Total Domination Number on Cartesian product of Simple Fuzzy Graphs

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Abstract:
In this paper, we introduce a concept of total domination number on Cartesian product of simple fuzzy graphs (G&H) such as fuzzy path, fuzzy cycle and complete fuzzy graph. We derive the sharp bounds for total domination number in fuzzy graph (G×H). We characterize the relationship between some domination parameters such as domination number, connected domination number, edge domination number and total domination number. Further, we present some general bounds that relating the total domination number with dominator coloring and chromatic number on cartesian product of simple fuzzy graphs.

Keywords: Fuzzy graph, Cartesian product of fuzzy graphs, Dominating set, Total dominating set, Edge dominating set, Connected dominating set, Dominator coloring, Chromatic number.

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1. INTRODUCTION

Systems can be represented by mathematical models of many different forms such as algebraic equations, differential equations, finite state achiness etc. The concept of system modeling and analysis by means of linguistic variables was introduced by zadeh(1965)[1]. Zadeh suitably chosen the input and output variables as numerical by fuzzy sets. So, fuzzy sets or their precise membership functions provide an interface between the input and output numerical values. Fuzzy set approaches have several advantages over other intelligent modeling technique such as neural networks, CMAC, radial basis function networks. The concept of fuzzy graph was first introduced by Kaufmann[2] from the fuzzy relation introduced by Zedah. Although
Rosenfield[3] introduced another elaborated definition, including fuzzy vertex and fuzzy edge, along with the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness were also proposed.

The theory of product graph has wide application in science and engineering such as model concurrency in multiprocessor systems, automata theory, designing fault-tolerant communication networks, multiprocessor computer systems, error correcting codes, drug designs etc. The cartesian product of two arbitrary fuzzy graphs was defined by Moderson.J.N. and Peng [4].

The study of dominating sets in graphs was begun by Ore and Berge. The domination number was introduced by Cockayne and Hedetniemi [5]. A.Somasundaram and S.Somasundaram [6] discussed domination with reference to degree in edges. Nagoorgani and Chandrasekaran [7] discussed domination in fuzzy graph using strong arcs. A.Somasundaram presented the concept of connected domination number of fizzy graphs[8]. The fuzzy vertex coloring of a fuzzy graph was introduced by the authors Eslahchi and Onagh [9] on fuzzy set of vertices.

R.Muthuraj and A.Sasireka introduced the concept of domination number, connected domination number, edge domination number and fuzzy dominator coloring on cartesian product of fuzzy graphs[10-13]. In this paper, we introduce the concept of total domination number on cartesian product of fuzzy graphs and characterize the graph attaining these bounds. We also discussed the domination, connected domination, edge domination and total domination on G×H and relations between them are characterized. The relationship between the fuzzy dominator chromatic number and total domination number are explained.

II. Preliminaries

In this section, basic concepts of fuzzy graph are discussed. Notation and more formal definitions which are followed as in [3], [7], [8].

Definition 2.1[7]

A fuzzy graph G=(V, σ, μ) is a pair of functions  σ : V→ [0,1] and μ : V×V→[0,1], where for all u, v ∈ V, we have μ(u,v) ≤ σ(u) ∧ σ(v).
Definition 2.2[3]

The order p and size q of a fuzzy graph $G = (V, \sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{xy \in V} \mu(x, y)$.

Notation:

Without loss of generality, let us simply use the letter ‘G’ to denote a fuzzy graph.

Definition 2.3[8]

Two vertices $u$ and $v$ in $G$ are called adjacent if $(\frac{1}{2})[\sigma(u) \land \sigma(v)] \leq \mu(u, v)$.

Definition 2.4[8]

An arc $(u, v)$ is said to be a strong arc or strong edge, if $\mu(u, v) \geq \mu^\infty(u, v)$ and the node $v$ is said to be the strong neighbor of $u$. A node $u$ is said to be isolated if $\mu(u, v) = 0$ for all $u \neq v$. In a fuzzy graph, every arc is a strong arc then the graph is called the strong arc fuzzy graph.

Definition 2.5[7]

If $u$ is a node in a fuzzy graph $G$ then $N(u) = \{v: (u, v) \text{ is a strong arc}\}$ is called as neighborhood of $u$ and $N[u] = N(u) \cup \{u\}$ is called as closed neighborhood of $u$.

Definition 2.6[3]

The strong neighborhood of an edge $e_i$ in a fuzzy graph $G$ is $N_s(e_i) = \{e_j \in E(G) / e_j \text{ is a strong arc in } G \text{ and adjacent to } e_i\}$.

