SUPPORT STRONG BONDAGE NUMBER IN FUZZY GRAPHS

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Abstract: Let G = (V,σ,µ) be a non empty undirected fuzzy graph .The vertex u in V is said to fuzzy support strong dominates a vertex v if µ(uv)∈E(G) and fuzzy supp(u)≥ fuzzy supp(v). The fuzzy support strong(weak) bondage number $b_f^{s/}(G)$ of G is the minimum scalar cardinality of the edges whose removal from G results in a fuzzy graph with larger fuzzy support strong (weak) domination number.

In this research work we introduce the concept of fuzzy support strong bondage number in fuzzy graphs. The fuzzy support bondage number is defined. We relate this parameter to other parameter of G and obtain nice results.

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1. Introduction and Definitions

Fuzzy concept is introduced in Graph theory. To work on domination in Fuzzy graphs, it is necessary to have a sound knowledge of fuzzy sets, Graph Theory and Domination Theory. Formally, a fuzzy graph $G = (V,\sigma,\mu)$ is a non-empty set V together with a pair of functions $\sigma: V \rightarrow [0,1]$ and $\mu: V \times V \rightarrow [0,1]$ such that $\mu(xy) \leq \sigma(x) \land \sigma(y)$, for all $x, y \in V$. $\sigma$ is called the fuzzy vertex set of G and $\mu$ is called the fuzzy edge set of G.
**Definition 1.1:** Let $G = (V, E)$ be a graph. The support of a vertex in a graph is defined as the sum of the degrees of its neighbors.

\[
(i.e.) \text{supp}(u) = \sum_{v \in N(u)} \deg(v).
\]

**Definition 1.2:** The fuzzy support of the fuzzy graph $G = (\sigma, \mu)$ is the sum of the degrees of its neighbors.

**Definition 1.3:** If all the vertices of the graph $G$ have the same degree then the graph $G$ is said to be regular.

**Definition 1.4:** The effective degree of a vertex $u$ is defined to be the sum of the weights of the effective edges incident at $u$ and is denoted by $dE(u)$.

**Definition 1.5:** Let $G = (V, E)$ be a graph. A subset $S$ of $V$ is called a dominating set in $G$ if every vertex in $V \setminus S$ is adjacent to some vertex in $S$.

**Remark 1.6:** The domination number of $G$ is the minimum cardinality taken overall dominating sets in $G$ and is denoted by $\gamma(G)$ or $\gamma$.

**Definition 1.7:** Let $x$ be an element of $V$. Let $N(x) = \{y \in V : \mu(xy) = \sigma(x) \wedge \sigma(y)\}$. The fuzzy support of $u$ is defined as the sum of the neighborhood degrees of the elements in $N(u)$. That is $\text{fuzzy supp}(v) = \sum_{u \in N(v)} \sigma(u)$.

**Example 1.8:**

\[
\begin{align*}
\text{Fuzzy supp}(v_1) &= \sum_{u \in N(v_1)} \sigma(u) = 1.2 \\
\text{Fuzzy supp}(v_2) &= \sum_{u \in N(v_2)} \sigma(u) = 1.2 \\
\text{Fuzzy supp}(v_3) &= \sum_{u \in N(v_3)} \sigma(u) = 1.2 \\
\text{Fuzzy supp}(v_4) &= \sum_{u \in N(v_4)} \sigma(u) = 1.2
\end{align*}
\]

**Example: 1.9**

I. Cycles $C_n$.

II. Complete graphs $K_n$.

**Definition 1.10:** A graph is said to be support regular if $\text{supp}(u)$ is constant for all $v \in V$.

**Definition 1.11:** Let $G = (\sigma, \mu)$ be a fuzzy graph. If $d_G(v) = k$ for all $v \in V$ (i.e.) if each vertex has the same degree, then $G$ is said to be regular fuzzy graph of degree $k$ or $k$-regular fuzzy graph.
Definition 1.12: Let \( u, v \in V(G) \). Then \( u \) is said to support strong dominates \( v \) if \( uv \in E(G) \) and \( \text{supp}(u) \geq \text{supp}(v) \). A subset \( D \) of \( V(G) \) is called a support strong dominating set if for every vertex \( v \in V-D \) there exist \( u \in D \) such that \( u \) support strongly dominates \( v \).

