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Results on the Existence of Solutions of Sobolev-Type Volterra-Fredholm Integro-Differential Equations

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Abstract

In this study, the semigroup theory and the Schauder fixed point theorem are applied to prove the existence and uniqueness of mild and strong solutions of a nonlinear Volterra-Fredholm integro-differential equation of Sobolev type with nonlocal condition.

Keywords Volterra-Fredholm integro-differential equation; nonlocal condition; mild and strong solutions; fixed point technique.

Mathematics Subject Classification (2010): 45J05, 34A12, 47H10.

1 Introduction

Integro-differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities and their rates of change in space and/or time are known or postulated [1–3, 13, 15–20]. The problem of existence of solutions of evolution equations with nonlocal conditions in Banach spaces has been studied first by Byszewski [10]. In that paper he has established the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$\begin{cases} u'(t) = -Au(t) + f(t, u(t)), \quad t \in (t_0, t_0 + a] \\ u(t_0) + g(t_1, t_2, ..., t_p, u(.)) = u_0 \end{cases} \tag{1}$$

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where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$ on a Banach space $X,\ 0 \leq t_0 < t_1 < t_2 < ... < t_p \leq t_0 + a, a > 0, u_0 \in X$ and $f : [t_0, t_0 + a] \times X \longrightarrow X, g : [t_0, t_0 + a]^p \times X \longrightarrow X$ are given functions. Subsequently several authors have investigated the same type of problem to different classes of abstract differential equations in Banach spaces [4–7,11,14,21–31]. Brill [9] and Showalter [30] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. This type of equations arise in various applications such as in the flow of fluid through fissured rocks [8], thermodynamics [12] and shear in second order fluids [22].

The purpose of this paper is to prove the existence of mild and strong solutions for Volterra-Fredholm integro-differential equation of Sobolev type with nonlocal condition of the form

$$(Bu(t))' + Au(t) = f(t, u(t)) + \int_0^t g(t, s, u(s))ds + \int_0^a h(t, s, u(s))ds, \tag{2}$$

$$u(0) + \sum_{k=1}^{p} c_k u(t_k) = u_0 \tag{3}$$

where $0 \leq t_1 < t_2 < ... < t_p \leq a$, B and A are linear operators with domains contained in a Banach space X and ranges contained in a Banach space Y and the nonlinear operators $f : I \times X \longrightarrow Y$ and $g, h : \Delta \times X \longrightarrow Y$ are given. Here $I = [0, a]$ and $\Delta = \{(s, t) : 0 \leq s \leq t \leq a\},\ t \in J := (0, a]$.

2 Auxiliary results

In order to prove our main theorem we assume certain conditions on the operators A and B. Let X and Y be Banach spaces with norm $|.|$ and $\|.\|$ respectively. The operators $A : D(A) \subset X \longrightarrow Y$ and $B : D(B) \subset X \longrightarrow Y$ satisfy the following hypothesis:

(H_1) A and B are closed linear operators,

(H_2) $D(B) \subset D(A)$ and B is bijective,

(H_3) $B^{-1} : Y \longrightarrow D(B)$ is bijective.

From the above fact and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \longrightarrow Y$. Further $-AB^{-1}$ generates a uniformly continuous semigroup $T(t), t \geq 0$ and so $\max_{t \in I} \|T(t)\|$ is finite. We denote $M = \max_{t \in I} \|T(t)\|, R = \|B^{-1}\|$. Let $B_r = \{x \in X : |x| \leq r\}$ and $c = \sum_{k=1}^{p} |c_k|$.

In this paper, we assume that there exists an operator E on $D(E) = X$ given by the formula

$$E = \Big[I + \sum_{k=1}^{p} c_k B^{-1} T(t_k) B\Big]^{-1}$$

and $Eu_0 \in D(B)$

$$E\int_0^{t_k} B^{-1}T(t_k-s)\Big[f(s,u(s))+\int_0^s g(s,\tau,u(\tau))d\tau+\int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \in D(B).$$

The existence of E can be observed from the following fact [11]. Suppose that $\{T(t)\}$ is a C_0 semigroup of operators on X such that $\|B^{-1}T(t_k)B\| \leq Ce^{-\delta t_k}(k = 1, 2, ..., p)$ where δ is a positive constant and $C \geq 1$. If $\sum_{k=1}^{p} |c_k| e^{-\delta t_k} < 1/C$ then $\| \sum_{k=1}^{p} c_k B^{-1} T(t_k) B\| < 1$. So such an operator E exists on X.

