

Mathematical Modeling of Disposal of Pollutant in Rivers

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Abstract

Analytical solutions for concentration of pollutant in a river is governed by an advection diffusion equation in steady state in one horizontal dimension are presented. We also provide a solution for time dependent source, instantaneous source and continuous discharge.

1. INTRODUCTION:

Water pollution problems are related to waste water discharge, which are dissolved oxygen, suspended solids, bacteria, nutrients, PH and toxic chemicals including volatile organics, acid/base, neutrals, metals and pesticides. The physical, chemical, and biological processes that control the fate of the water quality parameters. These processes, are numerous and varied. It is convenient to divide them into transport processes which affect all water quality parameters similarly, and transformation processes, which are constituent-specific. Many of these transformation processes, however, have comparable kinetics so that a different formulation is not required for each constituent.

This paper describes the basic theory of mathematical modeling of one-dimensional water bodies. The equations are presented for the spectrum of situations extending

from the stream to the large river, the tidal river and the saline estuary. Any natural body of water may be viewed as a mathematical system, composed of a number of complex interacting subsystems. The system receives, on the one hand, a series of external inputs such as rainfall, solar radiation, runoff, and winds which interact with the water body and its drainage basin to determine the natural background quality of the water. On the other hand, the system is subjected to a variety of manmade effects such as wastewater discharges, water diversions, and runoff from urban and land developments which also influence water quality. The response of the system to each of these inputs is the spatial and temporal distribution of the concentration of various substances which affect water use. Such substances include dissolved and suspended solids, various chemicals, dissolved oxygen, nutrients (nitrogen and phosphorus), bacteria and algae concentrations. The system is composed of a number of elements, with physical characteristics and corresponding mathematical descriptions. Physically, the concentration of these substances is determined by the dispersion and advection characteristics of the water body and by the various physical, chemical, biological, or radiological reactions which affect the substance. Franklin L. Burton (1995) and C. J. Harrish (1979) studied models related to water pollution, Barry, D. A., and Sposito, G. (1989) and Basha, H. A., and EI-Habel, F. S. (1993) studied analytical convection-dispersion model, R.N. Singh (2013) and Todd and Mays, (2011) also studied one dimensional water quality models, Jacob Bear, 1979, Carslaw, H. S., and Jaeger, J. C. (1971), Charles R. Fitts, (2002) and M. Necati Ozisic (1993) gives various analytical solution to one, two and three dimensional advection diffusion equations, Kitanidis, P. K. (1994) studied Particle-tracking equations for the solution of the advection-dispersion equation with variable coefficients, and Zoppou, C., Member, ASCE, and Knight, J. H., Analytical Solutions For Advection and Advection-Diffusion Equations With Spatially Variable Coefficients.

Mathematically, the system is described by a set of partial differential equations, with variable coefficients, each term of which corresponds to one of the basic characteristics. Here we give a analytical solution of concentration of pollutant in a river is governed by an advection diffusion equation in steady state in one horizontal dimension [19, 22]. We give a general mass conservation equation is averaged over the cross section of the stream giving, for constituents subjects to a single first-order decay-process. We also provide a solution for time dependent source [19, 20], instantaneous source and continuous discharge [8].

2. ONE-DIMENSIONAL MODELING:

Rivers and estuaries are generally many times longer than they are wide or deep. As a result, inputs from wastewater treatment plants or other sources rapidly mix over the

cross section, and a one-dimensional approach is often justified. In the one-dimensional approach, only longitudinal variations of constituent concentrations are resolved in the form of cross-section- averaged values. The general mass conservation equation is averaged over the cross section of the stream giving, for constituents subjects to a single first-order decay-process,

$$\frac{\partial c}{\partial t} = -u \frac{\partial c}{\partial x} + \frac{\partial}{\partial x} \left[(D_x + D_L) \frac{\partial c}{\partial x} \right] - KC + \Sigma I \quad (2.1)$$