Definition 2.7[7]

A path $\rho$ in a fuzzy graph is a sequence of distinct nodes $u_0, u_1, u_2, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $1 \leq i \leq n$; here $n \geq 0$ is called the length of the path $\rho$.

Definition 2.8[8]

A path in which every arc is a strong arc then the path is called strong path and the path contains $n$ strong arcs is denoted by $P_n$.

Definition 2.9[8]

A cycle in $G$ is said to be fuzzy cycle if it contains more than one weakest arc.
Definition 2.10[3]

A fuzzy graph \( G=(V, \sigma, \mu) \) is called complete fuzzy graph if \( \mu(u, v) = \sigma(u) \land \sigma(v) \), for all \( u, v \in V \) and is denoted by \( K_{\sigma} \).

Definition 2.11[7]

The cartesian product \( G = G_1 \times G_2 = (V, X) \) of graphs \( G_1 \) and \( G_2 \). Then \( V=V_1 \times V_2 \), and \( X=\{(u_1,u_2),(u_1,v_2)\mid u_1 \in V_1, (u_2,v_2) \in X_2\} \cup \{(u_1,w_2),(v_1,w_2)\mid w_2 \in V_2, (u_1,v_1) \in X_1\} \).

Let \( \sigma_i \) be a fuzzy subset of \( V_i \) and let \( \mu_i \) be a fuzzy subset of \( X_i \), \( i=1,2 \). Define the fuzzy subsets \( \sigma_1 \times \sigma_2 \) of \( V \) and \( \mu_1 \times \mu_2 \) of \( X \) as follows:

\[
(\sigma_1 \times \sigma_2)(u_1,u_2) = \min\{\sigma_1(u_1),\sigma_2(u_2)\} \text{ for all } (u_1,u_2) \in V
\]

\[
(\mu_1 \times \mu_2)((u_1,u_2),(u_1,v_2)) = \min\{\sigma_1(u_1),\mu_2(u_2,v_2)\} \text{ for all } u_1 \in V_1 \text{ and } (u_2,v_2) \in X_2
\]

\[
(\mu_1 \times \mu_2)((u_1,w_2),(v_1,w_2)) = \min\{\sigma_2(w_2),\mu_1(u_1,v_1)\} \text{ for all } w_2 \in V_2 \text{ and } (u_1,v_1) \in X_1
\]

Then the fuzzy graph \( G=(\sigma_1 \times \sigma_2, \mu_1 \times \mu_2) \) is said to be the cartesian product of \( G_1=(\sigma_1,\mu_1) \) and \( G_2=(\sigma_2,\mu_2) \).

Definition 2.12[7]

Let \( G=(\sigma, \mu) \) be a fuzzy graph then the distance \( d(\sigma(u),\sigma(v)) \) between two of its vertices \( \sigma(u) \) and \( \sigma(v) \) is the length of shortest path between them, i.e., \( d(\sigma(u),\sigma(v)) = \min \left[ \sum_{u,v \in V} \mu(u, v) \right] \).

Definition 2.13[3]

The eccentricity \( e(\sigma(v)) \) of a vertex \( v \) in a fuzzy graph \( G \) is the distance from \( v \) to a vertex farthest from \( v \), i.e., \( e(\sigma(v)) = \max \{ d(\sigma(u),\sigma(v)) \mid u \in V(G) \} \).

Definition 2.14[3]

The radius of a fuzzy graph \( G \) is defined as the minimum eccentricity of vertices and is denoted by \( \text{rad } G \), i.e., \( \text{rad } G = \min \{ e(\sigma(v)) \mid v \in V(G) \} \).

Definition 2.15[3]

The diameter of \( G \) is the maximum distance between two vertices of \( G \) and it is denoted by \( \text{diam } G \), i.e., \( \text{diam } G = \max \{ e(\sigma(v)) \mid v \in V(G) \} \).
III. Results on Total domination number on Cartesian product of fuzzy graphs

In this paper, we considered a fuzzy graph as the cartesian product on same types of two fuzzy graphs (say G and H) such as fuzzy path, fuzzy cycle and complete fuzzy graph. The order of H should be greater than or equal to the order of G. Let G=(V₁, σ₁, µ₁) be a fuzzy graph and H=(V₂, σ₂, µ₂) be a fuzzy graph. The cartesian product G×H=(V, X) of fuzzy graphs G and H.

where V=V₁×V₂, X= {((u₁,u₂), (u₁,v₂)) | u₁ ∈ V₁, (u₂,v₂) ∈ X₂ } ∪ {((u₁,w₂), (v₁,w₂)) | w₂ ∈ V₂, (u₁,v₁) ∈ X₁}. Let us consider |V(G)|=m, |V(H)|=n and N=mn. Using these notations, the following theorems are defined. For proving theorems, N can be partitioned into two factors. The factor values must be |G| & |H|. To derive the theorem, we use independent fuzzy path and fuzzy cycles which does not have unique strong neighbors in fuzzy path and fuzzy cycle of G×H.

