Subclasses of Univalent Functions Involving Modified Sigmoid Function

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Abstract

The authors obtained some geometric results on certain new classes of analytic functions involving sigmoid function defined by Fadipe-Joseph et al. 2016 as $T_\gamma (\lambda, \beta, \alpha, \mu)$. Extreme point property, radius of starlikeness and convexity, convolution property and Fekete-Szego inequality for the class were proved.

Keywords: Sǎlǎgean operator, modified sigmoid function, radius of starlikeness and convexity, convolution property and Fekete-Szego inequality.

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1. INTRODUCTION

The study of subclasses of analytic and univalent functions in geometric theory and applications is a link between geometry and analysis with wide range of interest among function theorists in recent time. Subclasses of analytic and univalent functions can be as many as researchers who are of interest in the area (see [1], [2], [3] and [4]). In this paper however, using Sǎlǎgean differential operator involving modified sigmoid function $\gamma(s)$, we investigate some geometric properties of the class $T_\gamma (\lambda, \beta, \alpha, \mu)$ introduced in [1].

Let $T_\gamma$ denote the class of functions of the form

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k; \quad a_k \geq 0, \quad \gamma(s) = \frac{2}{1 + e^{-s}}.$$  \hspace{1cm} (1.1)
and
\[ g_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) b_k z^k;\quad b_k \geq 0 \]  
which are analytic and univalent in the unit disk \( U = \{|z| : z \leq 1\} \).

Then \( f_\gamma(z) \) and \( g_\gamma(z) \) belong to class \( T_\gamma \).

If for convenience, we set \( T_\gamma = T_1 \) we see that \( T_1 = T \), is the usual class of the form
\[ f(z) = z - \sum_{k=2}^{\infty} a_k z^k;\quad a_k \geq 0 \]  which is analytic in the open unit disk \( U \).

We define an identity function as
\[ e_\gamma(z) = z;\quad a_k = 0 \quad for \ all \ k \geq 2 \ but \ \gamma \neq 0 \]  

1.1. Convolution or Hardamard Product:

Given two analytic functions \( f_\gamma(z) \) and \( g_\gamma(z) \) in \( T_\gamma \) where \( f_\gamma(z) \) and \( g_\gamma(z) \) are given by (1.1) and (1.2) respectively.

Convolution of \( f_\gamma(z) \) and \( g_\gamma(z) \) are defined as
\[ (f_\gamma * g_\gamma) = z - \sum_{k=2}^{\infty} \gamma(s) a_k b_k z^k = (g_\gamma * f_\gamma);\quad a_k b_k \geq 0 \]  

1.2. Starlikeness, Convexity and Close-to-convexity:

A function \( f(z) \) defined by (1.1) is said to be starlike of order \( \delta \) if
\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta; \quad z \in U \]  
for some \( \delta \ (0 \leq \delta \leq 1) \).

In the same way, a function \( f(z) \) defined by (1.1) is said to be convex of order \( \delta \) if and only if \( zf''(z) \) is starlike of order \( \delta \). In other words, if
\[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta; \quad z \in U \]  
for some \( \delta \ (0 \leq \delta \leq 1) \).

Furthermore, a function \( f(z) \) defined by (1.1) is said to be close-to-convex of order \( \delta \) if
\[ \text{Re} \left\{ zf'(z) \right\} > \delta; \quad z \in U \]  
for some \( \delta \ (0 \leq \delta \leq 1) \).
2. PRELIMINARY RESULTS

2.1. Salagean Differential Operator Involving Modified Sigmoid Function

Definition 2.1 Salagean Differential Operator Involving Modified Sigmoid Function

\[ D^n f_\gamma(z) = \gamma^n(s) z + \sum_{k=2}^{\infty} \gamma^m(s) k^n a_k z^k; \quad m = n + 1 \]  

(2.1)

for details see [1].

