A Study on 2-rainbow domination number of Honeycomb Networks and Honeycomb Cup Networks

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Abstract

The 2–rainbow domination is a variant of the classical domination problem in graphs and is defined as follows: Given an undirected graph G = (V, E) and a set of 2 colours numbered 1 and 2, we assign an arbitrary subset of these colours to each vertex of G. If a vertex is assigned the empty set, then the union of colour sets of its neighbors must be 2-colours. This minimum sum of numbers of assigned colours over all vertices of G is called the 2-rainbow domination number of G. In this paper, we studied the 2-rainbow domination number of Honeycomb networks and Honeycomb Cup Networks.

Keywords: Domination, domination number, 2-rainbow domination number, Honeycomb Networks, Honeycomb cup Networks.

1. INTRODUCTION:

Let G = (V, E) be a finite, simple, undirected and connected graphs. A non empty subset S of the vertex set V (G) of a graph G is called a dominating set if every vertex in V(G)−S is adjacent to at least one vertex in S. The domination number γ (G) is the minimum cardinality of a minimal dominating set of G.

Rainbow domination was first introduced by Brešar, Henning and Rall in 2005. The open neighborhood of v is the set N (v) = {u ∈ V (G) | uv ∈ E (G)} and its closed neighborhood is the set N[v] = N(v) ∪ {v}. Let f: V(G) → P{1,2,...,k} be a function...
that assigns to each vertex of $G$ a set of colours chosen from the power set of $\{1,2,\ldots,k\}$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset, \bigcup_{u \in N(v)} f(u) = \{1,2,\ldots,k\}$, then the function $f$ is called a $k$-rainbow dominating function (k-RDF) of $G$. The weight of the function $f$ is denoted by $w(f)$ and is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a k-RDF is called the $k$-rainbow domination number of $G$ and is denoted by $\gamma_{rk}(G)$.

A 2-rainbow dominating function of a graph $G$ is a particular case of k-RDF i.e., when $k=2$. We defined $f: V(G) \rightarrow P\{1,2\}$ such that for each vertex $v \in V(G)$ and $f(v) = \emptyset$, we have $\bigcup_{u \in N(v)} f(u) = \{1,2\}$. Such a function $f$ is called a 2-rainbow dominating function (2RDF) and minimum weight of such function is called the 2-rainbow domination number of $G$ and is denoted by $\gamma_{r2}(G)$.

For example, a 2-rainbow domination of path $P_5$ is shown in Figure 1. We assigned a colour set $\{1\}$ to $v_1$ and assigned a colour set $\{2\}$ to $v_4$. When $f(v_2) = \emptyset$ then $\bigcup_{u \in N(v_2)} f(u) = \{1,2\}$. Similarly, the union of colour sets of neighbors of $v_3$ and $v_5$ are $\{1,2\}$ respectively.

In fact $\gamma_{r2}(P_5) = 3$.

![Figure 1: Path $P_5$](image)

In Figure 1, Path $P_5$ depicted with 2-rainbow domination. The vertex with colour sets $\{1\}$ and $\{2\}$ are filled with colours red and blue respectively.

In this paper, we obtained the 2-rainbow domination number for Honeycomb network and Honeycomb Cup networks.

2. HONEYCOMB NETWORK AND HONEYCOMB CUP NETWORK

2.1 Honeycomb Network

Honeycomb networks are built recursively using hexagonal tessellations. The honeycomb network $HC(1)$ is a hexagon. The honeycomb network $HC(2)$ is obtained by adding six hexagons to the boundary edges of $HC(1)$. Inductively, honeycomb network $HC(n)$ is obtained from $HC(n-1)$ by adding a layer of hexagons around the boundary of $HC(n-1)$. The parameter $n$ of $HC(n)$ is determined as the number of hexagons between the center and boundary of $HC(n)$. The number of vertices and edges of $HC(n)$ are $6n^2$ and $9n^2-3n$ respectively. The diameter of $HC(n)$ is $4n-1$. A honeycomb network $HC(2)$ is shown in Figure 2 below.
In Graph Theory to study the honey comb network we use brick structure of the honey comb networks. Brick structure is obtained by shrinking one of the upper and lower vertices in the form of straight lines. Thus, in brick representation also there are equal number of vertices and edges. A brick representation of HC (2) is shown in Figure 3.

Honeycomb networks are widely used in computer graphics, cellular phone base station, image processing, and in chemistry as the representation of benzenoid hydrocarbons and carbon hexagons of carbon Nanotubes.

### 2.2 Honeycomb Cup Network

Honeycomb cup networks are built recursively using hexagonal tessellations. The honeycomb Cup network HCC (1) is obtained by adding two hexagons in adjacent and also adding one hexagon to the boundary of two hexagons. The parameter n of HCC (n) is determined as the number of hexagons between the center and boundary.
of HCC (n). The number of vertices and edges of HC (n) are $2(3n^2 + 4n + 1)$ and $(9n^2 + 9n + 1)$ respectively. A honeycomb Cup network HCC (1) and HCC (2) are shown in Figure 4 below.