Definition 3.1[6]

A subset D of V(G) is said to be a total dominating set if every vertex in V(G) is dominated by a vertex in D.

Definition 3.2[6]

Total dominating set D of a fuzzy graph G is called a minimal total dominating set if no proper subset of D is antotal dominating sets of G.

Definition 3.3[6]

The minimum cardinality taken over all a minimal total dominating set of a fuzzy graph is called the total domination number and is denoted by γₜ(G).

Definition 3.4[6]

The minimum fuzzy cardinality of a total dominating set is called the fuzzy total domination number and is denoted by γₕt(G).

Proposition 3.5

Let u₁ dominates u₂ in G and v₁ dominates v₂ in H. Cartesian product of these vertices is denoted by Vₜ={u₁v₁,u₂v₁,u₁v₂,u₂v₂}. The induced subgraph of Vₜ is one part of the total dominating set of G×H.
Example 3.6

\[ \gamma_t = 2 \text{ and } \gamma_{ft} = 1.1 \]

Proposition 3.7

The fuzzy graphs G & H have total dominating sets if and only if the cartesian product of these fuzzy graphs also has total dominating set.

Note:

\(|D|\) denotes the crisp cardinality of a set D.

Theorem 3.8

Let G and H be any two fuzzy graphs with m and n vertices, G×H be a cartesian product of fuzzy graphs with mn vertices. D1 and D2 are total dominating set of G and H respectively. Then

i. \( \gamma_t(G \times H) \geq \max \{ \gamma_t(G), \gamma_t(H) \} \)

ii. \( \gamma_t(G \times H) \leq \gamma_t(G) + \gamma_t(H) \).

Proof:

Let us consider a two fuzzy graphs G and H with m and n vertices. Their corresponding dominating sets are D1 and D2 respectively. G×H is a cartesian product of fuzzy graphs G and H.
D is a dominating set of $G \times H$. Since $D_1 \times D_2 \subseteq D \subseteq V(G \times H)$. Let $u_1 \in D_1$ and $v_1 \in D_2$ such that $u_1$ dominates $u_2$ and $v_1$ dominates $v_2$. Then there exists a induced subset of $V_d = \{u_1v_1, u_1v_2, u_2v_1, u_2v_2\}$ is a part of total dominating set of $G \times H$. i.e., $|D| \geq \max \{\gamma_t(G), \gamma_t(H)\}$.

$\therefore \gamma_t(G \times H) \geq \max \{\gamma_t(G), \gamma_t(H)\}$.

Similarly, we proved that $\gamma_t(G \times H) \leq \gamma_t(G) + \gamma_t(H)$.

**Remark:**

The arrangements of vertices in $G \times H$ look like a matrix format in xy plane. It allows us to easily construct a graph. The row represents $G$ fiber at $v_i$ and the column represents $H$ fiber at $u_i$. Let us consider $G$ contains $m$ vertices and $H$ contains $n$ vertices. They represent the label in x and y axis. The plotting points of $G \times H$ represents grid in graph.

**Theorem 3.9**

$G \times H$ is a fuzzy graph of cartesian product of fuzzy graphs $G$ and $H$ with $mn$ vertices. Where $m=|V(G)|$ and $n=|V(H)|$. Either in m-2 rows, $G$ fiber at $v_i$ contains atleast two total dominating vertices of $G \times H$ or in n-2 columns, $H$ fiber at $u_i$ contains atleast two total dominating vertices of $G \times H$.

**Proposition 3.10**

Let $u_1 \in D$ and $u_1$ has atleast one closed neighbor in $D$ and also in $V \setminus D$.

**Theorem 3.11**

If $S$ is a minimal total dominating set of a cartesian product of fuzzy graphs $G \& H$ then each $v \in S$ has atleast one of the following conditions holds.

i. There exists a vertex $w \in V \setminus S$ such that $N(w) \cap S = \{v\}$

ii. $<S \setminus \{v\}>$ contains an isolated vertex.

**Theorem 3.12**

Let $G \times H$ be a cartesian product of fuzzy graphs then $\alpha \geq \gamma_t(G \times H)$.