Definition 2.2 A function \( f_\gamma \in T_\gamma \) defined by (1.1) belongs to the class \( T_\gamma^\gamma(\lambda, \beta, \alpha, \mu) \) if

\[
\left| \frac{D^{n+1}f(z)}{D^n f(z)} + \lambda \right| - 2\alpha \left( \frac{D^{n+1}f(z)}{D^n f(z)} - \mu \right) > \beta \quad (z \in U)
\]

for \( |z| < 1, \ 0 < \lambda \leq 1, \gamma(s) = \frac{2}{1+e^{-s}} \) (i.e. \( \gamma \neq 0 \)), \( 0 < \beta \leq 1, \frac{1}{2} \leq \alpha \leq 1, \mu \geq 1, \ n \in N_0 = N \cup \{0\} \) See [1].

Lemma 2.1 [1]

Let \( f_\gamma(z) \in T_\gamma \) defined as \( f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k \) \( (a_k \geq 0 \ and \ \gamma \neq 0) \). If \( f_\gamma \in T_\gamma^\gamma(\lambda, \beta, \alpha, \mu) \), then

\[
\sum_{k=2}^{\infty} \gamma(s) k^n \left[ \gamma(s) k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right] |a_k| < \gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu.
\]

3. MAIN RESULTS

Theorem 3.1 Extreme points for class \( T_\gamma(\lambda, \beta, \alpha, \mu) \).

If a function \( f_\gamma(z) \) defined by (1.1) belongs to the class \( T_\gamma(\lambda, \beta, \alpha, \mu) \).

Let \( f_1(z) = z \) and

\[
f_\gamma(z) = \frac{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s) k^n \left[ \gamma(s) k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} z^k, \quad k \geq 2.
\]

The function \( f_\gamma \in T_\gamma^\gamma(\lambda, \beta, \alpha, \mu) \) if and only if it can be expressed in the form

\[
f_\gamma(z) = \sum_{k=1}^{\infty} \Psi_k f_k(z) \quad (3.1)
\]

where \( \Psi_k \geq 0 \) and \( \sum_{k=1}^{\infty} \Psi_k = 1 \).
Proof. Let \( f_\gamma(z) = \sum_{k=1}^{\infty} \Psi_k f_k(z), \Psi_k \geq 0, k = 1, 2, \ldots \) with

\[
\sum_{k=1}^{\infty} \Psi_k = 1
\]

\[
f_\gamma(z) = \sum_{k=1}^{\infty} \Psi_k f_k(z) = \Psi_1 f_1(z) + \sum_{k=2}^{\infty} \Psi_k f_k(z)
\]

\[
= \Psi_1(z) + \sum_{k=2}^{\infty} \Psi_k \left\{ z - \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)k^{\alpha} \left[ \gamma(s)k[1-\beta(1-2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} z^k \right\}
\]

that is,

\[
f_\gamma(z) = z - \sum_{k=2}^{\infty} \Psi_k \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)k^{\alpha} \left[ \gamma(s)k[1-\beta(1-2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} z^k.
\]

Then,

\[
f_\gamma(z) = \sum_{k=2}^{\infty} \Psi_k \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)k^{\alpha} \left[ \gamma(s)k[1-\beta(1-2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} \times \frac{\gamma(s)k^{\alpha} \left[ \gamma(s)k[1-\beta(1-2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}
\]

So that

\[
\sum_{k=2}^{\infty} \Psi_k = 1 - \Psi_1 \leq 1.
\]

In other words,

\[
f_\gamma(z) = \Psi_1 + \sum_{k=2}^{\infty} \Psi_k = 1 \Rightarrow 1 - \Psi_1 \leq 1.
\]

So by Lemma 2.1, \( f_\gamma(z) \in T_\gamma(\lambda, \beta, \alpha, \mu). \)

Conversely, assume that the function \( f_\gamma(z) \) defined by (1.1) belongs to the class \( T_\gamma(\lambda, \beta, \alpha, \mu), \) then by Lemma 2.1,

\[
a_k \leq \frac{\gamma(s)(1-\beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)k^{\alpha} \left[ \gamma(s)k[1-\beta(1-2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} (k \geq 2).
\]
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So that

Proof:
From Lemma 2.1,

\[ \Psi_k \leq \frac{\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu} \]

and

\[ \Psi_1 = 1 - \sum_{k=2}^{\infty} \Psi_k. \]

So that

\[ \sum_{k=1}^{\infty} \Psi_k f_k = \frac{\{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu\} \{\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]\}}{\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} \{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu\}. \]

We thus notice that we can express \( f_k(z) \) in the form (3.1).
Therefore, \( f_\gamma(z) = \sum_{k=1}^{\infty} \Psi_k f_k \) which completes the proof.

