New Oscillation Criteria for Half-Linear Second-Order Neutral Advanced Difference Equations

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Abstract

We obtained oscillation criteria for the second-order half-linear neutral advanced difference equations of the kind

$$\Delta(\alpha(\zeta)(\Delta w(\zeta))^{\delta}) + \eta(\zeta)y^{\delta}(\zeta + \kappa) = 0; \quad \zeta \ge \zeta_0,$$

where $w(\zeta) = y(\zeta) + p(\zeta)y(\zeta + \xi)$. We provide a new oscillation condition, which significantly improves the existing ones, by providing a new axiom bound for a non-oscillatory solution. The derived oscillation constant is unimprovable in a certain nonneutral case.

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1. INTRODUCTION

The goal of this research is to investigate the oscillation of the second-order neutral difference equation of advanced argument of the type

$$\Delta(\alpha(\zeta)(\Delta z(\zeta))^{\delta}) + \eta(\zeta)y^{\delta}(\zeta + \kappa) = 0; \quad \zeta \ge \zeta_0. \tag{1.1}$$

where $w(\zeta) = \alpha(\zeta) + p(\zeta)y(\zeta + \xi)$ and Δ is the forward difference operator defined by $\Delta y(\zeta) = y(\zeta + 1) - y(\zeta)$. Without further mention, we will assume that

- (C_1) $\delta > 0$ is a ratio of odd positive integers;
- (C_2) ξ is an integer and κ is a positive integer;
- $(C_3) \ \{\alpha(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is sequence of positive real numbers;
- (C_4) $\{\eta(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is non-increasing, non-negative real numbers and $\eta(\zeta) \not\equiv 0$ for sufficiently large values of ζ ;
- (C_5) $\{p(\zeta)\}_{\zeta=\zeta_0}^{\infty}$ is a non-negative real sequence;
- (C_6) The equation (1.1) is in so-called non-canonical form, i.e.,

$$\theta(\zeta) := \sum_{s=\zeta}^{\infty} \frac{1}{\alpha^{\frac{1}{\delta}}(s)} < \infty; \tag{1.2}$$

 (C_7) there exists a constant p_0 with $0 \le p_0 < 1$ and

$$p_0(\zeta) \ge \begin{cases} p(\zeta) \frac{\theta(\zeta + \xi)}{\theta(\zeta)} & \text{for } \xi \le 0, \\ p(\zeta) & \text{for } \xi \ge 0. \end{cases}$$
 (1.3)

Let $\phi = min\{0, \xi\}$. A solution of (1.1) is a real sequence $\{y(\zeta)\}$ which is defined for all $\zeta \geq \zeta_0 + \phi$ and satisfies equation (1.1) for all $\zeta \in N(\zeta_0) = \{\zeta_0, \zeta_0 + 1, \zeta_0 + 2, \ldots\}$. A nontrivial solution $\{y(\zeta)\}$ of equation (1.1) is said to be oscillatory if it a neither eventually positive nor eventually negative and it is non-oscillatory otherwise.

A large number of authors have studied into second-order linear difference equations. This is most likely due to the fact that such equations are functional relations using three consecutive terms of sequences, which are the next most difficult after functional relations involving two consecutive terms of sequences on the complexity scale.

Although there are a variety of dynamical behaviours for solutions to second-order difference equations, we will only consider conditions that are sufficient for all solutions

of [16] to be oscillatory in this paper. Several recent articles, particularly those by Zhang and Cheng [19], Wong and Agarwal [17, 18] as well as Thandapani et. al [16] have attracted our interest.

The study of oscillation for solutions of second-order neutral difference equations has received much interest. Second-order neutral difference equations are discussed in the articles [14, 15], whereas the second-order self-conjugate neutral difference equation is discussed in the work [4].

The application of the difference equation with advanced argument is in the progression of growth rate, which does not only rely on the present but also extends into the future. Incorporating a more advanced argument persuades the activities that are immediately available and useful for economic crises, population dynamics and decision-making, for example, are phenomenal problems that are expected to exhibit [1, 6]. The following equation can be used to describe the population of the future population growth limit

$$\Delta(a(\zeta))\Delta y(\zeta) + p(\zeta)y(g(\zeta)) = 0.$$

Despite the fact that delay difference equations have received a lot of attention, there have been few investigations on equations with advanced arguments [8, 10, 12, 13, 20].

Murugesan et al. [10] derived sufficient conditions for oscillation to the second-order advanced difference equation

$$\Delta(\alpha(\zeta)(\Delta y(\zeta))^{\delta}) + \eta(\zeta)y^{\delta}(\zeta + \kappa) = 0, \quad \zeta \ge \zeta_0.$$

Under the condition that $\sum_{\zeta=\zeta_0}^{\infty} \frac{1}{\alpha^{\frac{1}{\delta}}(\zeta)} < \infty$.

Dinakar et al. [5] derived sharp conditions for the oscillation of every solution of the second-order advanced noncanonical difference equation

$$\Delta(a(\zeta)(\Delta y(\zeta))^{\delta}) + \eta(\zeta)y^{\delta}(\kappa(\zeta)) = 0, \quad \zeta \ge \zeta_0.$$

In [8], we derived oscillatory conditions for the second-order noncanonical difference equation of the delay and advanced type

$$\Delta(\alpha(\zeta)\Delta y(\zeta)) + \eta(\zeta)y(\zeta + \kappa) = o; \quad \zeta \ge \zeta_0.$$

In [7], we derived single-condition criteria for the oscillation to the equation (1.1) for $p(\zeta) \equiv 0$.

