r,G)+N_2\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,G)\\&\leq 2\overline{N}(r,g)+2\overline{N}\left(r,\frac{1}{g}\right)+2N\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right)+2\overline{N}(r,f)+S(r,f)\\&\quad+S(r,g). \end{aligned}\tag{3.28}$$

If $A-B-1\neq 0$, then it follows from (3.25) that

$$\overline{N}\left(r,\frac{1}{F-\frac{B+1-A}{B}}\right)=\overline{N}\left(r,\frac{1}{G}\right). \tag{3.29}$$

Again by Nevanlinna Second Fundamental Theorem and (3.13)

$$\begin{aligned}T(r,F)&\leq \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{B+1-A}{B+1}}\right)+S(r,F)\\&\leq \overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)\\&\leq 2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)\\&\quad+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g}\right)+2k\overline{N}(r,g)+S(r,f)+S(r,g). \end{aligned}\tag{3.30}$$

From (3.16), (3.17), (3.28) and (3.30), we get

$$\begin{aligned}
&(n-2k+1)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)\\
&+(n-2k+1)dT(r,g)+kdN(r,g)+dN\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right)\\
&\leq 4\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+(2k+2)\overline{N}(r,g)+(k+3)\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g}\right)\\
&\quad+2N\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)+2N\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right)+S(r,f)+S(r,g).
\end{aligned}\tag{3.31}$$

Since $d\geq 2$

$$dN\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)\geq 2N\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)\tag{3.32}$$

$$dN\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right)\geq 2N\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right)\tag{3.33}$$

$$kdN(r,f)\geq 2\overline{N}(r,f)\tag{3.34}$$

$$kdN(r,g)\geq 2(k+1)\overline{N}(r,g).\tag{3.35}$$

Therefore

$$(nd-2kd+d-4)T(r,f)+(nd-2kd+d-k-4)T(r,g)\leq S(r,f)+S(r,g),\tag{3.36}$$

which contradicts with $n>2k-1+\frac{\frac{11}{2}k+10}{d}$. Hence $A-B-1=0$. Then by (3.25)

$$\overline{N}\left(r,\frac{1}{F+\frac{1}{B}}\right)=N\big(r,G\big).\tag{3.37}$$

Again by Nevanlinna Second Fundamental Theorem,

$$\begin{aligned}
T(r,F)&\leq\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F+\frac{1}{B}}\right)+S(r,f)\\
&\leq\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+S(r,f)\\
&\leq 2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)+2\overline{N}(r,g)+S(r,f)\\
&\quad+S(r,g).
\end{aligned}\tag{3.38}$$

Combining (3.16), (3.17), (3.28) and (3.38), we have

$$\begin{aligned}
&(n-2k+1)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)\\
&+(n-2k+1)dT(r,g)+kdN(r,g)+dN\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right)\\
&\leq 4\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+4\overline{N}(r,g)+2\overline{N}\left(r,\frac{1}{g}\right)+2N\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)\\
&\quad+2N\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right)+S(r,f)+S(r,g). \qquad (3.39)
\end{aligned}$$

Since $d\geq 2$

$$dN\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right)\geq 2N\left(r,\frac{f^{n-k+1}}{\big(f^nf(z+c)\big)^{(k)}}\right) \qquad (3.40)$$

$$dN\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right)\geq 2N\left(r,\frac{g^{n-k+1}}{\big(g^ng(z+c)\big)^{(k)}}\right) \qquad (4.41)$$

$$kdN(r,f)\geq 2\overline{N}(r,f) \qquad (3.42)$$

$$kdN(r,g)\geq 2\overline{N}(r,g). \qquad (3.43)$$

Therefore,

$$(nd-2kd+d-4)T(r,f)+(nd-2kd+d-4)T(r,g)\leq S(r,f)+S(r,g), \qquad (3.44)$$

which contradicts our assumption.

Subcase 2. $B=-1$. Then

$$G=\frac{A}{A+1-F} \qquad (3.45)$$

and

$$F=\frac{(1+A)G-A}{G}. \qquad (3.46)$$

If $A+1\neq 0$. We obtain

$$\overline{N}\left(r,\frac{1}{F-A-1}\right)=\overline{N}\big(r,G\big) \qquad (3.47)$$

$$\overline{N}\Big(r, \frac{1}{G-\frac{A}{A+1}}\Big) = \overline{N}\Big(r, \frac{1}{F}\Big). \tag{3.48}$$