**Fekete-Szegő inequality for the class** \( T_\gamma(\lambda, \beta, \alpha, \mu) \)

In this section, we use the values of \( a_2 \) and \( a_3 \) given by Lemma 2.1 to establish the Fekete-Szegő inequality for functions \( f_\gamma(z) \) belonging to the class \( T_\gamma(\lambda, \beta, \alpha, \mu) \).

**Theorem 3.2** Fekete-Szegő inequality for class \( T_\gamma(\lambda, \beta, \alpha, \mu) \).
If a function \( f(z) \in T_\gamma \) defined by (1.1) belongs to the class \( T_\gamma(\lambda, \beta, \alpha, \mu) \) and \( \mu \in R \). Then,

\[ |a_3 - \sigma a_2^2| \leq \frac{B_3[B_2^2 - \sigma B_3 B_1]}{B_1 B_2^2}. \]

**Proof:** From Lemma 2.1,

\[ a_2 \leq \frac{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)2^n \left[ 2\gamma(s)[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} \]

and

\[ a_3 \leq \frac{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)3^n \left[ 3\gamma(s)[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} \]

So that

\[ a_3 - \sigma a_2^2 = \frac{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)3^n \left[ 3\gamma(s)[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} \]

\[ -\sigma \left\{ \frac{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)2^n \left[ 2\gamma(s)[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} \right\}^2. \]
Set

\[ B_1 = \left\{ \gamma(s)3^n \left[ 3\gamma(s)[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right] \right\}; \quad \text{and} \]

\[ B_2 = \left\{ \gamma(s)2^n \left[ 2\gamma(s)[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right] \right\} \]

\[ B_3 = \{(\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu)\gamma(s)(k - \delta)\} \]

Thus,

\[ |a_3 - \sigma a_2^2| \leq \frac{B_3 B_2^2 - \sigma B_3 B_1}{B_1 B_2^2} = \frac{B_3[B_2^2 - \sigma B_3 B_1]}{B_1 B_2^2} \]

which completes the proof.

**Radius Properties for class** \( T_\gamma(\lambda, \beta, \alpha, \mu) \).

We now obtain the radii of starlikeness, convexity and close to convexity in this section as follows:

**Theorem 3.3 (Starlikeness):** Let the function \( f_\gamma(z) \) defined by (1.1) be in the class \( T_\gamma(\lambda, \beta, \alpha, \mu) \); then \( f_\gamma(z) \) is starlike of order \( \sigma \) (0 \( \leq \delta < 1 \)) in \( |z| < r_1 \), where

\[ r_1 = \inf_k \left\{ \left( \frac{1 - \delta}{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu} \right)^{\frac{1}{1 - \delta}} \gamma(s)[k - \delta] \left[ k^n \gamma(s)[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right] \right\}; \quad k \geq 2 \quad (3.2) \]

The result is sharp for

\[ f_\gamma(z) = z - \frac{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)k^n \left[ \gamma(s)[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]} z^k, \quad k \geq 2. \]

**Proof:** It suffices to show that \( \left| \frac{zf_\gamma'(z)}{f_\gamma(z)} - 1 \right| < 1 - \delta \).

That is,

\[ \left| \frac{zf_\gamma'(z)}{f_\gamma} - 1 \right| = \left| \frac{z - \sum_{k=2}^{\infty} k\gamma(s)a_k z^k - z + \sum_{k=2}^{\infty} \gamma(s)a_k z^k}{z - \sum_{k=2}^{\infty} \gamma(s)a_k z^k} \right| \]