We shall enhance the lower bound of the quantity $w(\zeta)/w(\zeta + \kappa)$ consecutively up to its limit value in this study, using a different technique. The oscillation criteria for (1.1)

is the major conclusion of the study, and it is a direct result of the observed lower and upper bounds of a nonoscillatory solution.

2. MAIN RESULTS

All functional inequalities that occur in the following are assumed to hold eventually, that is, they are satisfied for all ζ large enough. We only need to be concerned with positive solutions of (1.1) in the argument of the major conclusions, as usual and without compromising generality, because the proofs for eventually negative solutions are comparable.

Our results are based on the assumption that ω_* must be positive, where

$$\omega_* = \frac{1}{\delta} \liminf_{\zeta \to \infty} \alpha^{\frac{1}{\delta}}(\zeta + \kappa) \theta^{\delta + 1}(\zeta + \kappa + 1) \eta(\zeta).$$

Also, let us define

$$\sigma_* := \liminf_{\zeta \to \infty} \frac{\theta(\zeta)}{\theta(\zeta + \kappa)}$$

and notice that $\sigma_* \geq 1$. We will frequently use the fact that there exists a $\zeta_* \geq \zeta_0$, sufficiently large, with the property that for arbitrary fixed $\sigma \in [1, \sigma_*)$ and $\omega \in (0, \omega_*)$,

$$\eta(\zeta)\alpha^{\frac{1}{\delta}}(\zeta+\kappa)\theta^{\delta+1}(\zeta+\kappa+1) \ge \delta\omega \text{ and } \frac{\theta(\zeta)}{\theta(\zeta+\kappa)} \ge \sigma \text{ on } \zeta \ge \zeta_*.$$
 (2.1)

Let us define a sequence $\{\omega_k\}$ by

$$\omega_0 := (1 - p_0) \sqrt[\delta]{\omega_*},$$

$$\omega_{\zeta+1} := \frac{\omega_0 \sigma_*^{\omega_k}}{\sqrt[\delta]{1 - \omega_k}}, \quad k \in \mathbb{N}_0.$$

By induction, it is easy to check that if for some $k \in \mathbb{N}_0$, $\omega_i < 1$, $i = 0, 1, 2, \dots, k$, then ω_{k+1} exists and

$$\omega_{k+1} = l_k \omega_k > \omega_k, \tag{2.2}$$

where

$$l_0: = \frac{\sigma_*^{\omega_0}}{\sqrt[\delta]{1 - \omega_0}},$$

$$l_{k+1}: = \sigma_*^{\omega_0(l_k - 1)} \sqrt[\delta]{\frac{1 - \omega_k}{1 - l_k \omega_k}}, \quad k \in \mathbb{N}_0.$$

$$(2.3)$$

Lemma 2.1. Let $\omega_* > 0$ and $\{y(\zeta)\}$ be an eventually positive solution of (1.1). Then $\{w(\zeta)\}$ satisfies the followings, eventually:

(i)
$$w(\zeta) > 0$$
, $\Delta(\alpha(\zeta)(\Delta w(\zeta)))^{\delta} < 0$, and $y(\zeta) \geq w(\zeta) - p(\zeta)w(\zeta + \xi)$;

(ii)
$$\Delta w(\zeta) < 0$$
;

(iii)
$$\Delta\left(\frac{w(\zeta)}{\theta(\zeta)}\right) \geq 0;$$

(iv)
$$y(\zeta) \ge (1 - p_0)w(\zeta)$$
;

(v)
$$\lim_{\zeta \to \infty} w(\zeta) = 0$$
.

Proof. Let $\{y(\zeta)\}$ be an eventually positive solution of (1.1). Then, we can find $\zeta_1 \geq \zeta_0$ with $y(\zeta) > 0$, $y(\zeta + \xi) > 0$, and $y(\zeta + \kappa) > 0$ for $\zeta \geq \zeta_1$.

(i) Obviously, for all $\zeta \geq \zeta_1$, $w(\zeta) \geq y(\zeta) > 0$, and $\alpha(\zeta)(\Delta w(\zeta))^{\delta}$ is decreasing sequence and of one sign because

$$\Delta(\alpha(\zeta)(\Delta w(\zeta))^{\delta}) \le -\eta(\zeta)y^{\delta}(\zeta + \kappa) < 0.$$

Now,

$$y(\zeta) = w(\zeta) - p(\zeta)y(\zeta + \tau) \ge w(\zeta) - p(\zeta)w(\zeta + \tau), \quad \zeta \ge \zeta_1. \tag{2.4}$$

(ii) On the contrary, assume that $\Delta w(\zeta) > 0$ for $\zeta \geq \zeta_1$. First, we show that

$$y(\zeta) \ge \mu w(\zeta), \quad \zeta \ge \zeta_2 \ge \zeta_1,$$
 (2.5)

where

$$\mu := \begin{cases} 1 - p_0, & \xi \le 0, \\ 1 - \epsilon p_0, & \xi > 0, \end{cases}$$

 p_0 is given by (1.3) and $\epsilon \in (1, \frac{1}{p_0})$ which is arbitrary fixed.

If $\xi \leq 0$, (2.4) gives

$$y(\zeta) \ge w(\zeta)(1 - p(\zeta)) \ge w(\zeta) \left(1 - p(\zeta)\frac{\theta(\zeta + \xi)}{\theta(\zeta)}\right) \ge w(\zeta)(1 - p_0) > 0, \quad \zeta \ge \zeta_2,$$

where we made use of the fact that $\{\theta(\zeta)\}$ is decreasing and $\{w(\zeta)\}$ is increasing.