**Proof:**
Let I be a maximal independent set of $G \times H$ then for any $v \in V \setminus I$, $< I \cup \{v\} >$ has at least one edge $(u,v)$ for some $u \in I$ which implies that every $v \in V$ is either in $I$ or adjacent to a vertex in $I$. Then I is also a total dominating set. Therefore, we have $\alpha \geq \gamma_t(G \times H)$.

**Proposition 3.13**

If $G$ and $H$ is a fuzzy paths with $m(\geq 2)$ and $n(\geq m)$ vertices and $G \times H$ is a fuzzy graph of cartesian product of fuzzy paths with $N(=mn)$ vertices then

i) $m = 2 \Rightarrow \gamma_t(G \times H) = \frac{mn}{m+1}$

ii) $m = 3 \Rightarrow \gamma_t(G \times H) = n$

iii) $m = 4 \Rightarrow \gamma_t(G \times H) \leq m+n$

iv) $m = 5 \Rightarrow \gamma_t(G \times H) \geq m+n-1$

v) $m = 6 \Rightarrow \gamma_t(G \times H) \geq m+n+1$

vi) $m = 7 \Rightarrow \gamma_t(G \times H) \geq m+n+2$

vii) $m = 8 \Rightarrow \gamma_t(G \times H) \leq m+2n$

viii) $m = 9 \Rightarrow \gamma_t(G \times H) \leq m+2n$

**Theorem 3.14**

Let $G$ and $H$ be a fuzzy paths with $m$ and $n$ vertices and $G \times H$ is a fuzzy graph of cartesian product of fuzzy paths with $N(=mn)$ vertices then $\gamma_t(G \times H) \geq \frac{N}{\Delta(G)}$.

**Proof:**

Given that $G$ and $H$ is a fuzzy path with $m$ and $n$ vertices. The cartesian product of fuzzy paths $G \times H$ is a fuzzy graph with $N(=mn)$ vertices. Let $S$ be a total dominating set of $G \times H$. By the definition of total dominating set, every vertex in $G \times H$ is adjacent to some vertices in $S$. Since every vertex in $V$ is totally dominated by the set $S$. That is, every vertex belongs to the open neighbourhood of at least one vertex in $S$. Then $\gamma_t(G) = \bigcup_{v \in S} N_G(v)$. 

$\Rightarrow N = \left| \bigcup_{v \in S} N_G(v) \right|$

$\leq \sum_{v \in S} |N_G(v)|$

$\leq |S| \Delta(G)$

$N \leq \gamma_t(G) \Delta(G)$

$\therefore \gamma_t(G \times H) \geq \frac{N}{\Delta(G)}$. 

Proposition 3.15

Let $G$ and $H$ be any fuzzy cycles with $m(\geq 3)$ and $n(\geq m)$ vertices. $G \times H$ is a fuzzy graph of cartesian product of fuzzy cycles $G$ and $H$ with $N (=mn \geq 9)$ vertices then $\gamma_t(G \times H) \leq \frac{2N}{3}$.

Proposition 3.16

Let $G$ and $H$ be any fuzzy cycles with $m(\geq 3)$ and $n(\geq m)$ vertices. $G \times H$ is a fuzzy graph of cartesian product of fuzzy cycles $G$ and $H$ with $N (=mn \geq 9)$ vertices then

i) $m = 3 \implies \gamma_t(G \times H) = n.$

ii) $m = 4 \implies \gamma_t(G \times H) \geq \frac{m+n}{2}.$

iii) $m = 5 \implies \gamma_t(G \times H) \leq m+n.$

iv) $m = 6 \implies \gamma_t(G \times H) \leq m+n.$

v) $m = 7 \implies \gamma_t(G \times H) \geq m+n.$

vi) $m = 8 \implies \gamma_t(G \times H) \geq m+n.$

vii) $m = 9 \implies \gamma_t(G \times H) \leq \frac{3}{2}(m+n).$

Theorem 3.17

If $G$ and $H$ are complete fuzzy graphs with $m$ and $n$ vertices. $G \times H$ is a fuzzy graph of cartesian product of complete fuzzy graphs $G$ and $H$ then $K_{mn} \cong G \times H \cup \bar{G} \times \bar{H}$.

Example 3.18

$G$:

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig4}
\caption{Complete fuzzy graph ($K_3$)}
\end{figure}
\end{center}

$H$:

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig5}
\caption{Complete fuzzy graph ($K_3$)}
\end{figure}
\end{center}
In Fig.7, all edges consider as strong edges, so it assigns the minimum membership values from the end vertices. In Fig.8 represents the graph $K_9(V, \sigma, \mu)$, Where $V(K_9) = \{A, B, C, D, E, F, G, H, I\}$. The membership values of vertices and edges as follows.