Where, x =longitudinal distance along river or estuary, L

D_L =longitudinal dispersion coefficient, L^2/T

Equation 2.1 is almost identical to conservation of mass equation without terms containing y and z derivatives, except for the appearance of dispersion, which is distinct and separate from turbulent diffusion. The dispersion term arises during the averaging process (which is somewhat involved mathematically) due to the correlation of cross-sectional velocity and concentration variations [1, 2]. Dispersion in natural streams is predominantly due to lateral velocity variations, and the following formula can be used to estimate coefficient [1]:

$$D_L = 0.011 \frac{u^2 B^2}{Hu^*} \quad (2.2)$$

Where D_L =longitudinal dispersion coefficient, L^2/T

u =Cross- section-averaged velocity, L/T

B = Stream width, L

H = stream depth, L

u^* =shear velocity, $L/T = \sqrt{gHs}$, g =acceleration due to gravity, L^2/T and s =stream slope, L/L

Equation 2.2 remains approximate because it does not account for dead zones in which matter can get trapped, thereby increasing the effective dispersion coefficient. Bends can increase or decrease dispersion depending on their configuration; in particular, successive bends can increase dispersion if their separation is small. In estuaries, tidal flow reversals as well as secondary currents driven by salinity gradients tend to increase dispersion [3]. Dispersion is typically much larger than turbulent diffusion so that D_x can be neglected, compared to D_L in Eq. 2.1.

3. RIVER WATER POLLUTION, ADVECTION-DIFFUSION MODEL:

The concentration of pollutant in a river is governed by an advection diffusion equation in steady state in one horizontal dimension as [4]

$$D \frac{d^2C}{dx^2} - u \frac{dC}{dx} - kC = 0 \quad (3.1)$$

Here D , u and k are respectively dispersion coefficient, velocity of river and decay rate of pollutants. If the steady source pollutant is Q at $x = 0$, then the solution of this equation can be written a

$$C_1(x \geq 0) = A \exp(-m_1x) \quad (3.2)$$

$$C_2(x \leq 0) = B \exp(m_2x) \quad (3.4)$$

Here, A and B are constants and m_1 and m_2 are:

$$m_{1,2} = \frac{u}{2D} [1 \pm \sqrt{1 + 4kv/D^2}] \quad (3.5)$$

A and B can be obtained by using the following two conditions:

$$C_1(0) = C_2(0) \quad (3.6)$$

$$\int_{-\infty}^{\infty} C(x) dx = Q \quad (3.7)$$

The first condition gives

$$A = B \quad (3.8)$$

And the second condition gives

$$A = \frac{kQ}{u\sqrt{1+4ku/D^2}} \quad (3.9)$$

Thus we have

$$C_1(x \geq 0) = \frac{kQ}{u\sqrt{1+4ku/D^2}} \exp \left[-\frac{ux}{2D} [1 + \sqrt{1 + 4kv/D^2}] \right] \quad (3.10)$$

$$C_1(x \leq 0) = \frac{kQ}{u\sqrt{1+4ku/D^2}} \exp \left[\frac{ux}{2D} [1 - \sqrt{1 + 4kv/D^2}] \right] \quad (3.11)$$

Thus the steady state distribution can be determined for given values of source, velocity, and diffusion coefficient and decay rate.

4. SOURCE ($Q(t)$) IS TIME DEPENDENT:

In case the source, $Q(t)$ is time dependent, then the governing equation (3.1) is given by

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} - u \frac{\partial C}{\partial x} - kC \quad (4.1)$$

The following condition is needed to solve this equation:

$$C(x, 0) = C_0 \quad (4.2)$$

$$C(0, t) = C_s \quad (4.3)$$

$$C(\pm\infty, t) = 0 \tag{4.4}$$

The analytical solution of the distribution of concentration of pollutants is given by [5] as:

$$C(x, t) = C_0 + \frac{(C_s - C_0)}{2} \left[\operatorname{erfc} \left(\frac{x-ut}{\sqrt{4D}} \right) + \exp \left(-\frac{ux}{D} \right) \operatorname{erfc} \left(\frac{x+ut}{\sqrt{4D}} \right) \right] \tag{4.5}$$

These solutions for steady state and transient cases have been used extensively in the literature in predicting changes in the water quality in the rivers. For more general case of time dependent variations of the concentration in vertical and both horizontal directions and also in estimating the three dimensional distribution of the velocity field of rivers for use in concentration equation, numerical methods are applied for solving fluid flow Navier-Stokes equation. For fuller description of the river water quality, interactions of river water with atmosphere and river bed sediments need to be included. Temperature variation in the river and its underlying sediments also have signature past changes in the river dynamics and water quality. The advection diffusion equation for temperature can be solved following the same approach as for water quality.