Suppose that $\xi > 0$. Since, $\{\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta)\}$ is decreasing, we have

$$w(\zeta) = w(\zeta_1) + \sum_{s=\zeta_1}^{\zeta-1} \frac{1}{\alpha^{\frac{1}{\delta}}(s)} \alpha^{\frac{1}{\delta}}(s) \Delta w(s) \ge \alpha^{\frac{1}{\delta}}(\zeta) \Delta w(\zeta) \sum_{s=\zeta_1}^{\zeta-1} \frac{1}{\alpha^{\frac{1}{\delta}}(s)} = \alpha^{\frac{1}{\delta}}(\zeta) \Delta w(\zeta) R(\zeta, \zeta_1),$$

where $R(\zeta, \zeta_1) = \sum_{s=\zeta_1}^{\zeta_1} \frac{1}{\alpha^{\frac{1}{\delta}(s)}}$, and hence

$$\Delta\left(\frac{w(\zeta)}{R(\zeta,\zeta_1)}\right) = \frac{\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta)R(\zeta,\zeta_1) - w(\zeta)}{\alpha^{\frac{1}{\delta}}(\zeta)R(\zeta,\zeta_1)R(\zeta+1,\zeta_1)} \le 0.$$

Using this monotonicity in (2.4), we get

$$y(\zeta) \ge w(\zeta) \left(1 - \frac{R(\zeta + \xi, \zeta_1)}{R(\zeta, \zeta_1)} p(\zeta)\right).$$

Now, we see that

$$\lim_{\zeta \to \infty} \frac{R(\zeta + \xi, \zeta_1)}{R(\zeta, \zeta_1)} = 1,$$

we have, for any $\epsilon \in (1, \frac{1}{p_0})$ and $\zeta_2 \ge \zeta_1$ sufficiently large,

$$y(\zeta) \ge w(\zeta)(1 - \epsilon p_0), \quad \zeta \ge \zeta_2.$$

Thus, (2.5) is true in both cases. By using (2.5) in (1.1), we get

$$\Delta(\alpha(\zeta)(\Delta w(\zeta))^{\delta}) + \mu^{\delta}\eta(\zeta)w^{\delta}(\zeta + \kappa) \leq 0, \quad \zeta \geq \zeta_2$$

Summing the above inequality from ζ_2 to $\zeta-1$, and using that $\{w(\zeta)\}$ is increasing sequence, we obtain

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq \alpha(\zeta_2)(\Delta w(\zeta_2))^{\delta} - \mu^{\delta} \sum_{s=\zeta_2}^{\zeta-1} \eta(s) w^{\delta}(s+\kappa)$$

$$\leq \alpha(\zeta_2)(\Delta w(\zeta_2))^{\delta} - \mu^{\delta} w^{\delta}(\zeta_2 + \kappa) \sum_{s=\zeta_2}^{\zeta_1} \eta(s). \tag{2.6}$$

For any $\omega \in (0, \omega_*)$ satisfying (2.1), there exists $\zeta_3 \ge \zeta_2$ such that

$$\sum_{s=\zeta_{3}}^{\zeta-1} \eta(s) \ge \omega \sum_{s=\zeta_{3}}^{\zeta-1} \frac{\delta}{\alpha^{\frac{1}{\delta}}(s+\kappa)\theta^{\delta+1}(s+\kappa+1)}$$

$$\ge \omega \left[\frac{1}{\theta^{\delta}(\zeta+\kappa)} - \frac{1}{\theta^{\delta}(\zeta_{3}+\kappa)} \right], \quad \zeta \ge \zeta_{3}. \tag{2.7}$$

Using (2.7) along with the fact that $\lim_{\zeta\to\infty}\theta(\zeta)=0$ in (2.6), we see that

$$\begin{split} \alpha(\zeta)(\Delta w(\zeta))^\delta &\leq \alpha(\zeta_2)(\Delta w(\zeta_2))^\delta \\ &- \mu^\delta \omega w^\delta(\zeta_2 + \kappa) \left[\frac{1}{\theta^\delta(\zeta + \kappa)} - \frac{1}{\theta^\delta(\zeta_3 + \kappa)} \right] \to -\infty \text{ as } \zeta \to \infty, \end{split}$$

which contradicts the positivity of $\{\Delta w(\zeta)\}$.

(iii) Since $\{\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta)\}$ is a negative and decreasing sequence, we have

$$w(\zeta) \ge -\sum_{s=\zeta}^{\infty} \frac{1}{\alpha^{\frac{1}{\delta}}(s)} \alpha^{\frac{1}{\delta}}(s) \Delta w(s) \ge -\theta(\zeta) \alpha^{\frac{1}{\delta}}(\zeta) \Delta w(\zeta). \tag{2.8}$$

Hence

$$\Delta\left(\frac{w(\zeta)}{\theta(\zeta)}\right) = \frac{\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta)\theta(\zeta) + w(\zeta)}{\alpha^{\frac{1}{\delta}}(\zeta)\theta(\zeta)\theta(\zeta + 1)} \ge 0.$$