$\sigma(A)=0.4, \sigma(B)=0.6, \sigma(C)=0.5, \sigma(D)=0.2, \sigma(E)=0.3, \sigma(F)=0.3, \sigma(G)=0.4, \sigma(H)=0.8, \sigma(I)=0.5$.

$\mu(A,B)=0.4, \mu(A,C)=0.4, \mu(A,D)=0.2, \mu(A,E)=0.3, \mu(A,F)=0.3, \mu(A,G)=0.4, \mu(A,H)=0.4, \mu(A,I)=0.4, \mu(B,C)=0.5, \mu(B,D)=0.2, \mu(B,E)=0.3, \mu(B,F)=0.3, \mu(B,G)=0.4, \mu(B,H)=0.6, \mu(B,I)=0.5, \mu(C,D)=0.2, \mu(C,E)=0.3, \mu(C,F)=0.3, \mu(C,G)=0.4, \mu(C,H)=0.5, \mu(C,I)=0.5, \mu(D,E)=0.2, \mu(D,F)=0.2, \mu(D,G)=0.2, \mu(D,H)=0.2, \mu(D,I)=0.2, \mu(E,F)=0.3,$

In Fig.7, all edges consider as strong edges, so it assigns the minimum membership values from the end vertices. In Fig.8 represents the graph $K_9(V, \sigma, \mu)$, Where $V(K_9) = \{A, B, C, D, E, F, G, H, I\}$. The membership values of vertices and edges as follows.

$\sigma(A)=0.4, \sigma(B)=0.6, \sigma(C)=0.5, \sigma(D)=0.2, \sigma(E)=0.3, \sigma(F)=0.3, \sigma(G)=0.4, \sigma(H)=0.8, \sigma(I)=0.5$.

$\mu(A,B)=0.4, \mu(A,C)=0.4, \mu(A,D)=0.2, \mu(A,E)=0.3, \mu(A,F)=0.3, \mu(A,G)=0.4, \mu(A,H)=0.4, \mu(A,I)=0.4, \mu(B,C)=0.5, \mu(B,D)=0.2, \mu(B,E)=0.3, \mu(B,F)=0.3, \mu(B,G)=0.4, \mu(B,H)=0.6, \mu(B,I)=0.5, \mu(C,D)=0.2, \mu(C,E)=0.3, \mu(C,F)=0.3, \mu(C,G)=0.4, \mu(C,H)=0.5, \mu(C,I)=0.5, \mu(D,E)=0.2, \mu(D,F)=0.2, \mu(D,G)=0.2, \mu(D,H)=0.2, \mu(D,I)=0.2, \mu(E,F)=0.3,$
\( \mu(E,G)=0.3, \mu(E,H)=0.3, \mu(E,I)=0.3, \mu(F,G)=0.3, \mu(F,H)=0.3, \mu(F,I)=0.3, \mu(G,H)=0.4, \mu(G,I)=0.4, \mu(H,I)=0.5. \)

Let us consider \( F = K_3 \times K_3 \cup K_3 \times K_3 \). Here there exists a function from the vertices, \( \theta : V(F) \to V(K_9) \), and for the edges \( \phi : V(F) \to V(K_9) \). Where \( \theta : A \to (u_1, v_1), \ B \to (u_1, v_2), \ C \to (u_1, v_3), \ D \to (u_2, v_1), \ E \to (u_2, v_2), \ F \to (u_2, v_3), \ G \to (u_3, v_1), \ H \to (u_3, v_2), \ I \to (u_3, v_3). \)

Hence the graphs \( K_9 \) and \( K_3 \times K_3 \cup K_3 \times K_3 \) are isomorphic under the isomorphism \( (\theta, \phi) \).

**Theorem 3.19**

Let \( G \) and \( H \) be a fuzzy cycles with \( m \) and \( n \) (\( \geq 3 \)) vertices. \( G \times H \) is a fuzzy graph of cartesian product of fuzzy cycles with \( N(=mn) \) vertices.

i. If \( \Delta(G \times H) = 4 \) then \( \gamma_t(G \times H) \leq N - \Delta(G \times H) \).

ii. Order of \( G \times H \) is atleast 2 then \( \gamma_t(G \times H) \leq n \text{ rad}(G \times H) \).

iii. Order of \( G \times H \) is atleast 2 then \( \gamma_t(G \times H) \leq \frac{\text{diam}(G \times H) + 1}{2} \).