Definition 2.1 *[28] A continuous solution u of the integral equation*

$$\begin{aligned} u(t) &= B^{-1}T(t)BEu_0 - \sum_{k=1}^{p} c_k B^{-1}T(t)BE\int_0^{t_k} B^{-1}T(t_k-s) \\ &\times \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds + \int_0^t B^{-1}T(t-s) \\ &\times \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \qquad (4) \end{aligned}$$

is said to be a mild solution of the problem (2)-(3) *on I*

Definition 2.2 *[28] A function u is said to be a strong solution of the problem* (2)-(3) *on I if u is differentiable almost everywhere on I, $u' \in L^1(I, X)$, $u(0) + \sum_{k=1}^{p} c_k u(t_k) = u_0$ and*

$$(Bu(t))' + Au(t) = f(t,u(t)) + \int_0^t g(t,s,u(s))ds + \int_0^a h(t,s,u(s))ds, \ \ \textit{a.e on } I.$$

Remark. A mild solution of the nonlocal Cauchy problem (2)-(3) *satisfies the condition* (3). *From* (4)

$$\begin{aligned} u(0) &= Eu_0 - \sum_{k=1}^{p} c_k E\int_0^{t_k} B^{-1}T(t_k-s)\Big[f(s,u(s)) \\ &+ \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big] \qquad (5) \end{aligned}$$

and

$$\begin{aligned} u(t_i) &= B^{-1}T(t_i)BEu_0 - \sum_{k=1}^{p} c_k B^{-1}T(t_i)BE\int_0^{t_k} B^{-1}T(t_k-s) \\ &\times \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds + \int_0^{t_i} B^{-1}T(t_i-s) \\ &\times \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds. \end{aligned}$$

Therefore

$$\begin{aligned} & u(0) + \sum_{i=1}^{p} c_i u(t_i) \\ = & \Big[I + \sum_{i=1}^{p} c_i B^{-1}T(t_i)B\Big]Eu_0 \\ - & \Big[I + \sum_{i=1}^{p} c_k B^{-1}T(t_i)B\Big]\sum_{k=1}^{p} c_k E\int_0^{t_k} B^{-1}T(t_k-s) \\ \times & \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds + \sum_{i=1}^{p} c_i \\ \times & \int_0^{t_i} B^{-1}T(t_i-s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \\ \\ = & u_0 - \sum_{k=1}^{p} c_k \int_0^{t_k} B^{-1}T(t_k-s) \\ \times & \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds + \sum_{i=1}^{p} c_i \\ \times & \int_0^{t_i} B^{-1}T(t_i-s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \\ = & u_0. \end{aligned}$$

Further assume that,

(H_4) $g, h : \Delta \times B_r \longrightarrow Y$ *is continuous in* t *and there exist two constants* $K_1, K_2 > 0$ *such that* $\|g(t,s,u)\| \le K_1, \|h(t,s,u)\| \le K_2$ *for* $(s,t) \in \Delta$ *and* $u \in B_r$

(H_5) $f : I \times B_r \longrightarrow Y$ *is continuous in* t *on* I *and there exists a constant* $L > 0$ *such that* $\|f(t,u)\| \le L$ *for* $t \in I,$ *and* $u \in B_r$

(H_6) $RM\|BEu_0\| + (R^2M^2a\|BE\|c + RMa)(L + (K_1+K_2)a) \le r.$

3 Main Results

Theorem 3.1 *If the assumptions* $(H_1) \sim (H_6)$ *hold, then the problem* (2)-(3) *has a mild solution on* I.