- (iv) This can be proved directly from (C_7) and (2.5).
- (v) From (i) and (ii), $\{w(\zeta)\}$ is positive and decreasing sequence and so, $\lim_{\zeta\to\infty}w(\zeta)=l\geq 0$. On the contrary, suppose that $w(\zeta)\geq l>0$ for $\zeta\geq \zeta_2\geq \zeta_1$. Consequently, (1.1) becomes

$$\Delta(\alpha(\zeta)(\Delta w(\zeta))^{\delta}) \le -l^{\delta}(1 - p_0)^{\delta}\eta(\zeta), \quad \zeta \ge \zeta_2. \tag{2.9}$$

Summing (2.9) from ζ_2 to $\zeta-1$ and using (2.7) we have, for any $\omega \in (0, \omega_*)$ and $\zeta_3 \ge \zeta_2$ large enough,

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq -l^{\delta}(1-p_0)^{\delta} \sum_{s=\zeta_2}^{\zeta-1} \eta(s)$$

$$\leq -l^{\delta}(1-p_0)^{\delta} \omega \left[\frac{1}{\theta^{\delta}(\zeta+\kappa)} - \frac{1}{\theta^{\delta}(\zeta_2+\kappa)} \right]$$

$$\Delta w(\zeta) \leq -\frac{l(1-p_0)\sqrt[\delta]{\omega}}{\alpha^{\frac{1}{\delta}}(\zeta)} \left[\frac{1}{\theta^{\delta}(\zeta+\kappa)} - \frac{1}{\theta^{\delta}(\zeta_2+\kappa)} \right]^{\frac{1}{\delta}}.$$

Summing the above inequality from ζ_3 to $\zeta - 1$, we obtain

$$w(\zeta) \leq w(\zeta_3) - l(1 - p_0) \sqrt[\delta]{\omega} \sum_{s=\zeta_3}^{\zeta-1} \frac{1}{\alpha^{\frac{1}{\delta}}(s)} \left[\frac{1}{\theta^{\delta}(s+\kappa)} - \frac{1}{\theta^{\delta}(\zeta_2 + \kappa)} \right]^{\frac{1}{\delta}}.$$

$$= w(\zeta_3) - l(1 - p_0) \sqrt[\delta]{\omega} \sum_{s=\zeta_3}^{\zeta-1} \frac{1}{\alpha^{\frac{1}{\delta}}(s)\theta^{\delta}(s+\kappa)} \left[1 - \frac{\theta^{\delta}(s+\kappa)}{\theta^{\delta}(\zeta_2 + \kappa)} \right]^{\frac{1}{\delta}}$$

$$\leq w(\zeta_3) - l(1 - p_0) \sqrt[\delta]{\omega} \left[1 - \frac{\theta^{\delta}(\zeta_3 + \kappa)}{\theta^{\delta}(\zeta_2 + \kappa)} \right] \sum_{s=\zeta_3}^{\zeta-1} \frac{1}{\alpha^{\frac{1}{\delta}}(s)\theta^{\delta}(s+\kappa)}$$

$$w(\zeta) \leq w(\zeta_3) - l(1 - p_0) \sqrt[\delta]{\omega} \left[1 - \frac{\theta^{\delta}(\zeta_3 + \kappa)}{\theta^{\delta}(\zeta_2 + \kappa)} \right]^{\frac{1}{\delta}} \frac{R(\zeta, \zeta_3)}{\theta(\zeta_3 + \kappa)} \to -\infty \text{ as } \zeta \to \infty.$$

This is a contradiction. Hence l = 0. This completes the proof.

Lemma 2.2. Suppose that $\omega_* > 0$. If $\{y(\zeta)\}$ is an eventually positive solution of (1.1), then for any $\epsilon_0 \in (0,1)$,

(i)
$$\Delta\left(\frac{w(\zeta)}{\theta^{\epsilon_0\omega_0}(\zeta)}\right) < 0$$
 eventually;

(ii)
$$\epsilon_0 \omega_0 < 1$$
;

(iii)
$$\sigma_* < \infty$$
.

Proof. Pick $\zeta_1 \ge \zeta_0$ with $y(\zeta) > 0$, $y(\zeta + \xi) > 0$, and $y(\zeta + \kappa) > 0$ for $\zeta \ge \zeta_1$. $\{w(\zeta)\}$ satisfies Lemma 2.1 for $\zeta \ge \zeta_1$, and (2.1) is true.

Applying Lemma 2.1 (iv) in (1.1), we get

$$\Delta(\alpha(\zeta)(\Delta w(\zeta))^{\delta}) + (1 - p_0)^{\delta} w^{\delta}(\zeta + \kappa) \eta(\zeta) \le 0, \quad \zeta \ge \zeta_0,$$

which in view of (2.1) implies

$$\Delta(\alpha(\zeta)(\Delta w(\zeta))^{\delta}) + \frac{\omega\delta(1-p_0)^{\delta}}{\alpha^{\frac{1}{\delta}}(\zeta+\kappa)\theta^{\delta+1}(\zeta+\kappa+1)} w^{\delta}(\zeta+\kappa) \le 0.$$
 (2.10)

(i) Summing (2.10) from ζ_1 to $\zeta - 1$ and applying Lemma 2.1 (ii), we see that