**IV. Relations between Domination Parameters on Cartesian product of fuzzy graphs**

**Definition 4.1[3]**

A dominating set \( D \) of a fuzzy graph \( G \) is called the minimal dominating set if and only if for each vertex \( v \in V \), \( D \setminus \{u\} \) is not a dominating set of \( G \). The minimum cardinality taken over all minimal dominating sets in \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \). The minimum fuzzy cardinality taken over all minimal dominating sets in \( G \) is called the fuzzy domination number of \( G \) and is denoted by \( \gamma_f(G) \).

**Definition 4.2[3]**

A dominating set \( D \) of a fuzzy graph \( G=(V, \sigma, \mu) \) is connected dominating set if the induced subgraph \( \langle D \rangle \) is connected. The minimum cardinality of a connected dominating set of \( G \) is called the connected domination number of \( G \) and is denoted by \( \gamma_{cd}(G) \). The minimum fuzzy cardinality of a connected dominating set of \( G \) is called the fuzzy connected domination number of \( G \) and is denoted by \( \gamma_{fcd}(G) \).
Definition 4.3[6]

A subset $D$ of $E(G)$ is said to be an edge dominating set if for every $e_j \in E(G) \setminus D$, there exists $e_i \in D$ such that $e_i$ dominates $e_j$. An edge dominating set $D$ of a fuzzy graph $G$ is called a minimal edge dominating set if no proper subset of $D$ is an edge dominating sets of $G$. The smallest number of edges in any edge dominating set of a fuzzy graph $G$ is called its edge domination number and it is denoted by $\gamma'(G)$. The minimum fuzzy cardinality taken over all minimal edge dominating set of $G$ is called its fuzzy edge domination number and it is denoted by $\gamma'_f(G \times H)$.

Theorem 4.4

Let $G$ and $H$ be any two fuzzy graphs with $m$ and $n$ vertices. $G \times H$ be a cartesian product of fuzzy graphs with $mn$ vertices then $\gamma(G \times H) \leq \gamma_t(G \times H) \leq 2 \gamma(G \times H)$.

Proof:

First we prove that, $\gamma(G \times H) \leq \gamma_t(G \times H)$. By the definition of domination and total domination number which follows the inequality immediately. Let $S$ be a dominating set but not a total dominating set. Therefore, $<S>$ has isolated vertices. Each of these isolated vertices should have some private neighbors. Now, to construct a total dominating set, just add a private neighbor of each isolated vertex in $S$. we get a new set say $S'$ which yields at most twice the number of vertices in $S$. That is, $|S'| \leq 2|S|$. Since $S'$ is a total dominating set, $|S'| = \gamma_t(G \times H)$. Therefore, $\gamma_t(G \times H) \leq 2|S| = 2\gamma(G \times H)$.

Hence $\gamma(G \times H) \leq \gamma_t(G \times H) \leq 2 \gamma(G \times H)$.

Proposition 4.5

If $G$ and $H$ is a fuzzy paths with $m(\geq 2)$ and $n(\geq m)$ vertices and $G \times H$ is a fuzzy graph of cartesian product of fuzzy paths with $N(=mn)$ vertices then

i. $\gamma_t(G \times H) \leq \gamma + 2$, for $N \leq 36$

ii. $\gamma_t(G \times H) \leq \gamma + 4$, for $36 < N \leq 63$

iii. $\gamma_t(G \times H) \leq \gamma + 5$, for $N > 63$
Observation 4.6

If G and H are any fuzzy paths(cycles) on m and n vertices and G×H is a cartesian product of fuzzy paths(cycles) G & H on mn vertices then \( \gamma \leq \gamma_t \leq \gamma'_c \leq \gamma_c \).

Theorem 4.7

If G and H is a fuzzy paths with \( m(\geq 2) \) and \( n(\geq m) \) vertices and G×H is a fuzzy graph of cartesian product of fuzzy paths with \( N(=mn) \) vertices then

i. \( \gamma'_c = \gamma_t \) for \( N < 45 \)

ii. \( \gamma'_c > \gamma_t \) for \( N \geq 45 \)

Theorem 4.8

If G and H are any fuzzy cycles on m and n vertices and G×H is a cartesian product of fuzzy cycles G & H on \( N(=mn) \) vertices then \( \gamma'_c > \gamma_t \) for all \( N \).

Proposition 4.9

If G and H are a complete fuzzy graphs on m and n vertices and G×H is a cartesian product of complete fuzzy graphs G & H on \( mn \) vertices. D is a total dominating set and \( D' \) is a connected dominating set. Then \( D \cong D' \) and which are isomorphic to \( K_m \).

Theorem 4.10

If G and H are a complete fuzzy graphs on m and n vertices and G×H be a cartesian product of complete fuzzy graphs on \( mn \) vertices then \( \gamma(G×H) = \gamma_t(G×H) = m \).

Proof:

Let D be a minimal dominating set of a fuzzy graph G×H. \( D_t \) be a minimal totaldominating set of a fuzzy graph G×H. The vertex set of G×H can be partitioned into disjoint m components. Such as \( (u_1,v_1),(u_1,v_2),(u_1,v_3),\ldots,(u_1,v_n), (u_2,v_1),(u_2,v_2),(u_2,v_3),\ldots,(u_2,v_n),\ldots,(u_m,v_n) \). Since each component should be a complete on itself. In a component, \( (u_1,v_1) \) is dominated by \( (u_1,v_j) \), \( (u_2,v_1) \) is dominated by \( (u_2,v_j) \), and so on \( (u_m,v_1) \) is dominated by \( (u_m,v_j) \) for all \( 1 \leq j \leq n \). Therefore, every vertex is dominated by exactly one vertex in each component. \( \gamma(G×H) = m \). By the definition of cartesian product of fuzzy graph, there exists some edges between the components. So the vertices in dominating set should be adjacent to each
other and also it may contain complete graph which yields a total dominating set. Therefore, 
$$\gamma_t(G \times H) = m.$$ 
Hence $$\gamma(G \times H) = \gamma_t(G \times H) = m.$$ 

**Theorem 4.11**

If G and H are a complete fuzzy graphs on m and n vertices and G \times H is a cartesian product of complete fuzzy graphs G & H on mn vertices then

i. \( \gamma = \gamma_t = \gamma_c \) for all \( N \).

ii. \( \gamma'_c = \gamma_t \) for \( m = n \)

iii. \( \gamma'_c > \gamma_t \) for \( m < n \)

**V. Relations between \( \gamma_t \) and \( \chi_{fd} \) on Cartesian product of fuzzy graphs**

**Definition 5.1[9]:**

A family \( \Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \) of fuzzy sets on \( V \) is called a k-fuzzy coloring of \( G = (V, \sigma, \mu) \) if

a) \( \forall \Gamma = \sigma \)

b) \( \gamma_i \land \gamma_j = 0 \)

c) For every strong edge \((u,v)\) of \( G \), \( \gamma_i(u) \land \gamma_j(v) = 0 \) for \( 1 \leq i \leq k \).

A fuzzy graph \( G \) is k-fuzzy colorable if \( G \) has a proper k-fuzzy coloring. The fuzzy chromatic number \( \chi_t(G) \), of a fuzzy graph \( G \), is the minimum \( k \) for which \( G \) is k-fuzzy colorable.

The above definition of k-fuzzy coloring was defined by the authors Eslahchi and Onagh [9] on fuzzy set of vertices.

**Definition 5.2[13]:**

A fuzzy dominator coloring (FDC) of a fuzzy graph \( G \) is a proper fuzzy coloring in which each vertex of \( G \) dominates every vertex of at least some color class.

**Definition 5.3[13]:**

Fuzzy dominator chromatic number of a fuzzy graph is the minimum number of color classes in a dominator fuzzy coloring of \( G \). It is denoted by \( \chi_{fd}(G) \).

**Remark**
In a cartesian product of fuzzy paths \( G \times H \) has a minimal fuzzy dominator coloring in which at least two color classes are non dominated.

**Theorem 5.4**

If \( G \) and \( H \) is a fuzzy paths with \( m(\geq 2) \) and \( n(\geq m) \) vertices and \( G \times H \) is a fuzzy graph of cartesian product of fuzzy paths with \( N(=mn) \) vertices then \( \chi_{fd}(G \times H) = \gamma_t(G \times H) + 2 \).

**Proof:**

By the definition of dominator coloring to form color class, every dominated vertex in minimal total dominating set assign different colors say \( C_1, C_2, C_3, \ldots, C_{\gamma_t} \). The neighbors of dominated vertices assign color \( C_{\gamma_t+1} \) but within the neighbor of different dominated vertices, they adjacent to each other so these vertices assign distinct colors as \( C_{\gamma_t+1} \) and \( C_{\gamma_t+2} \). Therefore, totally its need only \( \gamma_{t+2} \) color class to color a cartesian product of fuzzy graphs.