Example 1.13:

![Diagram](image)

\( V_2 \) is fuzzy Support strong dominate vertex.

Definition 1.14: Let \( G = (\sigma, \mu) \) be a fuzzy graph. The vertex \( u \) in \( G=(\sigma, \mu) \) is said to fuzzy support strongly dominates a vertex \( v \) if \( (uv) \in E(G_f) \) and fuzzy \( \text{supp}(u) \geq \text{supp}(v) \). A fuzzy subset \( D \) of \( V(G) \) is called a fuzzy support strongly dominating set if for every \( v \in V-D \) there exists \( u \in D \) such that \( u \) fuzzy support strongly dominates \( v \).

Definition 1.15: Let \( u, v \in V(G) \). Then \( u \) is said to support weak dominates \( v \) if \( uv \in E(G) \) and \( \text{supp}(u) \leq \text{supp}(v) \). A subset \( D \) of \( V(G) \) is called a support weak dominating set if for every vertex \( v \in V-D \) there exist \( u \in D \) such that \( u \) support weakly dominates \( v \).

Definition 1.16: Let \( G_f = (\sigma, \mu) \) be a fuzzy graph. The vertex \( u \) in \( G=(\sigma, \mu) \) is said to fuzzy support weakly dominates a vertex \( v \) if \( \mu(uv) \in E(G) \) and fuzzy \( \text{supp}(u) \leq \text{supp}(v) \). A fuzzy subset \( D \) of \( V(G) \) is called a fuzzy support weakly dominating set if for every \( v \in V-D \) there exists \( u \in D \) such that \( u \) fuzzy support weakly dominates \( v \).

Definition 1.17: The minimum fuzzy cardinality of a minimal fuzzy support strong (weak) dominating set is called the fuzzy support strong (weak) domination number. It is denoted by \( \gamma_f^{\text{supp}} \) for fuzzy support, \( \gamma_f^{w} \).

Definition 1.18: Let \( G=(\sigma, \mu) \) be a nonempty undirected fuzzy graph. The fuzzy bondage number \( b_f(G) \) of \( G \) is the minimum scalar cardinality of the edges whose removal from \( G \) results in a fuzzy graph with larger fuzzy domination number.

\[
b_f(G) = \min \left\{ |B| : B \subseteq E(G), \gamma_f(G-B) \geq \gamma_f(G), |B| = \sum_{uv} \mu(uv) \right\}
\]

Definition 1.19: Let \( G=(\sigma, \mu) \) be a nonempty undirected fuzzy graph. The fuzzy support strong(weak) bondage number \( b_f^{\text{supp}}(G) \) (\( b_f^{w}(G) \)) of \( G \) is the minimum scalar cardinality of the edges whose removal from \( G \) results in a fuzzy graph with larger fuzzy support strong (weak) domination number.
\[ b^s_{\text{supp}}(G) = \land \{|B| : B \subseteq E(G), \gamma^s_{f\text{supp}}(G - B) > \gamma^s_{f\text{supp}}(G), |B| = \sum_{u \neq v} \mu(uv)\} \]

\[ b^s_{\text{supp}}(G) = \land \{|B| : B \subseteq E(G), \gamma^s_{f\text{supp}}(G - B) > \gamma^s_{f\text{supp}}(G), |B| = \sum_{u \neq v} \mu(uv)\} \]

**Result 1.20:** For any complete fuzzy graph \( K_\sigma \) the fuzzy support strong domination number \( \gamma^s_{f\text{supp}}(K_\sigma) = \alpha \), where \( \alpha \) is the vertex membership value.