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq \alpha(\zeta_{1})(\Delta w(\zeta_{1}))^{\delta} - \omega(1 - p_{0})^{\delta} \sum_{s=\zeta_{1}}^{\zeta-1} \frac{\delta}{\alpha^{\frac{1}{\delta}}(s+\kappa)\theta^{\delta+1}(s+\kappa+1)} w^{\delta}(s+\kappa)$$

$$\leq \alpha(\zeta_{1})(\Delta w(\zeta_{1}))^{\delta} - \omega(1 - p_{0})^{\delta} w^{\zeta}(\zeta+\kappa) \sum_{s=\zeta_{1}}^{\zeta-1} \frac{\delta}{\alpha^{\frac{1}{\delta}}(s+\kappa)\theta^{\delta+1}(s+\kappa+1)}$$

$$\leq \alpha(\zeta_{1})(\Delta w(\zeta_{1}))^{\delta} - \omega(1 - p_{0})^{\delta} w^{\delta}(\zeta+\kappa) \left[\frac{1}{\theta^{\delta}(\zeta+\kappa)} - \frac{1}{\theta^{\delta}(\zeta_{1}+\kappa)} \right]. \tag{2.11}$$

By Lemma 2.1 (v), there is $\zeta_2 \ge \zeta_1$ such that

$$\alpha(\zeta_1)(\Delta w(\zeta_1))^{\delta} + \frac{\omega(1-p_0)^{\delta}w^{\delta}(\zeta+\kappa)}{\theta^{\delta}(\zeta_1+\kappa)} < 0, \quad \zeta \ge \zeta_2.$$

Hence, using (2.11) and the non-decreasing nature of $\{w(\zeta)/\theta(\zeta)\}\$, we obtain

$$\alpha(\zeta)(\Delta w(\zeta)^{\delta} < -\omega(1-p_0)^{\delta} \frac{w^{\delta}(\zeta)}{\theta^{\delta}(\zeta)},$$

that is

$$\theta(\zeta)\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta) < -\sqrt[\delta]{\omega}(1-p_0)w(\zeta) = -\epsilon_0\omega_0 w(\zeta),$$

where $\epsilon_0 = \sqrt[\delta]{\omega/\omega_*}$. Therefore,

$$\Delta\left(\frac{w(\zeta)}{\theta^{\epsilon_0\omega_0}(\zeta)}\right) \leq \frac{\theta(\zeta)\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta) + \epsilon_0\omega_0w(\zeta)}{\alpha^{\frac{1}{\delta}}(\zeta)\theta(\zeta)\theta^{\epsilon_0\omega_0}(\zeta+1)} < 0, \quad \zeta \geq \zeta_2.$$

- (ii) The result is based on the reality that $\left\{\frac{w(\zeta)}{\theta(\zeta)}\right\}$ is increasing and $\left\{\frac{w(\zeta)}{\theta^{\epsilon_0\omega_0}(\zeta)}\right\}$ is decreasing.
- (iii) Suppose that $\sigma_* = \infty$. By (i), for arbitrary fixed $\sigma \in [1, \infty)$, there exists $\zeta_3 \geq \zeta_2$ such that

$$w(\zeta + \kappa) \ge \left(\frac{\theta(\zeta + \kappa)}{\theta(\zeta + 2\kappa)}\right)^{\epsilon_0 \omega_0} w(\zeta + 2\kappa) \ge \sigma^{\epsilon_0 \omega_0} w(\zeta + 2\kappa), \quad \zeta \ge \zeta_3.$$
 (2.12)

Let us choose σ such that

$$\sigma^{\epsilon_0 \omega_0} > \frac{1}{\epsilon_0 \omega_0}.\tag{2.13}$$

Summing (2.10) from ζ_3 to $\zeta - \kappa - 1$ and using (2.12) we get

$$\alpha(\zeta - \kappa)(\Delta w(\zeta - \kappa))^{\delta} \le \alpha(\zeta_3)(\Delta w(\zeta_3))^{\delta} - \omega(1 - p_0)^{\delta} \sum_{s = \zeta_3}^{\zeta - \kappa - 1} \frac{\delta w^{\delta}(s + \kappa)}{\alpha^{\frac{1}{\delta}}(s + \kappa)\theta^{\delta + 1}(s + \kappa + 1)}$$

Using the decreasing nature of $\{\alpha(\zeta)(\Delta w(\zeta))^{\delta}\}$, we get

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq \alpha(\zeta_3)(\Delta w(\zeta_3))^{\delta} - \omega(1 - p_0)^{\delta} w^{\delta}(\zeta) \sum_{s=\zeta_3}^{\zeta - \kappa - 1} \frac{\delta}{\alpha^{\frac{1}{\delta}}(s + \kappa)\theta^{\delta + 1}(s + \kappa + 1)}.$$

As in part (i) of the proof, we attained

$$\theta(\zeta+\kappa)\alpha^{\frac{1}{\delta}}(\zeta+\kappa)\Delta w(\zeta+\kappa)<-\sigma^{\epsilon_0\omega_0}\sqrt[\delta]{\omega}(1-p_0)w(\zeta+\kappa)=-\sigma^{\epsilon_0\omega_0}\epsilon_0\omega_0w(\zeta+\kappa),$$
 which by virtue of (2.13) implies

$$\theta(\zeta + \kappa)\alpha^{\frac{1}{\delta}}(\zeta + \kappa)\Delta w(\zeta + \kappa) < -w(\zeta + \kappa),$$

and therefore,

$$\Delta\left(\frac{w(\zeta+\kappa)}{\theta(\zeta+\kappa)}\right) < 0.$$

This contradicts the Lemma 2.1 (iii) and hence the proof.

The following first main result of the paper has been proved using Lemma 2.2.

Theorem 2.3. If $\omega_* > 0$ and $\sigma_* = \infty$ then (1.1) is oscillatory.