Proof. Let $Z = C(I, X)$ and $Z_0 = \{u \in Z : u(t) \in B_r, t \in I\}$. Clearly, Z_0 is a bounded closed convex subset of Z. We define a mapping $F : Z_0 \longrightarrow Z_0$ by

$$\begin{aligned}(Fu)(t) &= B^{-1}T(t)BEu_0 - \sum_{k=1}^{p} c_k B^{-1}T(t)BE \int_0^{t_k} B^{-1}T(t_k - s) \\ &\times \Big[f(s, u(s)) + \int_0^s g(s, \tau, u(\tau))d\tau + \int_0^a h(s, \tau, u(\tau))d\tau\Big]ds + \int_0^t B^{-1}T(t - s) \\ &\times \Big[f(s, u(s)) + \int_0^s g(s, \tau, u(\tau))d\tau + \int_0^a h(s, \tau, u(\tau))d\tau\Big]ds, t \in I.\end{aligned}$$

Obviously F is continuous and maps Z_0 into itself. Moreover, F maps Z_0 into a precompact subset of Z_0. Note that the set $Z_0(t) = \{(Fu)(t) : u \in Z_0\}$is precompact in X, for every fixed $t \in I$. We shall show that $F(Z_0) = S = \{Fu : u \in Z_0\}$ is an equicontinuous family of functions. For $0 < s < t$, we have

$$\begin{aligned}&\|(Fu)(t) - (Fu)(s)\| \\ \leq\ & \|B^{-1}T(t) - T(s)BEu_0\| + R^2Ma\|BE\|(L + (K_1 + K_2)a)\sum_{k=1}^{p} c_k\|T(t) - T(s)\| \\ +\ & \int_0^t \|B^{-1}\|\|T(t-\theta) - T(s-\theta)\|\Big[\|f(\theta, u(\theta))\| + \int_0^\theta g(\theta, \tau, u(\tau))d\tau + \int_0^a h(\theta, \tau, u(\tau))d\tau\Big]d\theta \\ +\ & \int_0^t \|B^{-1}\|\|T(s-\theta)\|\Big[\|f(\theta, u(\theta))\| + \int_0^\theta g(\theta, \tau, u(\tau))d\tau + \int_0^a h(\theta, \tau, u(\tau))d\tau\Big]d\theta \\ \leq\ & (R\|BEu_0\| + R^2Ma\|BE\|(L + (K_1 + K_2)a)c)\|T(t) - T(s)\| \\ +\ & R(L + (K_1 + K_2)a)\int_0^t \|B^{-1}\|\|T(t-\theta) - T(s-\theta)\|d\theta + RM(L + (K_1 + K_2)a)|t - s|.\end{aligned}$$

The right hand side of the above inequality is independent of $u \in Z_0$ and tends to zero as $s \longrightarrow t$ as a consequence of the continuity of $T(t)$ in the uniform operator topology for $t > 0$. It is also clear that S is bounded in Z. Thus by $Arzela\check{\ }Ascoli's$theorem, S is precompact. Hence by the Schauder fixed point theorem, F has a fixed point in Z_0 and any fixed point of F is a mild solution of (2)-(3) on I such that $u(t) \in X$ for $t \in I$. Next we prove that the problem (2)-(3) has a strong solution.

Theorem 3.2 . *Assume that*

(i) *Conditions* $(H_1) \sim (H_6)$ *hold;*

(ii) Y *is a reflexive Banach space with norm* $\|.\|$*;*

(iii) $f : I \times B_r \longrightarrow Y$ *is Lipschitz continuous in* t *that is, there exists a constant* $L_1 > 0$ *such that*
$\|f(t,u) - f(s,v)\| \leq L_1[|t-s| + \|u-v\|]$ *for* $s,t \in I$ *and* $u,v \in B_r$*;*

(iv) $g, h : \Delta \times B_r \longrightarrow Y$ *is Lipschitz continuous in* t *that is, there exists a constant* $L_2 > 0$ *such that*
$\|g(t,\tau,u) - g(s,\tau,u)\| \leq L_g|t-s|,\ \|h(t,\tau,u) - h(s,\tau,u)\| \leq L_h|t-s|$ *for* $(t,\tau),\ (s,t) \in \Delta$ *and* $u \in B_r$*;*

(v) $Eu_0 \in D(AB^{-1})$ *and*

$$E\int_0^{t_k} B^{-1}T(t_k-s)\Big[f(s,u(s))+\int_0^s g(s,\tau,u(\tau))d\tau+\int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \in D(B);$$

(vi) u *is the unique mild solution of the problem* (2)-(3)*. Then* u *is a unique strong solution of the problem* (2)-(3) *on* I.