**Theorem 1.21:** Let \( K_\sigma \) be fuzzy complete graph. Then the fuzzy support strong bondage number of \( K_\sigma \) is

\[ \text{(i.e.) } b^s_{f\text{supp}}(K_\sigma) = \left\lfloor \frac{p}{0.2} \right\rfloor \]

**Proof:**

**Case (i):** If \( \sigma(u_n) = \alpha \) (constant), \( n = 1, 2, 3, \ldots \) (i.e.) Every vertex of \( K_\sigma \) is constant. Let \( H \) is fuzzy spanning subgraph of a fuzzy complete graph \( K_\sigma \) obtained by removing fewer than \( \left\lfloor \frac{p}{0.2} \right\rfloor \) edges from \( K_\sigma \). Then \( H \) contains a vertex of degree \( p - \alpha \) whose fuzzy support is greater than or equal to the fuzzy support value of the remaining vertices. Then \( \gamma^s_{f\text{supp}}(H) = \alpha \). Therefore

\[ b^s_{f\text{supp}}(K_\sigma) = \left\lfloor \frac{p}{0.2} \right\rfloor \]

If \( p \) is even, the removal of \( p/2 \) independent edges from \( K_\sigma \) reduces the degree of each vertex to \( p - 2\alpha \), fuzzy support of each vertex is \( (p - 2\alpha)^2 \) and therefore yields a graph \( H_t \) with fuzzy support strong domination number \( \gamma^s_{f\text{supp}}(H_t) = (0.2)\alpha \). If \( p \) is odd, the removal of \( \left\lfloor \frac{p - \alpha}{0.2} \right\rfloor \) independent edges from \( K_\sigma \) leaves a graph having exactly one vertex of degree \( p - \alpha \), say \( v \) by removing one edge incident with this vertex. We obtained a graph \( H_t \), fuzzy support value of the vertex \( v \) is greater than or equal to the support value of the remaining vertices. Therefore \( \gamma^s_{f\text{supp}}(H_t) = (0.2)\alpha \). In both cases, \( (p \text{ is even, } p \text{ is odd}) \) the graph \( H_t \) resulted from the removal of \( \left\lfloor \frac{p}{0.2} \right\rfloor \) edges from \( K_\sigma \). Then \( \gamma^s_{f\text{supp}}(H) = \left\lfloor \frac{p}{0.2} \right\rfloor \).

**Case (ii):** If \( \alpha \) is distinct.

If \( \alpha \) is distinct then the removal of single edge can increase the fuzzy support strong domination number. From the above cases the fuzzy support strong bondage number is less than or equal to

\[ \left\lfloor \frac{p}{0.2} \right\rfloor \] (i.e.) \( b^s_{f\text{supp}}(H)(K_\sigma) = \left\lfloor \frac{p}{0.2} \right\rfloor \).

**Theorem 1.22:** Let \( K_\sigma \) be fuzzy complete graph. Then the fuzzy support weak bondage number of \( K_\sigma \) is \( b^w_{f\text{supp}}(K_\sigma) = \alpha \).

**Proof:** Let \( K_\sigma \) be fuzzy complete graph. Let \( H \) be a spanning subgraph obtained by removing any edge from \( K_\sigma \). Since the domination number of the complete graph is \( \alpha \). Since for any complete graph \( \mu(xy) = \sigma(x) \land \sigma(y) \). Therefore the removing of a single edge can increase the support...
weak domination number of $K_\alpha$. Hence the support weak domination number of $(H|E|) \geq (G|E|)$. Therefore the support weak domination number of $G = \alpha$.

**Theorem 1.24:** The fuzzy support strong domination number of fuzzy cycles $C_n (n \geq 3)$ with constant vertex membership value $\alpha$ and $\mu (xy) = \sigma (x) \land \sigma (y)$ is $\gamma_f^{\text{supp}} (C_n) = \left\lceil \frac{p}{0.3} \right\rceil$ for $n \geq 3$, where $p$ is the order of a fuzzy graph.

**Proof:** Since the membership value of the fuzzy cycles is constant and $\mu (xy) = \sigma (x) \land \sigma (y)$. Then the fuzzy cycles are totally regular. For any totally regular fuzzy cycle each vertex have the same fuzzy support. Hence the fuzzy support strong domination is equal to the domination number of fuzzy cycles. From this we get that,

$$\gamma(C_n) = \gamma_f^{\text{supp}} (C_n) = \left\lceil \frac{p}{0.3} \right\rceil$$

for $n \geq 3$.