By similar arguments we can obtain a contradiction. Therefore $A+1=0$, then $FG \equiv 1$, that is $\big((f^n f(z+c))^{(k)}\big)^d \big((g^n g(z+c))^{(k)}\big)^d = 1$, we have $\big((f^n f(z+c))^{(k)}\big)\big((g^n g(z+c))^{(k)}\big) = h$, wher $h^d = 1$. Suppose z_0 is a zero of f with multiplicity p, then z_0 is a pole of g with multiplicity q such that $np-k = nq+k$. So $n(p-q)-2k = 0$. Since $n > 2k-1+\frac{\frac{11}{2}k+10}{d}$, we can deduce a contradiction. So $f(z) \neq 0$. Similarly, we can prove $f(z) \neq \infty$, $g(z) \neq 0$ and $g(z) \neq \infty$. So $f(z)$ and $g(z)$ are two non-constant entire functions. According to Lemma 2.5, we obtain $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$ for three non-zero constants c_1, c_2 and c such that $(-1)^{kd}(c_1c_2)^{(n+1)d}((n+1)c)^{2kd} = 1$.

Subcase 3. $B = 0$. Then (3.25) and (3.26) gives $G = \frac{F+A-1}{A}$ and $F = AG+1-A$. If $A-1 \neq 0$, then

$$\overline{N}\Big(r, \frac{1}{F+A-1}\Big) = \overline{N}\Big(r, \frac{1}{G}\Big) \tag{3.49}$$

and

$$\overline{N}\Big(r, \frac{1}{G+\frac{1-A}{A}}\Big) = \overline{N}\Big(r, \frac{1}{F}\Big). \tag{3.50}$$

Proceeding similarly as in subcase 1, we get a contradiction. Therefore $A-1=0$, then $F \equiv G$, that is, $\big((f^n f(z+c))^{(k)}\big)^d = \big((g^n g(z+c))^{(k)}\big)^d$. We have $\big(f^n f(z+c)\big)^{(k)} = h\big(g^n g(z+c)\big)^{(k)}$ with $h^d = 1$. This completes the proof of Theorem 1.1.

3.2. Proof of Theorem 1.2. Let

$$F = \Big(\big(f^n f(z+c)\big)^{(k)}\Big)^d, G = \Big(\big(g^n g(z+c)\big)^{(k)}\Big)^d \tag{3.51}$$

Since $E_{(f^n f(z+c))^{(k)}}(S,0) = E_{(g^n g(z+c))^{(k)}}(S,0)$, we see that F and G share (1, 0).
If $H \not\equiv 0$, then by Lemma 2.3

$$\begin{aligned} T(r,F) \leq{}& N_2\Big(r,\frac{1}{F}\Big) + N_2\big(r,F\big) + N_2\Big(r,\frac{1}{G}\Big) + N_2\big(r,G\big) + 2\overline{N}\Big(r,\frac{1}{F}\Big) \\ &+ 2\overline{N}\big(r,F\big) + \overline{N}\Big(r,\frac{1}{G}\Big) + \overline{N}\big(r,G\big) + S(r,F) + S(r,G). \end{aligned} \tag{3.52}$$

By Lemma 2.4, we obtain

$$\begin{aligned} (n-2k+1)T(r,f) \leq T(r,(f^n f(z+c))^{(k)}) + S(r,f) \leq (k+1)(n+1)T(r,f) \\ + S(r,f) \end{aligned} \tag{3.53}$$

$$(n-2k+1)T(r,g) \leq T(r,(g^n g(z+c))^{(k)}) + S(r,g) \leq (k+1)(n+1)T(r,g) + S(r,g) \tag{3.54}$$

$$2\overline{N}\big(r,F\big) = 4\overline{N}(r,f) \tag{3.55}$$

$$\overline{N}\big(r,G\big) = 2\overline{N}\big(r,g\big) \tag{3.56}$$

$$\begin{aligned}2\overline{N}\left(r,\frac{1}{F}\right) &= 2\overline{N}\left(r,\frac{1}{\left(\left(f^n f(z+c)\right)^{(k)}\right)^d}\right) = 2\overline{N}\left(r,\frac{1}{\left(f^n f(z+c)\right)^{(k)}}\right)\\ &\leq 2(k+1)\overline{N}\left(r,\frac{1}{f}\right) + 2N\left(r,\frac{1}{f}\right) + 4k\overline{N}(r,f) + S(r,f)\end{aligned} \tag{3.57}$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{G}\right) &= \overline{N}\left(r,\frac{1}{\left(\left(g^n g(z+c)\right)^{(k)}\right)^d}\right) = \overline{N}\left(r,\frac{1}{\left(g^n g(z+c)\right)^{(k)}}\right)\\ &\leq (k+1)\overline{N}\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g}\right) + 2k\overline{N}(r,g) + S(r,g).\end{aligned} \tag{3.58}$$