In Graph Theory, to study the honeycomb network we use brick structure of the honeycomb cup networks. Brick structure is obtained by shrinking one of the upper and lower vertices in the form of straight lines. Thus, in brick representation also there are equal number of vertices and edges. A brick representation of HCC (1) and HCC (2) are shown in Figure 5.
3. 2-RAINBOW DOMINATION NUMBER FOR HONEYCOMB NETWORK AND HONEYCOMB CUP NETWORK:

Theorem 3.1.
If HC (n) is a honeycomb network, then \( \gamma_{r2}(HC (n)) = 3n^2 + 1 \).

Proof:
Case (i): The proof is given by constructing a 2-RDF for any given honeycomb network of dimension n, when n is even.

First we found the dominating sets of HC (n).
By definition of honeycomb network, there are \( 3n^2 \) dominating sets in HC (n).
(i.e.,) \( \gamma(HC (n)) = 3n^2 \)
Consider n = 4.
The dominating sets in HC (4) are 48. (by definition of honeycomb network)
(i.e.,) \( \gamma(HC (4)) = 3(4)^2 = 48 \).
Let \( S \) be the dominating set such that \( S = \{ x \in V(HC(4)) : f(x) \neq \emptyset \} \). Then for every \( u \in V(HC (4)) - S \), we have \( |f(N(u))| \geq 2 \).
Let \( f: V (HC (4)) \rightarrow \mathcal{P} \{1, 2\} \) be defined as
\[
f(x_i) = \begin{cases} 
\{1, 2\}, & \text{if } x_i \in S \\
\emptyset, & \text{if } x_i \notin S 
\end{cases}
\]
Where \( i = 1, 2, \ldots, n \).
\[
\bigcup_{u_i \in N(x_i)} f(u_i) = \{1, 2\} \quad \text{Where } x_i \notin S \text{ and } i = 1, 2, \ldots, n.
\]
Such a function \( f \) is a 2–rainbow dominating function and since each vertex in the dominating set is assigned with the set of two colours.
Thus \( \gamma_{r2}(HC (4)) = 49 = 3(4)^2 + 1 \).
The 2-rainbow domination number of HC (4) is given in figure 6.
Here we assigned the colour set \{1,2\} in the center of HC (4) and assigned the colour sets \{1\} and \{2\} to the other dominating sets in the HC (4) alternately. Clearly all the vertices that are assigned \( \emptyset \) has the colour set \{1,2\} in its neighborhood.
Case (ii):

When \( n \) is odd.

First we obtained the dominating sets of \( HC(n) \).

By the definition of honeycomb network, there are \( 3n^2 \) dominating sets in \( HC(n) \).

(i.e.,) \( \gamma(HC(n)) = 3n^2 \)
Consider $n = 7$.

The dominating sets in $HC(7)$ are 147.

(i.e.,) $\gamma(HC(7)) = 3(4)^2 = 147$.

Let $S$ be the dominating set such that $S = \{x \in V(HC(7)) : f(x) \neq \emptyset \}$.

Then for every $u \in V(HC(7)) - S$, we have $|f(N(u))| \geq 2$.

Let $f : V(HC(7)) \rightarrow \mathcal{P}\{1, 2\}$ be defined as

$$f(x_i) = \begin{cases} \{1,2\}, & \text{if } x_i \in S \\ \emptyset, & \text{if } x_i \notin S \end{cases} \quad \text{Where } i = 1, 2, \ldots, n.$$  

$$U_{u_i \in N(x)} f(u_i) = \{1,2\} \quad \text{Where } x_i \notin S \text{ and } i = 1, 2, \ldots, n.$$  

Such a function $f$ is a 2–rainbow dominating function and since each vertex in the dominating set is assigned with the set of two colours.

Thus

$$\gamma_{r2}(HC(7)) = 148 = 3(7)^2 + 1.$$  

The 2-rainbow domination number of $HC(7)$ is given in figure 7.

Here we assigned the colour set $\{1,2\}$ in the centre of $HC(7)$ and assigned the colour sets $\{1\}$ and $\{2\}$ to the other dominating sets in the $HC(7)$ alternately.

Clearly all the vertices that are assigned $\emptyset$ has the colour set $\{1,2\}$ in its neighborhood.
Figure – 7 (HC (7))
In figure 7, the brick representation of honeycomb network HC (7) is shown with 2-rainbow domination number. The vertex with the colour set \{1,2\} is filled with green colour. The vertex with colour sets \{1\} and \{2\} are filled with the colours red and blue respectively.