**Theorem 1.25:** The fuzzy support strong domination number of fuzzy paths $P_n (n \geq 2)$ with constant vertex membership value $\alpha$ and $\mu (xy) = \sigma (x) \land \sigma (y)$ is $\gamma_f^{\text{supp}} (P_n) = \left\lceil \frac{p}{0.3} \right\rceil$ for $n \geq 3$, where $p$ is the order of a fuzzy graph.

**Proof:** Since the membership value of each vertex is constant with $\mu (xy) = \sigma (x) \land \sigma (y)$. If $n=2,3$ then the fuzzy support of each is always same. Therefore the support strong domination number of $P_n (n= 2,3)$ is $\alpha$. Since $n \geq 4$ then the end vertices of each path $P_n$ have the fuzzy support less than from other vertices. If the number of vertices is odd then the middle vertex belongs to the dominating set and the number of vertices in the dominating set is also odd. Then the fuzzy support strong domination number $\gamma_f^{\text{supp}} (P_n) = \left\lceil \frac{p}{0.3} \right\rceil$.

If the number of vertices is even then the open neighbourhood of the end vertices in path is belongs to the dominating set and the number vertices in the dominating set is also even. Then the fuzzy support strong domination number of even paths is

$$\gamma_f^{\text{supp}} (P_n) = \left\lceil \frac{p}{0.2} \right\rceil$$

Hence from the above results we get that the fuzzy support strong domination of paths $P_n (n \geq 2)$ is

$$\gamma_f^{\text{supp}} (P_n) = \left\lceil \frac{p}{0.3} \right\rceil$$

**Theorem 1.26:** Let $G$ be a fuzzy cycle with $n(n \geq 3)$ vertices. If the vertex membership value of each vertex $\alpha$ is constant with $\mu (xy) = \sigma (x) \land \sigma (y)$ then the fuzzy support strong bondage number $b_f^{\text{supp}} (C_3) = 0.2 \alpha$

$b_f^{\text{supp}} (C_4) = 0.3 \alpha$
\[ 0.2\alpha \quad \text{if} \quad p \equiv \alpha \mod 0.3 \\
\beta^{f}_{\text{supp}}(C_{n}) = \begin{cases} 
0.1\alpha \quad \text{if} \quad p \equiv 0, 0.2\alpha \mod 0.3 
\end{cases} \quad n \geq 5 \]

**Proof:** Let \( C_{3} \) be a fuzzy cycle. If \( n=3 \) then the removing of a single edge does not increase the domination number of \( C_{3} \). Since \( \mu(xy) = \sigma(x) \land \sigma(y) \) and the membership value of each vertex is constant. Therefore, we have to remove at least two edges from \( C_{3} \) to increase the fuzzy support strong domination number. Hence, we obtained the result \( \beta^{f}_{\text{supp}}(C_{3}) = 0.2\alpha \). Similarly, we get the result for \( n=4 \). Since removing of at least three edges can only increase the fuzzy support strong domination number. Since \( C_{n} \) is a fuzzy cycle with \( \mu(xy) = \min\{\sigma(x), \sigma(y)\} \) and the membership value of these graphs are constant. Thus, the fuzzy support strong domination (fuzzy scalar cardinality of the removing edges) of the graphs has \( p \equiv 0 \) (mod 0.3) is 0.2 \( \alpha \). Similarly, if \( n=6, 7, 9, 10, ..., \) then the removing of a single edge can increase the fuzzy support strong domination number. Thus, all the graphs of these type require removing of only one edge to increase the fuzzy support strong domination number. Since \( \mu(xy) = \sigma(x) \land \sigma(y) \) and the membership value of these graphs are constant. Therefore, the fuzzy support strong bondage number of \( C_{n} \) with 0 \( \alpha \), if \( p \equiv 0 \) (mod 0.3) is \( \alpha \). Hence, the proof.