\[ \left| -\sum_{k=2}^{\infty} \frac{\gamma(s)(k - 1)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} \gamma(s)a_k z^{k-1}} \right| \leq \frac{\sum_{k=2}^{\infty} \gamma(s)(k - 1)a_k |z|^{k-1}}{(1 - \sum_{k=2}^{\infty} \gamma(s)a_k |z|^{k-1})} < 1 - \delta \]
Hence by Lemma 2.1, the above inequality holds if
\[
\sum_{k=2}^{\infty} \gamma(s)(k - 1)a_k|z|^{k-1} \leq (1 - \delta)(1 - \sum_{k=2}^{\infty} \gamma(s)a_k|z|^{k-1})
\]
\[
\sum_{k=2}^{\infty} \gamma(s)(k - 1)a_k|z|^{k-1} \leq (1 - \delta) - \sum_{k=2}^{\infty} \gamma(s)(1 - \delta)a_k|z|^{k-1}
\]
\[
\sum_{k=2}^{\infty} \gamma(s)(k - 1 + \delta)a_k|z|^{k-1} \leq (1 - \delta)
\]
\[
\sum_{k=2}^{\infty} \gamma(s)(k + \delta)a_k|z|^{k-1} \leq (1 - \delta)
\]
\[
\sum_{k=2}^{\infty} \gamma(s) \left( \frac{k - \delta}{1 - \delta} \right) a_k|z|^{k-1} \leq 1.
\]
Hence by Lemma 2.1, the above inequality holds if
\[
\sum_{k=2}^{\infty} \gamma(s) \frac{k - \delta}{1 - \delta} a_k|z|^{k-1} \leq 1.
\]
It follows that
\[
\sum_{k=2}^{\infty} \gamma(s) \frac{(k - \delta)|z|^{k-1}}{(1 - \delta)} \leq \frac{1}{a_k}
\]
\[
\frac{\gamma(s)(k - \delta)|z|^{k-1}}{(1 - \delta)} \leq \frac{\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda - \mu]} ; \ (k = 2, 3, \ldots)
\]
By multiplying both sides by the inverse of \(\frac{\gamma(s)(k - \delta)}{(1 - \delta)}\), we have that
\[
|z|^{k-1} \leq \frac{(1 - \delta)\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda - \mu]} \gamma(s)(k - \delta)
\]
We find the \((k - 1)th\) root of both sides, so that
\[
|z| \leq \left\{ \frac{(1 - \delta)\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{\gamma^2(s)(1 - \beta) + \gamma(s)\beta[2\alpha(\gamma(s) - \mu) - \lambda - \beta\gamma(s)\mu] \gamma(s)(k - \delta)}} \right\}^{\frac{1}{k-1}} \text{ where } |z| < r_1.
\]
Thus,
\[
r_1 = \inf_k \left\{ \frac{(1 - \delta)\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda - \beta\gamma(s)(k - \delta)]} \right\}^{\frac{1}{k-1}} ; \ k \geq 2
\]
which completes the proof.
Theorem 3.4 (Convexity): Let the function $f_\gamma(z)$ defined by (1.1) be in the class $T_\gamma(\lambda, \beta, \alpha, \mu)$; then $f_\gamma(z)$ is convex of order $\delta$ ($0 \leq \delta < 1$) in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{(1 - \delta)\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{k(k - \delta)\gamma(s)[\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu]} \right\}^{\frac{1}{k-1}}; \quad k \geq 2$$

(3.3)

The result is sharp for

$$f_\gamma(z) = z - \frac{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu}{\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}z^k, \quad k \geq 2.$$

Proof: It suffices to show that $\left| \frac{zf''_\gamma(z)}{f'_\gamma(z)} \right| < 1 - \delta$, $|z| < r_2$.

Since

$$\left| \frac{zf''_\gamma(z)}{f'_\gamma(z)} \right| = \left| \frac{\sum_{k=2}^{\infty} \gamma(s)k(k-1)a_kz^{k-1}}{1 - \sum_{k=2}^{\infty} \gamma(s)ka_kz^{k-1}} \right| \leq \frac{\sum_{k=2}^{\infty} \gamma(s)k(k-1)a_k|z|^{k-1}}{1 - \sum_{k=2}^{\infty} \gamma(s)ka_k|z|^{k-1}} < 1 - \delta$$