Combining (3.52), (3.5), (3.9), (3.10), (3.55), (3.56), (3.57), (3.58) and (3.12), we deduce

$$\begin{aligned}T\left(r,\left(\left(f^n f(z+c)\right)^{(k)}\right)^d\right) &\leq (4k+4)\overline{N}\left(r,\frac{1}{f}\right) + (8k+8)\overline{N}\big(r,f\big) + 4N\left(r,\frac{1}{f}\right)\\ &\quad + (k+3)\overline{N}\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g}\right) + (2k+2)\overline{N}(r,g)\\ &\quad + 2N\left(r,\frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right) + S(r,f) + S(r,g).\\ &\leq (10k+16)T(r,f) + 2k\overline{N}(r,f) + (3k+6)T(r,g)\\ &\quad + 2N\left(r,\frac{g^{n-k+1}}{\left(g^n g(z+c)\right)^{(k)}}\right) + S(r,f) + S(r,g).\end{aligned} \tag{3.59}$$

Similarly, we have

$$\begin{aligned}T\left(r,\left(\left(g^n g(z+c)\right)^{(k)}\right)^d\right) &\leq (10k+16)T(r,g) + 2k\overline{N}(r,g) + (3k+6)T(r,f)\\ &\quad + 2N\left(r,\frac{f^{n-k+1}}{\left(f^n f(z+c)\right)^{(k)}}\right) + S(r,f) + S(r,g).\end{aligned} \tag{3.60}$$

By Lemma 2.4, we have,

$$(n-2k+1)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k+1}}{\left(f^nf(z+c)\right)^{(k)}}\right)$$
$$\leq dT\left(r,\left(f^nf(z+c)\right)^{(k)}\right)+S(r,f) \tag{3.61}$$
$$(n-2k+1)dT(r,g)+kdN(r,g)+dN\left(r,\frac{g^{n-k+1}}{\left(g^ng(z+c)\right)^{(k)}}\right)$$
$$\leq dT\left(r,\left(g^ng(z+c)\right)^{(k)}\right)+S(r,g). \tag{3.62}$$

From (3.59), (3.60), (3.61) and (3.62), we have

$$(n-2k+1)dT(r,f)+kdN(r,f)+dN\left(r,\frac{f^{n-k+1}}{\left(f^nf(z+c)\right)^{(k)}}\right)$$
$$+(n-2k+1)dT(r,g)+kdN(r,g)+dN\left(r,\frac{g^{n-k+1}}{\left(g^ng(z+c)\right)^{(k)}}\right)$$
$$\leq(10k+16)T(r,f)+(3k+6)T(r,g)+2k\overline{N}(r,f)+2N\left(r,\frac{g^{n-k+1}}{\left(g^ng(z+c)\right)^{(k)}}\right)$$
$$+(10k+16)T(r,g)+(3k+6)T(r,f)+2k\overline{N}(r,g)+2N\left(r,\frac{f^{n-k+1}}{\left(f^nf(z+c)\right)^{(k)}}\right)$$
$$+S(r,f)+S(r,g). \tag{3.63}$$

Since $d\geq 2$

$$dN\left(r,\frac{f^{n-k+1}}{\left(f^nf(z+c)\right)^{(k)}}\right)\geq 2N\left(r,\frac{f^{n-k+1}}{\left(f^nf(z+c)\right)^{(k)}}\right) \tag{3.64}$$

$$dN\left(r,\frac{g^{n-k+1}}{\left(g^ng(z+c)\right)^{(k)}}\right)\geq 2N\left(r,\frac{g^{n-k+1}}{\left(g^ng(z+c)\right)^{(k)}}\right) \tag{3.65}$$

$$kdN(r,f)\geq 2k\overline{N}(r,f) \tag{3.66}$$

$$kdN(r,g)\geq 2k\overline{N}(r,g). \tag{3.67}$$

Therefore,

$$(nd-2kd+d-13k-22)T(r,f)+(nd-2kd+d-13k-22)T(r,g)$$
$$\leq S(r,f)+S(r,g), \tag{3.68}$$

which contradicts with $n > 2k - 1 + \frac{13k+22}{d}$. Hence $H \equiv 0$. Similar to the arguments in Theorem 1.1, we see that Theorem 1.2 holds.

3.3. Proof of Theorem 1.3. Since f and g are entire functions, we have $\overline{N}\big(r, f\big) = \overline{N}\big(r, g\big) = 0$. Proceeding as in the proof of Theorem 1.1 and applying Lemma 2.4 we shall obtain that Theorem 1.3 holds.

3.4. Proof of Theorem 1.4. Since f and g are entire functions, we have $\overline{N}\big(r, f\big) = \overline{N}\big(r, g\big) = 0$. Proceeding as in the proof of Theorem 1.1 and applying Lemma 2.4 we shall obtain that Theorem 1.4 holds.

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