Hence, \( \chi_{fd}(G \times H) = \gamma_t(G \times H) + 2 \).

**Observation 5.5**

If \( G \) and \( H \) are any fuzzy paths(cycles) on \( m \) and \( n \) vertices and \( G \times H \) is a cartesian product of fuzzy paths(cycles) \( G \) & \( H \) on \( mn \) vertices then \( \chi_t \leq \gamma_t \).

**Proposition 5.6**

If \( G \) and \( H \) are any fuzzy paths(cycles) on \( m \) and \( n \) vertices and \( G \times H \) is a cartesian product of fuzzy paths(cycles) \( G \) & \( H \) on \( mn \) vertices then

\[
\min\{\chi_t(G \times H), \gamma_t(G \times H)\} \leq \chi_{fd}(G \times H) \leq \chi_t(G \times H) + \gamma_t(G \times H).
\]

**Theorem 5.7**

If \( G \) and \( H \) are any fuzzy cycles on \( m \) and \( n \) vertices and \( G \times H \) is a cartesian product of fuzzy cycles \( G \) & \( H \) on \( mn \) vertices then

\[
\chi_{fd} = \begin{cases} 
\gamma_t + 2, & \text{if } N \equiv 0 \mod 4 \text{ and } m, n \text{ are even numbers} \\
\gamma_t + 3, & \text{otherwise}
\end{cases}
\]

**Proof:**

\( G \times H \) is a cartesian product of fuzzy cycles \( G \) & \( H \) on \( mn \) vertices where \( m \) is \( |V(G)| \) and \( n \) is \( |V(H)| \). \( G \times H \) must contains \( m \) cycles and \( n \) vertices. Every \( H \) fiber at \( u_i \) must contain atleast one total dominating vertices. So it assigns \( \gamma_t \) color classes. Let \( (u_2,v_1) \in D \) and \( (u_1,v_1) \in V \setminus D \) then
N[(u_1, v_1)] \cap D = \{(u_2, v_1)\}. Such vertices assign new colors say \(\gamma_{t+1}\) color. Except \((u_1, v_1)\), some other neighborhoods of \((u_1, v_1)\) in V\D assign \(\gamma_{t+2}\) color and which are adjacent to \(\gamma_{t+1}\) colored vertex. Since the adjacent vertices assign distinct colors. That is, for a cartesian product of fuzzy graphs have proper fuzzy coloring.

i. If \(N \equiv 0 \ mod \ 4\) where \(m\) and \(n\) are even numbers then \((u_2, v_1)\) dominates itself and the vertices \((u_1, v_1), (u_3, v_1)\) and \((u_2, v_2)\). Any one of these vertices must contain in total dominating set. Since in \(G \times H\), origin and end vertices of \(H\) fiber must assign distinct colors say \(\gamma_{t+1}, \gamma_{t+2}\). Every \(H\) fiber must contains even number of elements. So, the neighborhood of element in \(D\) may assign same colors. Hence \(\chi_{fd}(G \times H) = \gamma_{t+2}\).

ii. Suppose \(N \not\equiv 0 \ mod \ 4\) and any one of \(m\) and \(n\) are not even number. Consider that \(m\) is an odd number. Similar argument also done here, every \(H\) fiber at \(u_i\) must contain atleast one total dominating element. So it’s enough with \(\gamma_i\) color classes for forming dominator coloring set. The neighborhoods of these elements assign distinct colors so its need \(\gamma_{t+2}\) color classes. Here we consider \(m\) is odd then \(G\) fiber at \(v_i\) contains odd number of elements. Either origin and end vertices might assign same color or some element occur the neighborhood of \(\gamma_{t+1}\) and \(\gamma_{t+2}\) color class vertices. Such element may not assign any one of these colors, choose \(\gamma_{t+3}\) color. Therefore, odd number of elements in \(G \times H\), the dominator coloring is \(\gamma_{t+3}\). Hence \(\chi_{fd}(G \times H) = \gamma_{t+3}\).

**Observation 5.8**

For cartesian product of complete fuzzy graphs \(G \times H\), \(\gamma_t(G \times H) \leq \chi_{fd}(G \times H)\).

**VI. Conclusion**

The concept of total domination number on cartesian product of fuzzy graphs is introduced. The graphs attaining these bounds are characterized. The relationship between total domination and domination parameter on \(G \times H\) are distinguished such as domination, connected domination and edge domination. We described the bounds on them also. Finally, the relationship between the fuzzy dominator chromatic number and total domination number are explained.

**References**


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