To prove the Theorem, we must show that

$$\sum_{k=2}^{\infty} \gamma(s)k(k-1)a_k|z|^{k-1} \leq 1 - \delta$$

$$\sum_{k=2}^{\infty} \gamma(s)k(k-\delta)a_k|z|^{k-1} \leq 1 - \delta.$$

And by Lemma 2.1, we obtain

$$|z|^{k-1} \leq \frac{(1 - \delta)\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{k(k - \delta)\gamma(s)[\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu]}$$

or

$$r_2 = \inf_k \left\{ \frac{(1 - \delta)\gamma(s)k^n \left[ \gamma(s)k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha\mu + \frac{1}{\beta} \right) \right]}{k(k - \delta)\gamma(s)[\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu]} \right\}^{\frac{1}{k-1}}$$

which completes the proof.
The result is sharp if
Thus,

**Proof:** It suffices to show that \(|f'_\gamma(z) - 1| = 1 - \delta,\; (0 \leq \delta < 1)\) for \(|z| < r_3\).
Thus,

Since \(|f'_\gamma(z) - 1| \leq \sum_{k=2}^{\infty} \gamma(s) k \alpha_k |z^{k-1} - 1| \leq 1 - \delta\) if we divide both by \((1 - \delta)\), then,

By coefficient estimates of \(f_\gamma(z) \in T_\gamma(\lambda, \beta, \alpha, \mu)\) given by Lemma 2.1 above, (3.5) holds if

Hence,

\[
|z| \leq \left\{ \frac{(1 - \delta) \gamma(s) k^n \left[ \gamma(s) k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha \mu + \frac{1}{3} \beta \right) \right]}{k \gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu} \right\}^{\frac{1}{n-1}}; \; k \geq 2.
\]

**Theorem 3.5** (Close-to-convex): Let the function \(f_\gamma(z)\) defined by (1.1) be in the class \(T_\gamma(\lambda, \beta, \alpha, \mu)\). Then \(f_\gamma(z)\) is closed-to-convex of order \(\delta\) \((0 \leq \delta < 1)\) in \(|z| < r_3\), where

\[
r_3 \leq \left\{ \frac{(1 - \delta) \gamma(s) k^n \left[ \gamma(s) k[1 - \beta(1 - 2\alpha)] - \beta \left( \lambda + 2\alpha \mu + \frac{1}{3} \beta \right) \right]}{k \gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu) - \lambda] - \mu} \right\}^{\frac{1}{n-1}}; \; k \geq 2.
\]
Neighbourhood Property for class $T_γ(λ, β, α, µ)$.
We define the $(n, ς)$-neighborhood of a function $f_γ \in T_γ$ by

$$N_{(n,ς)}(e) = \left\{ g_γ : g_γ \in T_γ, g_γ(z) = z - \sum_{k=t+1}^{∞} \gamma(s)b_k z^k \text{ and } \sum_{k=2}^{∞} \gamma(s)k|b_k| \leq ς \right\}.$$  (3.6)

And for the identity function $e_γ(z)$ defined by (1.3), we have the following result.
Let $T_γ$ denote the class of functions $f_γ(z)$ defined by (1.1) which is analytic in the open unit disk $U = \{|z| : z \in C \text{ and } |z| < 1\}.$

**Theorem 3.6** Neighbourhood Property for class $T_γ(λ, β, α, µ)$.
Let

$$ς = \frac{\gamma(s)(1 - \beta) + \beta[2α(γ(s) - μ) - λ] - μ}{\gamma(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right]}.$$  (3.7)

Then $T_γ(λ, β, α, µ) \subset N_{(n,ς)}e(z)$.

**Proof:** Suppose $f_γ \in T_γ(λ, β, α, µ)$, then from Lemma 2.1, we have that

$$\sum_{k=2}^{∞} γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right] |a_k| < γ(s)(1 - β) + \beta[2α(γ(s) - μ) - λ] - μ$$

and thus for $k = 2$, we have that

$$γ(s)2^n \left[ γ(s)2[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right] |a_2| \leq \sum_{k=2}^{∞} γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right] |a_k| < γ(s)(1 - β) + \beta[2α(γ(s) - μ) - λ] - μ$$

$$\sum_{k=2}^{∞} γ(s)|a_k| \leq \frac{γ(s)(1 - β) + \beta[2α(γ(s) - μ) - λ] - μ}{γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right]}.$$  (3.8)

