The Solution of Exponential Growth and Exponential Decay by Using Laplace Transform

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Abstract

We consider the solution of exponential growth and exponential decay by using Laplace transform. In that an arbitrary constant can be specifically expressed as the initial value, the proposed method has a meaning.

Keywords: initial values, exponential decay, extinction of an infectious disease

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1. INTRODUCTION

Exponential growth and exponential decay can be solved in the usual way. However, in order to avoid including vague arbitrary constants, it is intended to appropriately limit arbitrary constants to initial values. In this case, the method used is an integral transform. Since the solutions by several integral transforms is almost similar, we would just like to approach the Laplace transform method.

The exponential growth and exponential decay can usually be solved by the method of separable or linear. Typically, since the time rate of change \( y'(t) \) is proportional to \( y(t) \), this gives

\[
dy/dt = ky,
\]

where \( k \) is a constant. Applying the method of a separating variable, we get \( dy/y = kdt \). Integrating both sides, we get

\[
y = e^{kt+c*} = ce^{kt} \ (c = e^{c*}).
\]
On the one hand, this equation is a linear ODE. It is well-known that the solution of linear ODE \( y' + p(t)y = 0 \) is \( y(t) = ce^{-\int p(t)dt} \). Hence, the given equation \( dy/dt = ky \) has a solution

\[
y(t) = ce^{-\int (-k)dt} = ce^{kt}.
\]

Since the arbitrary constant \( c \) is too vague, this method is somewhat unsatisfactory. For that reason, we would like to specify this constant \( c \) more by using the initial value.

On the one hand, various approaches to integral transforms can be found in [1-14, 16]. As a recent study, Wang et al.[15] modeled a differential equation on COVID-19. In this article, the solutions of the rate of changes in [15] were obtained as

\[
s = s(0)e^{-\frac{\beta s}{N}t}, \quad i = i(0)e^{(\frac{\beta s}{N} - \alpha v)t},
\]

and

\[
r = \alpha vi t + r(0)
\]

by initial values, where \( s \): susceptible, \( i \): infectious, \( \beta \): contagion rate, \( s \): susceptible, \( v \): transfer rate, \( r \): removal, \( N \): total population, and \( \alpha \): removal rate for quarantine.

2. THE SOLUTION OF EXPONENTIAL GROWTH AND EXPONENTIAL DECAY BY USING LAPLACE TRANSFORM

We would like to consider exponential growth and exponential decay by using Laplace transform.

**Theorem 2.1.** (Exponential growth and exponential decay) The solution of \( dy/dt = ky \) is \( y = y(0)e^{kt} \), where \( k \) is a constant.

Proof. By using Laplace transform, we get \( sY - y(0) = kY \), where \( Y = \mathcal{L}(y) = F(s) \).

Organizing this equality, we have

\[
Y = \frac{y(0)}{s - k}.
\]

Since

\[
\mathcal{L}(e^{at}) = \frac{1}{s - a},
\]

the solution is \( y = y(0)e^{kt} \), where \( k \) is a constant. Of course, if \( k \) is positive constant, it corresponds to exponential growth, and if \( k \) is negative constant, exponential decay.

Wang et al.[15] modeled a differential equation on COVID-19 as follows:

\[
\frac{ds}{dt} = -\frac{\beta is}{N}, \quad \frac{di}{dt} = \frac{\beta is}{N} - \alpha vi,
\]

\[
\frac{dr}{dt} = \alpha vi,
\]

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where $s$: susceptible, $i$: infectious, $\beta$: contagion rate, $s$: susceptible, $v$: transfer rate, $r$: removal, $N$: total population, and $\alpha$: removal rate for quarantine.

Let us solve the above rate of changes by using integral transform.

**Theorem 2.2. (Extinction of an infectious disease)** The solutions of the above rate of changes can be represented as

$$s = s(0)e^{-\frac{\beta i}{N}t}, \quad i = i(0)e^{\left(\frac{\beta s}{N} - \alpha v\right)t},$$

and

$$r = \alpha vi t + r(0)$$

by initial values.

Proof. Taking the Laplace transform on $ds/dt$, we get

$$uS - s(0) = -\frac{\beta i}{N}S,$$

where $\mathcal{L}(s) = S$. Simplification gives

$$S = \frac{s(0)}{u + \beta i/N}.$$  

Since $\mathcal{L}(e^{at}) = \frac{1}{u-a}$, we have the solution

$$s = s(0)e^{-\frac{\beta i}{N}t}.$$  

By the same way,

$$uI - i(0) = \frac{\beta s}{N}I - \alpha v I$$

for $\mathcal{L}(i) = I$. Thus

$$I = \frac{i(0)}{u - \left(\frac{\beta s}{N} - \alpha v\right)},$$

hence

$$i = i(0)e^{\left(\frac{\beta s}{N} - \alpha v\right)t}.$$  

Similarly, we get

$$u^2 R - r(0)u - \alpha vi = 0,$$

hence

$$R = \frac{\alpha vi}{u^2} + \frac{r(0)}{u}.$$  

Thus $r = \alpha vi t + r(0)$ because $\mathcal{L}(t) = 1/ u^2$. 


Of course, these equations can be solved in basic ways such as the method of separable or linear. However, if an integral transform is used in this way, there is an advantage in that an arbitrary constant is expressed as an initial value. Arbitrary constant can be specified as an initial value.

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**REFERENCES**