Lemma 2.4. Suppose that $\omega_* > 0$ and $\{y(\zeta)\}$ is an eventually positive solution of (1.1). Then, for any $k \in \mathbb{N}_0$, we have

$$\Delta\left(\frac{w(\zeta)}{\theta^{\omega_k}(\zeta)}\right) < 0$$

eventually and

$$\omega_* \le \frac{\max\left\{\mu^{\delta}(1-\mu)\sigma_*^{-\delta\omega} : 0 < \mu < 1\right\}}{(1-p_0)^{\delta}}.$$
 (2.14)

Proof. Pick $\zeta_1 \geq \zeta_0$ such that $y(\zeta) > 0$, $y(\zeta + \xi) > 0$, and $y(\zeta + \kappa) > 0$ for $\zeta \geq \zeta_1$, $\{w(\zeta)\}$ satisfies Lemma 2.1 for $\zeta \geq \zeta_1$, and (2.1) is true. The proof is split into two portions.

First, we establish by means of introduction that for arbitrary $\epsilon_k \in (0,1)$ and ζ large enough,

$$\theta(\zeta)\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta) < -\epsilon_k\omega_k w(\zeta),$$

which implies

$$\Delta\left(\frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)}\right) < 0, \quad k \in \mathbb{N}_0, \tag{2.15}$$

where $\epsilon_k \in (0,1)$ is defined by

$$\begin{split} \epsilon_0 &:= \sqrt[\delta]{\frac{\omega}{\omega_*}}, \\ \epsilon_{k+1} &:= \sqrt[\epsilon_0]{\frac{1-\omega_k}{1-\epsilon_k\omega_k}} \frac{\sigma^{\epsilon_k\omega_k}}{\sigma_*^{\omega_k}}, \quad k \in \mathbb{N} \end{split}$$

for σ and ω given by (2.1). The value of ϵ_k is arbitrary and depends on values of σ and ω . It is easy to verify that

$$\lim_{(\omega,\sigma)\to(\omega_*,\sigma_*)} \epsilon_k = 1.$$

By Lemma 2.1 (iii) and (2.15), we have

$$1 - \epsilon_k \omega_k > 0.$$

From Lemma 2.2 (i), (2.15) is true for k=0. Next, suppose that (2.15) for some k>0 and $\zeta \geq \zeta_k \geq \zeta_1$. Summing (2.10) from ζ_k to $\zeta - \kappa - 1$, we have

$$\alpha(\zeta - \kappa)(\Delta w(\zeta - \kappa))^{\delta} \le \alpha(\zeta_k)(\Delta w(\zeta_k))^{\delta} - \omega(1 - p_0)^{\delta} \sum_{s = \zeta_k}^{\zeta - \kappa - 1} \eta(s) w^{\delta}(s + \kappa)$$

or

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \le \alpha(\zeta_k)(\Delta w(\zeta_k))^{\delta}$$

$$-\omega(1-p_0)^{\delta} \sum_{s=\zeta_k}^{\zeta-\kappa-1} \theta^{\delta\epsilon_k\omega_k}(s+\kappa) \frac{w^{\delta}(s+\kappa)}{\theta^{\delta\epsilon_k\omega_k}(s+\kappa)} \eta(s)$$

or

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq \alpha(\zeta_k)(\Delta w(\zeta_k))^{\delta} - \omega(1 - p_0)^{\delta} \left(\frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)}\right)^{\delta} \sum_{s=\zeta_k}^{\zeta - \kappa - 1} \theta^{\delta \epsilon_k \omega_k}(s + \kappa)\eta(s)$$

or

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq \alpha(\zeta_k)(\Delta w(\zeta_k))^{\delta} - \omega(1 - p_0)^{\delta} \left(\frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)}\right)^{\delta} \sum_{s=\zeta_k}^{\zeta - \kappa - 1} \left(\frac{\theta(s + \kappa)}{\theta(s + 2\kappa)}\right)^{\delta \epsilon_k \omega_k} \theta^{\delta \epsilon_k \omega_k}(s + 2\kappa) \eta(s + \kappa)$$

where we applied the mathematical induction hypothesis (2.15) in the last two inequalities and applying the decreasing nature of $\{\alpha(\zeta)(\Delta w(\zeta))^{\delta}\}$. Using (2.1) in the last inequality, we arrive at

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq \alpha(\zeta_k)(\Delta w(\zeta_k))^{\delta} - \omega \sigma^{\delta \epsilon_k \omega_k} (1 - p_0)^{\delta} \left(\frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)} \right) \sum_{s=\zeta_k}^{\zeta - \kappa - 1} \frac{\delta}{\alpha^{\frac{1}{\delta}}(s + 2\kappa)\theta^{\delta + 1 - \delta \epsilon_k \omega_k}(s + 2\kappa + 1)},$$

or

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq \alpha(\zeta_k)(\Delta w(\zeta_k))^{\delta} - \frac{\omega \sigma^{\delta \epsilon_k \omega_k} (1 - p_0)^{\delta}}{1 - \epsilon_k \omega_k} \left(\frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)}\right)^{\delta} \left[\frac{1}{\theta^{\delta(1 - \epsilon_k \omega_k)}(\zeta + \kappa)} - \frac{1}{\theta^{\delta(1 - \epsilon_k \omega_k)}(\zeta_k + \kappa)}\right]$$