But $|z| < r$ which implies that

$$|f'(z)| \leq 1 - |z| \sum_{k=2}^{∞} kγ(s)a_k \leq 1 - r \sum_{k=2}^{∞} kγ(s)a_k$$  (3.9)
It thus follows that
\[ |f'_γ(z)| \leq 1 - r \sum_{k=2}^{∞} kγ(s)a_k \leq k \frac{γ(s)(1 - β) + β[2α(γ(s) - μ) - λ] - μ}{γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right] k^n} \]

\[ \Rightarrow \sum_{k=2}^{∞} kγ(s)a_k \leq \sum_{k=2}^{∞} γ(s)a_k \leq \frac{γ(s)(1 - β) + β[2α(γ(s) - μ) - λ] - μ}{γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right] k^n} = γ \] by equation (3.6)

Thus, \( f_γ \in N_{(n, γ)}(e). \)

**Theorem 3.7 Convolution Property for class \( T_γ(λ, β, α, μ). \)**

Let \( f_γ(z) \) and \( g_γ(z) \) defined by (1.1) and (1.2)) respectively be members of \( T_γ(λ, β, α, μ) \) such that \( h_γ(z) = z - \sum_{k=2}^{∞} γ(s)a_kb_kz^k \) where \( a_kb_k \geq 0 \) as defined by (1.4).

Then, \( h_γ(z) \) is in the subclass of \( T_γ(λ, β, α, μ) \) where

\[ \frac{γ(s)(1 - β) + β[2α(γ(s) - μ) - λ] - μ_1}{γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right]} \leq \frac{γ(s)(1 - β) + β[2α(γ(s) - μ) - λ] - μ_1}{γ(s)(1 - β) + β[2α(γ(s) - μ) - λ] - μ_2} \]

**Proof:** From Lemma 2.1,

\[ \sum_{k=2}^{∞} γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right] a_k \leq 1 \] \hspace{1cm} (3.10)

Similarly,

\[ \sum_{k=2}^{∞} γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right] b_k \leq 1 \] \hspace{1cm} (3.11)

We however need to determine the largest \( μ_2 \) such that

\[ \sum_{k=2}^{∞} γ(s)k^n \left[ γ(s)k[1 - β(1 - 2α)] - β \left( λ + 2αμ + \frac{1}{β} \right) \right] a_kb_k \leq 1. \]
By Cauchy-Schwarz inequality,
\[
\sum_{k=2}^{\infty} |a_kb_k| \leq \sqrt{\sum_{k=2}^{\infty} |a_k|^2} \sqrt{\sum_{k=2}^{\infty} |b_k|^2}.
\]
Using the Cauchy-Schwarz inequality, we obtain
\[
\sum_{k=2}^{\infty} \frac{\gamma(s) k^n}{\gamma(s)(1 - \beta(1 - 2\alpha))} \left( \gamma(s) - \beta \left( \lambda + 2\alpha \mu + \frac{1}{\beta} \right) \right) \leq 1.
\]
It suffices to show what
\[
\frac{\gamma(s) k^n}{\gamma(s)(1 - \beta(1 - 2\alpha))} \left( \gamma(s) - \beta \left( \lambda + 2\alpha \mu + \frac{1}{\beta} \right) \right) \leq \frac{\gamma(s)(1 - \beta) + \beta(2\alpha(\gamma(s) - \mu)) - \lambda - \mu_1}{\gamma(s)(1 - \beta) + \beta[2\alpha(\gamma(s) - \mu)] - \lambda - \mu_2}.
\]
equivalently,
\[
\sqrt{a_kb_k} \leq \frac{\gamma(s)(1 - \beta) + \beta(2\alpha(\gamma(s) - \mu)) - \lambda - \mu_1}{\gamma(s)(1 - \beta) + \beta(2\alpha(\gamma(s) - \mu)) - \lambda - \mu_2}.
\]
But from (3.12) we have
\[
\sqrt{a_kb_k} \leq \frac{\gamma(s)(1 - \beta) + \beta(2\alpha(\gamma(s) - \mu)) - \lambda - \mu_1}{\gamma(s)(1 - \beta) + \beta(2\alpha(\gamma(s) - \mu)) - \lambda - \mu_2}.
\]
Consequently, we show that
\[
\frac{\gamma(s)(1 - \beta) + \beta(2\alpha(\gamma(s) - \mu)) - \lambda - \mu_1}{\gamma(s)(1 - \beta) + \beta(2\alpha(\gamma(s) - \mu)) - \lambda - \mu_2} \leq 1.
\]
REFERENCES