**Theorem 1.27:** If \( k \) edges can be removed from a graph \( G \) to yield a subgraph \( H \) with \( \beta^{f}_{\text{supp}}(H) = \alpha \) and \( \gamma^{f}_{\text{supp}}(H) \geq \gamma^{f}_{\text{supp}}(G) \), then \( \beta^{f}_{\text{supp}}(G) \leq k + \alpha \).

**Proof:** Since \( \beta^{f}_{\text{supp}}(H) = \alpha \) there exist a subgraph \( H \) obtained by deleting an edge from \( H \) such that \( \gamma^{f}_{\text{supp}}(H_{1}) \geq \gamma^{f}_{\text{supp}}(H) \). But \( \gamma^{f}_{\text{supp}}(H) \geq \gamma^{f}_{\text{supp}}(G) \). Therefore \( \gamma^{f}_{\text{supp}}(H_{1}) \geq \gamma^{f}_{\text{supp}}(G) \). Therefore \( \beta^{f}_{\text{supp}}(G) \leq k + \alpha \).

**Theorem 1.28:** Let \( G \) be a fuzzy graph. Let \( u \) be a vertex in \( G \). Suppose there exists a minimum fuzzy support strong dominating set \( D \) not containing \( u \). Suppose no two vertices in \( N^{s}_{\text{supp}}(u) \) are adjacent. Then \( \beta^{f}_{\text{supp}}(G) \leq \Delta^{f}_{\text{supp}}(G) + \alpha \).

**Proof:** Let \( u \) be a vertex such that \( \text{deg}f^{w}_{\text{supp}}(u) = \Delta^{w}_{\text{supp}}(G) \). Let \( S - N^{s}_{\text{supp}}(u) = \{u_{1}, u_{2}, ..., u_{r}\} \). Remove the edges uu \( (1 \leq i \leq r) \) in the resulting graph \( H \), \( f(\text{supp})_{H}(u) = f(\text{supp})_{G}(x) \) for all \( x \in N_{G}(u) - N^{s}_{\text{supp}}(u) \). Since \( f(\text{supp})_{H}(u) = f(\text{supp})_{G}(u) \) and \( f(\text{supp})_{H}(u) = f(\text{supp})_{G}(x) - r \). Since there are \( r \) neighbors of \( u \) each is losing one degree, each is in \( H \) and \( u \) is a neighbor of \( x \). Which lose \( r \) degree in \( H \). Therefore, \( u \) is a support strong isolate in \( H \). Hence, \( D \) support strong dominates every vertex of \( H \) expect \( u \). Therefore \( (D \cup \{u\}) \) is support dominating set of \( H \). Let \( v \)
be the vertex in H which support strong dominates u in G. Then it may happen that \((D \cup \{u\}) - v\) is a support strong dominating set of H. Let w support strongly dominates v in H. Remove the edge vw. Therefore \((D \cup \{u\})\) is a minimum support strongly dominating set of H. Therefore \(\gamma_{f(supp)}^s(H) \geq \gamma_{f(supp)}^s(G)\). Therefore

\[
\begin{align*}
\beta_{f(supp)}^s(G) & \leq |S| + \alpha \\
& = |N_{f(supp)}^s(u)| + \alpha \\
& = deg_{f(supp)}^w(u) + \alpha \\
& \leq \Delta_{f(supp)}^w(G) + \alpha
\end{align*}
\]

2. Applications:
A measure of the efficiency of a domination in graphs was first given by Bauer et al [7] in 1983, who called this measure as domination line-stability defined as the minimum number of lines (i.e. edges) which when removed from G increases. In 1990, Fink et al [8] formally introduced the bondage number as a parameter for measuring the vulnerability of the interconnection network under link failure. The minimum dominating set of sites plays an important role in the network for it dominates the whole network with minimum cost. So we must consider whether its function remains good when the network is attacked. Suppose that someone such as a saboteur does not know which sites in the network take part in the dominating role, but does know that the set of these special sites corresponds to a minimum dominating set in the related graph. Then how many links does he have to attack so that the cost cannot remain the same in order to dominate the whole network? That minimum number of links is just the fuzzy bondage number.

References: