[1,2]-COMPLEMENTARY CONNECTED DOMINATION IN GRAPHS

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Abstract

A set \( S \subseteq V(G) \) in a graph \( G \) is said to be \([1,2]\)-complementary connected dominating set if for every vertex \( v \in V - S \), \( 1 \leq |N(v) \cap S| \leq 2 \) and \( < V - S > \) is connected. The minimum cardinality of \([1,2]\)-complementary connected dominating set is called \([1,2]\)-complementary connected domination number and is denoted by \( \gamma_{[1,2]}(G) \). In this paper, we initiate the study of this parameter.

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1 Introduction

The graph \( G = (V, E) \) we mean a finite, undirected, connected graph with neither loops nor multiple edges. The order and size of \( G \) are denoted by \( n \) and \( m \) respectively. The degree of a vertex \( u \) in \( G \) is the number of edges incident with \( u \) and is denoted by \( d_G(u) \), simply \( d(u) \). The minimum and maximum degree of a graph \( G \) is denoted by \( \delta(G) \) and \( \Delta(G) \), respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [2].

A set \( S \subseteq V(G) \) is a dominating set if every vertex in \( V(G) - S \) is adjacent to atleast one vertex in \( S \). The minimum cardinality of a dominating set is called the domination number and is denoted by \( \gamma(G) \). In [7], T.Tamizh Chelvam and B.Jayaprasad introduced the concept of

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complementary connected domination in graphs. A dominating set \( S \) is a complementary connected dominating set if it induces a connected subgraph in \( G \). The minimum cardinality of a complementary connected dominating set of \( G \) is called the complementary connected domination number and is denoted as \( \gamma_{cc} (G) \). In [6], V.R.Kulli and B.Janakiraman introduced the concept of nonsplit domination number of a graph. In [4],[5], Paulraj Joseph.J and Arumugam.S proved that \( \gamma(G) + \chi(G) \leq n + 1 \) and \( \gamma(G) + \kappa(G) \leq n \). Also they characterized the corresponding extremal graphs.

In [3], Mustapha Chellali et.al first studied the concept of \([1,2]\)-sets. A subset \( S \subseteq V \) in a graph \( G \) is a \([j,k]\)-set if, for every vertex \( v \in V \setminus S \), \( j \leq |N(v) \cap S| \leq k \) for any non-negative integer \( j \) and \( k \). In [9], Xiaojing Yang and Baoyindureng Wu, extended to the study of the parameter. A vertex set \( S \) of a graph \( G \) is a \([1,2]\)-set if, \( 1 \leq |N(v) \cap S| \leq 2 \) for every vertex \( v \in V \setminus S \), that is, every vertex \( v \in V \setminus S \) is adjacent to either one or two vertices in \( S \). The minimum cardinality of a \([1,2]\)-set of \( G \) is denoted by \( \gamma_{[1,2]}(G) \) and is called \([1,2]\)-domination number of \( G \).

Motivated by the above, in this paper we introduce the concept of \([1,2]\)-complementary connected domination number of graphs.

2 Main Result

Definition 2.1 A set \( S \subseteq V(G) \) in a graph \( G \) is said to be \([1,2]\)-complementary connected dominating set if for every vertex \( v \in V \setminus S \), \( 1 \leq |N(v) \cap S| \leq 2 \) and \( \langle V \setminus S \rangle \) is connected. The minimum cardinality of \([1,2]\)-complementary connected dominating set is called \([1,2]\)-complementary connected domination number and is denoted as \( \gamma_{[1,2]cc}(G) \).

Figure 2.1

In figure 1.1, \( S=\{v_1,v_3\} \) be the \([1,2]\)-complementary connected dominating set and the
complement $V - S = \{v_2, v_3, v_4, v_6, v_7\}$ is connected. Hence $\gamma_{[1,2]c}(G) = 2$.

**Observation 2.1**

1. For $P_n$, $\gamma_{[1,2]c}(P_n) = \begin{cases} n - 1 & \text{if } n \leq 3; \\ n - 2 & \text{otherwise}; \end{cases}$
2. For $C_n$, $\gamma_{[1,2]c}(C_n) = n - 2$, for any $n \geq 3$
3. If $G$ is $K_n$, $W_n$ for any $n \geq 2$, then $\gamma_{[1,2]c}(G) = 1$.
4. For $K_{m,n}$, $\gamma_{[1,2]c}(K_{m,n}) = 2$, for any $n, m \geq 2$.

**Observation 2.2** For any connected graph of order $n$, $1 \leq \gamma_{[1,2]c}(G) \leq n - 1$ and the bounds are sharp. For $K_{1,n}$ the bound is sharp.

**Observation 2.3** For any graph $G$, $\gamma(G) \leq \gamma_{cc}(G) \leq \gamma_{[1,2]c}(G)$

In figure 2.2, the dominating set $S = \{v_1, v_3, v_4\}$ and hence $\gamma(G) = 3$, the complementary connected dominating set $S = \{v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$ and hence $\gamma_{cc}(G) = 8$ and the $[1,2]$-complementary connected dominating set $S = \{v_2, v_3, v_4, v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$ and hence $\gamma_{[1,2]c}(G) = 10$. Hence $\gamma(G) \leq \gamma_{cc}(G) \leq \gamma_{[1,2]c}(G)$.

**Theorem 2.1** For any graph $G$, $\left\lfloor \frac{n}{\Delta + 1} \right\rfloor \leq \gamma_{[1,2]c}(G)$

**Proof.** Since $G$ be a connected graph, by observation 2.3 $\gamma(G) \leq \gamma_{[1,2]c}(G)$. Hence $\left\lfloor \frac{n}{\Delta + 1} \right\rfloor \leq \gamma_{[1,2]c}(G) \leq n - \Delta$. 

283
In figure 2.3, the maximum degree is three. The number of vertices is ten. The $[1,2]$ complementary connected dominating set is $S = \{v_1, v_4, v_9, v_7\}$ and $\gamma_{[1,2]cc}(G) = 4$.

$$\left\lceil \frac{n}{\Delta + 1} \right\rceil \leq \gamma_{[1,2]cc}(G) \leq n - \Delta,$$

here $\left\lceil \frac{10}{3 + 1} \right\rceil \leq 4 \leq 10 - 4 \Rightarrow 3 \leq 4 \leq 6$.

**Observation 2.4** Complement of $[1,2]–$ complementary connected dominating set need not be $[1,2]–$ complementary connected dominating set.

In figure 2.4, the $[1,2]$ complementary connected dominating set is $S = \{v_5, v_2\}$ and $\gamma_{[1,2]cc}(G) = 2$.

And the complement of the set $S$ is not $[1,2]$ complementary connected dominating set.

**Observation 2.5** Every $[1,2]–$ complementary connected dominating set is a dominating set but not conversely. Figure 2 shows that every $[1,2]–$ complementary connected dominating set is a dominating set but not conversely.
Observation 2.6 Every \([1,2]-\)complementary connected dominating set is a connected dominating set but the converse is not necessarily true.

Theorem 2.2 Let \(G\) be a connected cubic graph. Then \(\gamma_{[1,2c]}(G) = \chi(G) = 2\) if and only if \(G \cong C_4\).

Theorem 2.3 There does not exist a connected 3-regular graph on six vertices, whose chromatic number equal to \([1,2]-\)complementary connected domination number equal to 3.

Theorem 2.4 There does not exist a connected 3-regular graph on eight vertices, whose chromatic number equal to \([1,2]-\)complementary connected domination number equal to 3.

3 Relationship with other domination parameter

Theorem 3.1 Let \(G\) be a graph. Then \(\gamma_{[1,2c]}(G) + \kappa(G) \leq 2n - 2\) and the equality holds if and only if \(G \cong K_2\).

Proof. Let \(\gamma_{[1,2c]}(G) + \kappa(G) \leq 2n - 2\). Assume that \(\gamma_{[1,2c]}(G) + \kappa(G) = 2n - 2\). Then the only possible case is \(\gamma_{[1,2c]}(G) = n\) and \(\kappa(G) = n - 1\). If \(\kappa(G) = n - 1\), then \(G\) is Complete graph. But for Complete graph \(\gamma_{[1,2c]}(G) = 1\) therefore \(n = 2\) and hence \(G \cong K_2\). Converse is obvious.

Theorem 3.2 Let \(G\) be a graph. Then \(\gamma_{[1,2c]}(G) + \kappa(G) = 2n - 3\) if and only if \(G \cong K_3\).

Proof. Let \(\gamma_{[1,2c]}(G) + \kappa(G) = 2n - 3\). Then there are two possible cases to consider.

(i) \(\gamma_{[1,2c]}(G) = n - 1\) and \(\kappa(G) = n - 2\)

(ii) \(\gamma_{[1,2c]}(G) = n - 2\) and \(\kappa(G) = n - 1\)

Case 1. \(\gamma_{[1,2c]}(G) = n - 1\) and \(\kappa(G) = n - 2\)

If \(\kappa(G) = n - 2\), then we have \(n - 2 \leq \delta(G)\). If \(\delta(G) = n - 1\), then \(G\) is \(K_n\) which is a contradiction. Hence \(\delta(G) = n - 2\). Then \(G\) is \(K_n - Q\), where \(Q\) is the matching in \(G\) is \(C_4\). Therefore \(\gamma_{[1,2c]}(G) = 2 \leq 3\) which is a contradiction.

Case 2. \(\gamma_{[1,2c]}(G) = n - 2\) and \(\kappa(G) = n - 1\)

Since \(\kappa(G) = n - 1\), \(G\) is a Complete graph and \(\gamma_{[1,2c]}(K_n) = 1\) we have \(n = 3\) therefore \(G \cong K_3\).

The converse is obvious.

Theorem 3.3 Let \(G\) be a graph. Then \(\gamma_{[1,2c]}(G) + \chi(G) \leq 2n - 1\) and the equality holds if and only if \(G \cong K_2\).
Proof. Let $\gamma_{[1,2]c}(G) + \chi(G) = 2n - 1$. Therefore the only possible case is $\gamma_{[1,2]c}(G) = n - 1$ and $\chi(G) = n$. If $\chi(G) = n$, then $G$ is Complete. But for Complete graph $\gamma_{[1,2]c}(G) = 1$ and $n = 2$ therefore $G \cong K_2$. Converse is obvious.

**Theorem 3.4** Let $G$ be a graph. Then $\gamma_{[1,2]c}(G) + \chi(G) = 2n - 2$ if and only if $G \cong P_3$ or $K_2$.

Proof. Let $\gamma_{[1,2]c}(G) + \chi(G) = 2n - 2$. Then the possible cases are, (i) $\gamma_{[1,2]c}(G) = n - 1$ and $\chi(G) = n - 1$ or (ii) $\gamma_{[1,2]c}(G) = n - 2$ and $\chi(G) = n$.

**Case 1** $\gamma_{[1,2]c}(G) = n - 1$ and $\chi(G) = n - 1$

Since $\chi(G) = n - 1$, $G$ is contains a clique $K$ on $n - 1$ vertices or does not contains a clique $K$ on $n - 1$ vertices. Let $G$ is contains a clique $K$ on $n - 1$ vertices. Let $u$ be the vertex other than clique. Since $G$ is connected, $u$ be adjacent to some $v_i$ in the clique. Here $\{u, v_i\}$ forms $[1,2]cc$-set. Therefore $\gamma_{[1,2]c}(G) = 2$ so that $n = 3$ and hence $K = K_2$. Let $\{v_1, v_2\}$ be the vertices of $K_2$. Without loss of generality let $u$ be adjacent to $v_1$ in the clique. Then $G$ is $P_3$.

**Case 2** $\gamma_{[1,2]c}(G) = n - 2$ and $\chi(G) = n$

Since $\chi(G) = n$, $G$ is Complete. But for a complete graph $\gamma_{[1,2]c}(G) = 1$ and hence $n = 3$. Thus $G$ is $K_3$.

The converse is obvious.

**Theorem 3.5** Let $G$ be a graph. Then $\gamma_{[1,2]c}(G) + \chi(G) = 2n - 3$ if and only if $G \in K_4$ or $K_{1,3}$.

Proof. Let $\gamma_{[1,2]c}(G) + \chi(G) = 2n - 3$. Then the possible cases are

(i) $\gamma_{[1,2]c}(G) = n - 1$ and $\chi(G) = n - 2$, (ii) $\gamma_{[1,2]c}(G) = n - 2$ and $\chi(G) = n - 1$,

(iii) $\gamma_{[1,2]c}(G) = n - 3$ and $\chi(G) = n$.

**Case 1** $\gamma_{[1,2]c}(G) = n - 1$ and $\chi(G) = n - 2$

Since $\chi(G) = n - 2$, $G$ contains a clique $K$ on $n - 2$ vertices or $G$ does not contain a clique $K$ on $n - 2$ vertices. Suppose $G$ contains a clique $K$ on $n - 2$ vertices. Let $S = V - V(K) = \{v_1, v_2\}$. Then either $< S > = K_2$ or $< S > = K_2$

**Subcase 1** $< S > = K_2$

Since $G$ is connected either $v_1$ or $v_2$ is adjacent to a vertex in $K$. Let $v_1$ be adjacent to $u_1 \in V(K)$. Then $\{v_1, v_2, u_1\}$ is a $[1,2]$-complementary connected dominating set of $G$. Hence $\gamma_{[1,2]c}(G) \leq 3$ so
that $n \leq 4$. If $n = 4$, then $G$ is $P_4$ which is a contradiction.

**Subcase: 2** $\langle S \rangle = \overline{K}_2$

Since $G$ is connected, we have two cases to consider.

**Subcase: 2.1** $N(v_1) \cup N(v_2) \neq \emptyset$

Let $u \in N(v_1) \cup N(v_2)$. Then $\{v_1, v_2, u\}$ is a $\gamma_{[1,2]cc}(G)$-set of $G$ and hence $\gamma_{[1,2]cc}(G) = 3$ which gives $n = 4$. Thus $G$ is isomorphic to $K_{1,3}$. On increasing the degree of $v_i$ we get contradiction.

**Subcase: 2.2** $N(v_1) \cup N(v_2) = \emptyset$

Let $v_1 u_1, v_2 u_2 \in E(G)$ for some $u_1, u_2 \in V(K)$. Then $\{u_1 v_1, v_1, u_2\}$ is a $\gamma_{[1,2]cc}(G)$-set of $G$. Thus $\gamma_{[1,2]cc}(G) = 3$ and hence $n = 4$. Thus $K = K_2$ which gives $G = P_4$ which is a contradiction.

**Case: 2** $\gamma_{[1,2]cc}(G) = n - 2$ and $\chi(G) = n - 1$

Since $\chi(G) = n - 1$, $G$ contains a clique $K$ on $n - 1$ vertices or does not contain a clique $K$ on $n - 1$ vertices. Let $G$ contains a clique $K$ on $n - 1$ vertices and let $v \in V(K)$. Since $G$ is connected, without loss of generality we may assume that $v$ be adjacent to $u \in V(K)$. Then $\{u, v\}$ is a $\gamma_{[1,2]cc}(G)$-set of $G$ which gives $\gamma_{[1,2]cc}(G) = 2$ and hence $n = 3$. Thus $K = K_1$ and hence $G$ is $P_2$ which is a contradiction.

**Case: 3** $\gamma_{[1,2]cc}(G) = n - 3$ and $\chi(G) = n$

Since $\chi(G) = n$, $G$ is Complete and hence $\gamma_{[1,2]cc}(G) = 1$. Thus $n = 4$ so that $G = K_4$.

The converse is obvious.

**Conclusion:**

In this paper, we introduced the concept of $[1,2]$-connected domination number of graphs and characterized its bounds. We also showed the relation between $[1,2]cc$ sets with connectivity and chromatic number of graphs. The authors characterize the results related to sum of connectivity and $[1,2]$-complementary connected domination number, chromatic number and $[1,2]$-complementary connected domination number of order $2n-4, 2n-5, \ldots$ in subsequent papers.

**References**