Proof. Since all the assumptions of Theorem 3.1 are satisfied, then the problem (2)-(3) has a mild solution belonging to $C(I, B_r)$. By assumption (vi), u is the unique mild solution of the problem (2)-(3). Now, we shall show that u is a unique strong solution of the problem (2)-(3) on I.

For any $t \in I$, we have

$$\begin{aligned}
& u(t+h) - u(t) \\
= \;& B^{-1}[T(t+h) - T(t)]BEu_0 - \sum_{k=1}^{p} c_k B^{-1}[T(t+h) - T(t)]BE \\
\times \;& \int_0^{t_k} B^{-1}T(t_k - s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \\
+ \;& \int_0^{h} B^{-1}T(t+h-s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \\
+ \;& \int_0^{t+h} B^{-1}T(t+h-s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \\
- \;& \int_0^{t} B^{-1}T(t-s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds
\end{aligned}$$

$$\begin{aligned}
&= B^{-1}T(t)[T(h)-I]BEu_0 - \sum_{k=1}^{p} c_k B^{-1}[T(t+h)-T(t)]BE \\
&\times \int_0^{t_k} B^{-1}T(t_k-s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \\
&+ \int_0^h B^{-1}T(t+h-s)\Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \\
&+ \int_0^t B^{-1}T(t-s)[f(s+h,u(s+h)) - f(s,u(s))]ds \\
&+ \int_0^t B^{-1}T(t-s)\Big[\int_0^{s+h} g(s+h,\tau,u(\tau))d\tau - \int_0^s g(s,\tau,u(\tau))d\tau\Big]ds \\
&+ \int_0^t B^{-1}T(t-s)\Big[\int_0^a h(s+h,\tau,u(\tau))d\tau - \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds.
\end{aligned}$$

Using our assumptions we observe that

$$\begin{aligned}
&\|u(t+h)-u(t)\| \\
\le\ & R\|BEu_0\|Mh\|AB^{-1}\| + cM^2R^2a\|BE\|(L+(K_1+K_2)a)h\|AB^{-1}\| \\
+\ & hRM(L+(K_1+K_2)a) + RM\int_0^t L_1[h + \|u(s+h)-u(s)\|]ds \\
+\ & RM\int_0^t \Big[\int_0^s \|g(s+h,\tau,u)-g(s,\tau,u)\|d\tau + \int_0^{s+h}\|g(s+h,\tau,u)\|d\tau\Big]ds \\
+\ & RM\int_0^t \Big[\int_0^a \|h(s+h,\tau,u)-h(s,\tau,u)\|d\tau + \int_0^a \|h(s+h,\tau,u)\|d\tau\Big]ds \\
\le\ & R\|BEu_0\|Mh\|AB^{-1}\| + [cM^2R^2a\|BE\|h\|\|AB^{-1}\| + hRM](L+(K_1+K_2)a) \\
+\ & RML_1\int_0^t [h+\|u(s+h)-u(s)\|]ds + RMah((K_1+K_2)+(L_g+L_h)a) \\
\le\ & Ph + Q\int_0^t |u(s+h)-u(s)\|ds,
\end{aligned}$$

where

$$\begin{gathered}
P = R\|BEu_0\|M\|AB^{-1}\| + cM^2R^2a\|BE\|\|AB^{-1}\| + RM(L+(K_1+K_2)a) \\
+MRL_1a + RM(K_1+K_2)a + RM(L_g+L_h)a^2
\end{gathered}$$

and $Q = RML_1$. By Gronwallis inequality

$$\|u(t+h)-u(t)\| \le Phe^Q, \ \text{ for } t \in J.$$

Therefore, u is Lipschitz continuous on I. The Lipschitz continuity of u on I combined with (iii) and (iv) imply that

$$t \longrightarrow f(t,u(t)),\ t \longrightarrow \int_0^t g(t,s,u(s))ds \ \text{ and } t \longrightarrow \int_0^a h(t,s,u(s))ds,\ t \in (0,a]$$

are Lipschitz continuous on I. Using the Corollary 2.11 [28] and the definition of strong solution we observe that the linear Cauchy problem:

$$(Bv(t))' + Av(t) = f(t,u(t)) + \int_0^t g(t,s,u(s))ds + \int_0^a h(t,s,u(s))ds,\ t \in (0,a],$$

$$v(0) = u_0 - \sum_{k=1}^{p} c_k u(t_k),$$

has a unique strong solution v satisfying the equation

$$\begin{aligned} v(t) &= B^{-1}T(t)BEv(0) + \int_0^t B^{-1}T(t-s) \\ &\times \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds,\ \ t \in I \end{aligned} \tag{6}$$

Now, we will show that $v(t) = u(t)$ for $t \in I$. Observe that

$$\begin{aligned} v(0) &= u(0) = Eu_0 - \sum_{k=1}^{p} c_k E \int_0^{t_k} B^{-1}T(t_k - s) \\ &\times \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds. \end{aligned}$$

So

$$\begin{aligned} B^{-1}T(t)BEv(0) &= B^{-1}T(t)BEu_0 - \sum_{k=1}^{p} c_K B^{-1}T(t)BE \int_0^{t_k} B^{-1}T(t_k - s) \\ &\times \Big[f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau + \int_0^a h(s,\tau,u(\tau))d\tau\Big]ds \end{aligned}$$

Substituting this in the equation (6) we see that $v(t) = u(t)$. Consequently, u is a strong solution of the problem (2)-(3) on I.

4 Example

Consider the following differential equation

$$\frac{\partial}{\partial t}(z(t,x) - z_{xx}(t,x)) = \mu(t,z(t,x)) + \int_0^t \eta_1(t,s,z(t,x))ds + \int_0^\pi \eta_2(t,s,z(t,x))ds, \tag{7}$$

$$z(t,0) = z(t,\pi) = 0,\ t \in J,$$

$$z(0,x)+\sum_{k=1}^{p} z(t_k,x)=z_0(x),\ x\in J:=[0,\pi],\ 0<t_1<t_2<...<t_p\leq a. \tag{8}$$

Let us take $X = Y = L^2[0,\pi]$. Define the operators $A : D(A) \longrightarrow X \longrightarrow Y$, $B : D(B) \longrightarrow X \longrightarrow Y$ by

$$Az = -z_{xx},$$

$$Bz = z - z_{xx},$$

respectively, where each domain $D(A), D(B)$ is given by

$$\{z \in X : z, z_x \text{ are absolutely continuous}, \ z_{xx} \in X, z(0) = z(\pi) = 0\}.$$

Define the operators $f : J \times X \longrightarrow Y,\ g, h : \Delta \times X \longrightarrow Y$ by

$$f(t,z)(x)=\mu(t,z(t,x)),\ \ g(t,s,z)(x)=\eta_1(t,s,z(t,x)),\ \ h(t,s,z)(x)=\eta_2(t,s,z(t,x)),$$

and satisfy the conditions (H_4) and (H_5) on a bounded closed set $B_r \subset X$. Here r satisfies the condition (H_6) Then the above problem (7) can be formulated abstractly as

$$(Bz(t))' + Az(t) = f(t,z) + \int_0^t g(t,s,z(s))ds + \int_0^\pi h(t,s,z(s))ds, \ \text{ a.e on } J.$$

Also, A and B can be written as

$$Az = \sum_{n=1}^{\infty} n^2 \Big\langle z, z_n \Big\rangle z_n, \quad z \in D(A)$$

$$Bz = \sum_{n=1}^{\infty} (1+n^2) \Big\langle z, z_n \Big\rangle z_n \ \ z \in D(B)$$

where $z_n(x) = \sqrt{2/\pi}\sin nx,\ n = 1, 2, ...$ is the orthogonal set of eigenfunctions of A. Furthermore, for $z \in X$ we have

$$B^{-1}z = \sum_{n=1}^{\infty} \frac{1}{(1+n^2)} \Big\langle z, z_n \Big\rangle z_n,$$

$$-AB^{-1}z = \sum_{n=1}^{\infty} \frac{-n^2}{(1+n^2)} \Big\langle z, z_n \Big\rangle z_n,$$

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t/(1+n^2)} \Big\langle z, z_n \Big\rangle z_n.$$

It is easy to see that $-AB^{-1}$ generates a strongly continuous semigroup $T(t)$ on Y and $T(t)$ is compact such that $\|T(t)\| \leq e^{-t}$ for each $t > 0$. For this $T(t), B, B^{-1}$ we assume that the operator E exists. So all the conditions of the above theorem are satisfied. Hence the equation (7) with nonlocal condition (8) has a mild solution.

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