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} \leq \alpha(\zeta_{k})(\Delta w(\zeta_{k}))^{\delta} - \frac{\omega \sigma^{\delta \epsilon_{k} \omega_{k}} (1 - p_{0})^{\delta}}{1 - \epsilon_{k} \omega_{k}} \left(\frac{w(\zeta)}{\theta^{\epsilon_{k} \omega_{k}}(\zeta)}\right)^{\delta} \left[\frac{1}{\theta^{\delta(1 - \epsilon_{k} \omega_{k})}(\zeta)} - \frac{1}{\theta^{\delta(1 - \epsilon_{k} \omega_{k})}(\zeta_{k} + \kappa)}\right]. \quad (2.16)$$

In view of (2.15), the sequence $\{\frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)}\}$ is bounded from above. We claim that

$$\lim_{\zeta \to \infty} \frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)} = 0.$$

To prove the assert, it is enough to show that there exists $\epsilon > 0$ with

$$\Delta\left(\frac{w(\zeta)}{\theta^{\epsilon_k\omega_k+\epsilon}}\right) < 0. \tag{2.17}$$

Indeed, if

$$\lim_{\zeta \to \infty} \frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)} = c > 0,$$

then

$$\frac{w(\zeta)}{\theta^{\epsilon_k \omega_k + \epsilon}(\zeta)} \ge \frac{c}{\theta^{\epsilon}(\zeta)} \to \infty \text{ as } \zeta \to \infty, \tag{2.18}$$

which is a contradiction to (2.17). By $\lim_{\zeta \to \delta} \theta(\zeta) = 0$, we see that there are

$$l \in \left(\sigma^{\omega_{k-1}(\epsilon_{k-1}-1)} \sqrt[\delta]{\frac{1-\omega_{k-1}}{1-\epsilon_{k-1}\omega_{k-1}}}, 1\right)$$

and $\zeta_k' \geq \zeta_k$ such that

$$\frac{1}{\theta^{\delta(1-\epsilon_k\omega_k)}(\zeta)} - \frac{1}{\theta^{\delta(1-\epsilon_k\omega_k)}(\zeta_k+\kappa)} > l^{\delta} \frac{1}{\theta^{\delta(1-\epsilon_k\omega_k)}(\zeta)}, \quad \zeta \geq \zeta_k'.$$

Using the above estimate in (2.16), we obtain

$$\alpha(\zeta)(\Delta w(\zeta))^{\delta} < \frac{l^{\delta}\omega\sigma^{\delta\epsilon_k\omega_k}(1-p_0)^{\delta}}{1-\epsilon_k\omega_k} \left(\frac{w(\zeta)}{\theta(\zeta)}\right)^{\delta}, \quad \zeta \ge \zeta_k',$$

that is

$$\theta(\zeta)\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta) < -l(1-p_0)\sigma^{\epsilon_k\omega_k}\sqrt[\delta]{\frac{\omega}{1-\epsilon_k\omega_k}}w(\zeta).$$

Simple computation shows that

$$l(1-p_0)\sigma^{\epsilon_k\omega_k}\sqrt[\delta]{\frac{\omega}{1-\epsilon_k\omega_k}}-\epsilon_k\omega_k=\epsilon_0\omega_0\left(\frac{l\sigma^{\epsilon_k\omega_k}}{\sqrt[\delta]{1-\epsilon_k\omega_k}}-\frac{\sigma^{\epsilon_{k-1}\omega_{k-1}}}{\sqrt[\delta]{1-\epsilon_{k-1}\omega_{k-1}}}\right).$$

Since ϵ_k is arbitrary large, in view of (2.2), we see that

$$\epsilon_k \omega_k > \omega_{k-1},$$
 (2.19)

and hence

$$l(1-p_0)\sigma^{\epsilon_k\omega_k}\sqrt[\delta]{\frac{\omega}{1-\epsilon_k\omega_k}} - \epsilon_k\omega_k \ge \epsilon_0\omega_0\left(\frac{l\sigma^{\omega_{k-1}}}{\sqrt[\delta]{1-\omega_{k-1}}} - \frac{\sigma^{\epsilon_{k-1}\omega_{k-1}}}{\sqrt[\delta]{1-\epsilon_{k-1}\omega_{k-1}}}\right)$$

$$=: \epsilon > 0.$$

Therefore,

$$\theta(\zeta)\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta) < -(\epsilon_k\omega_k + \epsilon)w(\zeta),$$

and (2.17) is true. Hence, we prove the claim and hence there is $\zeta_k'' \geq \zeta_k'$ such that

$$\alpha(\zeta_k)(\Delta w(\zeta_k))^{\delta} + \frac{\omega \sigma^{\delta \epsilon_k \omega_k} (1 - p_0)^{\delta}}{1 - \epsilon_k \omega_k} \left(\frac{w(\zeta)}{\theta^{\epsilon_k \omega_k}(\zeta)}\right)^{\delta} \frac{1}{\theta^{\delta(1 - \epsilon_k \omega_k)}(\zeta_k + \kappa)} < 0, \quad \zeta \ge \zeta_k''.$$
(2.20)

Using (2.20) in (2.16) implies that

$$\theta(\zeta)\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta) < -(1-p_0)\sigma^{\epsilon_k\omega_k}\sqrt[\delta]{\frac{\omega}{1-\epsilon_k\omega_k}}w(\zeta) = -\epsilon_{k+1}\omega_{k+1}w(\zeta) \qquad (2.21)$$

and

$$\Delta\left(\frac{w(\zeta)}{\theta^{\epsilon_{k+1}\omega_{k+1}}(\zeta)}\right) < 0,$$

which complete the induction step.