6. INSTANTANEOUS SOURCE:

Instantaneous release of a constituent at a point in the stream may occur as a result of an accident. It can also be used as a means of determining dispersion coefficients, with a deliberate release of a tracer constituent such as dye. The solution of equation 2.1 for an instantaneous release at $x = 0$ is [1, 8].

$$C = \frac{Me^{-Kt}}{A\sqrt{4\pi D_L t}} e^{-[(x-ut)^2/4D_L t]} \tag{6.1}$$

Where M =mass released, M

A =stream cross-sectional area, L^2

Corresponding longitudinal concentration distributions at different times is a function of distance after the release. Each instantaneous concentration distribution has the shape of a Gaussiann curve, the general form of which is

$$C = C_m e^{-x^2/2\sigma^2} \tag{6.2}$$

Where C_m =maximum concentration value, obtained at $x = 0$.

σ =standard deviation =half width of curve at point where $C=0.61 C_m$

The similarity between eqs.17-28 and 17-29 is evident. The maximum concentration is

$$C_m = \frac{Me^{-Kt}}{A\sqrt{4\pi D_L t}} \tag{6.3}$$

The maximum concentration decreases with time due to decay (exponential term in the numerator) and due to dispersion (square root term in the denominator). The centre of the patch is located at $x = ut$, thus moving downstream at the speed of the flow. The width of the patch, measured by its standard deviation, $\sigma = \sqrt{2D_L t}$, increase with time.

In a field dye study, the dispersion coefficient can be determined from measured maximum concentrations and Eq.17-30, with $K = 0$. Alternatively, D_L can be determined by matching the standard deviation of measured concentration profiles with the expression given above. In tidal estuaries, the measurements should preferably correspond to the same time during the tidal cycle, although Eq. 17-28 is valid for time varying currents also.

7. CONTINUOUS DISCHARGE:

The solution of Eq. 3.1 for continuous discharge at $x = 0$ is [17].

$$C = \frac{M'}{A\sqrt{u^2 + 4KE_L}} e^{(xu/2D_L)(1 \pm \sqrt{1 + 4D_L/u^2})} \quad (7.1)$$

Where M' =Discharge rate, $M/T = Q_D C_D$

Q_D =Discharge flowrate, L^3/T

C_D =discharge concentration, M/L^3

$\pm = +$ for $x < 0$ and $-$ for $x > 0$

Equation 7.1 is plotted in Fig. 17-6. Note that for a conservative substance ($K = 0$) the concentration is uniform and equal to M'/Au downstream of the discharge point. The upstream intrusion is not greatly affected by the decay coefficient. In many cases, the value of the term $4KD/u^2$ is small compared to others. For example, the term $4KD_L/u$ equals 0.0028 for the following typical values: $u=1$ ft/s, $K=0.30$ d⁻¹ = 3.5×10^{-6} s⁻¹, and $D_L = 200$ ft²/s. In this case, concentrations downstream of the source are very closely given by

$$C = \frac{M'}{Au} e^{-Kx/u}; (x > 0) \quad (7.2)$$

This is independent of the dispersion coefficient. Thus, it is generally true that dispersion can be neglected for continuous discharges in rivers.

Conclusion: In the one-dimensional approach, only longitudinal variations of constituent concentrations are resolved in the form of cross-section-averaged values. The general mass conservation equation is averaged over the cross section of the stream giving, for constituents subjects to a single first-order decay-process. In Instantaneous release of a constituent at a point in the stream the maximum

concentration decreases with time due to decay and due to dispersion.

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