**Theorem 3.2.**

If HCC (n) is a honeycomb Cup network, then \(\gamma_{r2}(HCC (n)) = 3n^2 + 4n + 2\).

**Proof:**

Case (i) :

The proof is given by constructing a 2-RDF for any given honeycomb cup network of dimension n, when n is even.

First we obtained find the dominating sets of HCC (n).

By definition of honeycomb cup network, there are \(2(3n^2 + 4n + 2)\) vertices in HCC (n) and then we have \(3n^2 + 4n + 1\) dominating sets in HCC (n).

(i.e.,) \(\gamma(HCC(n)) = 3n^2 + 4n + 1\).

Consider \(n = 2\).

The dominating sets in HCC (2) are 21. (by definition of honeycomb cup network)

(i.e.,) \(\gamma(HCC(2)) = 3(2)^2 + 4(2) + 1 = 21\).

Let \(S\) be the dominating set such that \(S = \{x \in V(HCC(2)) : f(x) \neq \emptyset\}\). Then for every \(u \in V(HCC(2)) - S\), we have \(|f(N(u))| \geq 2\).

Let \(f : V(HCC(2)) \rightarrow \mathcal{P}\{1, 2\}\) be defined as

\[
f(x_i) = \begin{cases} 
\{1, 2\}, & \text{if } x_i \in S \\
\emptyset, & \text{if } x_i \notin S
\end{cases} \quad \text{Where } i = 1, 2, \ldots, n.
\]

\[
\bigcup_{x_i \in N(x_j)} f(u_i) = \{1, 2\} \quad \text{Where } x_i \notin S \text{ and } i = 1, 2, \ldots, n.
\]

Such a function \(f\) is a 2- rainbow dominating function and since each vertex in the dominating set is assigned with the set of two colours.

Thus \(\gamma_{r2}(HCC(2)) = 22 = 3(2)^2 + 4(2) + 2\).

The 2-rainbow domination number of HCC (2) is given in figure 8.
Here we assigned the colour set \{1,2\} in the centre of HCC (2) and assigned the colour sets \{1\} and \{2\} to the other dominating sets in the HCC (2) alternatively.

Clearly all the vertices that are assigned \emptyset has the colour set \{1,2\} in its neighborhood.

In figure 8, the brick representation of honeycomb cup network HCC (2) is shown with 2-rainbow domination number. The vertex with the colour set \{1,2\} is filled with green colour. The vertex with colour sets \{1\} and \{2\} are filled with the colours red and blue respectively.

**Case(ii)**

When \(n\) is odd.
First we found the dominating sets of HCC (n). By the definition of honeycomb network, there are \(3n^2 + 4n + 1\) dominating sets in HCC (n).

(i.e.,) \(\gamma(HCC (n)) = 3n^2 + 4n + 1\)
Consider \(n = 3\).
The dominating sets in HC (3) are 40.
(i.e.,) \(\gamma(HC (3)) = 3(3)^2 + 4(3) + 1 = 40\).
Let \(S\) be the dominating set such that \(S = \{x \in V(HCC (3)) : f(x) \neq \emptyset\}\). Then for every \(u \in V(HCC (3)) - S\), we have \(|f(N(u))| \geq 2\).
Let \( f: V(\text{HCC}(3)) \rightarrow \mathcal{P}\{1, 2\} \) be defined as
\[
f(x_i) = \begin{cases} 
{\{1, 2\}}, & \text{if } x_i \in S \\
\emptyset, & \text{if } x_i \notin S
\end{cases}
\]
Where \( i = 1, 2, \ldots, n \).
\[
\bigcup_{u_i \in N(x_i)} f(u_i) = \{1, 2\} \quad \text{Where } x_i \notin S \text{ and } i = 1, 2, \ldots, n.
\]
Such a function \( f \) is a 2-rainbow dominating function and since each vertex in the dominating set is assigned with the set of two colours.

Thus
\[
\gamma_{r2}(\text{HCC}(3)) = 41 = 3(3)^2 + 4(3) + 2.
\]

The 2-rainbow domination number of \( \text{HCC}(3) \) is given in figure 9.

Here we assigned the colour set \{1,2\} in the center of \( \text{HCC}(3) \) and assigned the colour sets \{1\} and \{2\} to the other dominating sets in the \( \text{HCC}(3) \) alternatively.

Clearly all the vertices that are assigned \( \emptyset \) has the colour set \{1,2\} in its neighborhood.
In figure 9, the brick representation of honeycomb cup network HCC (3) is shown with 2-rainbow domination number. The vertex with the colour set \{1,2\} is filled with green colour. The vertex with colour sets \{1\} and \{2\} are filled with the colours red and blue respectively.

4. CONCLUSION:

In this paper, we obtained the 2-rainbow domination number for n dimensional Honeycomb network and Honeycomb cup network. We also attained exact bounds for this new parameter.

REFERENCES