Now, for some $k \in \mathbb{N}_0$, $\epsilon_{k+1} \in (0,1)$ is arbitrary large, we assume that $\epsilon_{k+1} < \frac{1}{l_k}$, where l_k is given by (2.3). Applying (2.19) in (2.21), we have

$$\theta(\zeta)\alpha^{\frac{1}{\delta}}(\zeta)\Delta w(\zeta) < -\epsilon_{k+1}\omega_{k+1}w(\zeta) < -\epsilon_{k+1}l_k\omega_k w(\zeta) < \omega_k w(\zeta),$$

which immediate implies.

$$\Delta\left(\frac{w(\zeta)}{\theta^{\omega_k}(\zeta)}\right) < 0.$$

By Lemma 2 (iii), we derive that $\omega_k < 1$ for any $k \in \mathbb{N}_0$. This fact together with (2.2), we have $\{\omega_k\}$ is bounded above and increasing sequence. So, $\lim_{k\to\infty}\omega_k = \mu$, where μ is the smallest positive root of

$$\omega_*(1-p_0)^{\delta} = \mu^{\delta}(1-\mu)\sigma_*^{-\delta\mu},$$

which implies (2.14) and hence the proof.

Corollary 2.5. Assume $\omega_* > 0$ and $\sigma_* < \infty$. If $\{y(\zeta)\}$ is an eventually positive solutions of (1.1) with $w(\zeta) > 0$ with $\Delta w(\zeta) < 0$, then for any $\mu \in (0,1)$

$$\frac{w(\zeta)}{w(\zeta + \kappa)} \ge \mu \sigma_*^{\omega_k}, \quad k \in \mathbb{N}_0.$$
 (2.22)

Proof. From Lemmas 2.1 (i) and 2.4, we have $w(\zeta) > 0$ and $\Delta\left(\frac{w(\zeta)}{\theta^{\omega_k}(\zeta)}\right) < 0$ eventually. Therefore,

$$\frac{w(\zeta)}{\theta^{\omega_k}(\zeta)} > \frac{w(\zeta + \kappa)}{\theta^{\omega_k}(\zeta + \kappa)},\tag{2.23}$$

which implies (2.22).

Applying Lemma 2.4, we prove our second main result.

Theorem 2.6. Suppose that $\omega_* > 0$ and $\sigma_* < \infty$. If

$$\omega_* > \frac{\max\{\mu^{\delta}(1-\mu)\sigma_*^{-\delta\mu} : 0 < \mu < 1\}}{(1-p_0)^{\delta}},\tag{2.24}$$

then (1.1) is oscillatory.

3. EXAMPLE

Example 3.1. Let us consider the second-order difference equation

$$\Delta \left(\zeta^{\frac{1}{3}} (\zeta + 1)^{\frac{1}{3}} \left(\Delta \left(y(\zeta) + \frac{1}{2\zeta} y(\zeta - 1) \right) \right)^{\frac{1}{3}} \right) + \frac{1}{\zeta^{\frac{2}{3}}} y^{\frac{1}{3}} (\zeta + 1) = 0; \quad \zeta \ge 2. \quad (3.1)$$

Here, we have $\alpha(\zeta) = \zeta^{\frac{1}{3}}(\zeta+1)^{\frac{1}{3}}$, $p(\zeta) = \frac{1}{2\zeta}$, $\eta(\zeta) = \frac{1}{\zeta^{\frac{2}{3}}}$, $\xi = -1$, $\kappa = 1$ and $\delta = \frac{1}{3}$.

We can easily show that $\theta(\zeta) = \frac{1}{\zeta} < \infty$, $\omega_* > 0$, and $\sigma_* < \infty$. Also we can choose $p_0 = \frac{1}{2}$. We can easily check that

$$\omega_* > \max \left\{ \frac{\mu^{\frac{1}{3}}(1-\mu): 0 < \mu < 1}{\left(\frac{1}{2}\right)^{\frac{1}{3}}} \right\}.$$

So, all the assumptions of the Theorem 2.6 one verified and hence every solution of (3.1) is oscillatory.

Example 3.2. Consider the following second-order neutral advanced difference equation

$$\Delta \left(\zeta^{\frac{1}{5}} (\zeta + 1)^{\frac{1}{5}} \left(\Delta \left(y(\zeta) + \frac{1}{\zeta + 1} y(\zeta + 1) \right) \right)^{\frac{1}{5}} \right) + \frac{1}{\zeta^{\frac{4}{5}}} y^{\frac{1}{5}} (\zeta + 1) = 0; \quad \zeta \ge 1.$$
(3.2)

Here, $\alpha(\zeta)=\zeta^{\frac{1}{5}}(\zeta+1)^{\frac{1}{5}}$, $p(\zeta)=\frac{1}{\zeta+1}$, $\eta(\zeta)=\frac{1}{\zeta^{\frac{4}{5}}}$, $\xi=1$, $\kappa=1$ and $\delta=\frac{1}{5}$. We can easily show that $\theta(\zeta)=\frac{1}{\zeta}<\infty$, $\omega_*=5$, and $\sigma_*=1$. Also, we can easily check that

$$\omega_* > \max \left\{ \frac{\mu^{\frac{1}{5}}(1-\mu) : 0 < \mu < 1}{\left(\frac{1}{2}\right)^{\frac{1}{5}}} \right\}.$$

All the assumptions of the Theorem 2.6 are verified and hence every solution of (3.2) is oscillatory.

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