Edge – Odd Gracefulness of Different Types of Shell Graphs by Removing Two Chords

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INTRODUCTION

Choudum and Kishore [1999] found graceful labeling of the union of paths and cycles, Bhat-Nayak and Selvam [2003] got graceful labeling for n-cone $C_m \vee K_n$. Barrientos [2005] obtained the graceful labeling for unions of cycles and complete bipartite graphs. Cheng et. al. [2008] analyzed graceful labeling for generalized spiders and caterpillars. Guo [1994, 1995] investigated graceful labelings for bipartite graph $B(m, n)$ and $B(m, n, p)$. Liu [1995] proved that the star graph with top sides is graceful. Seoud and Youssef [2000] showed that some classes of families in terms of disconnected from paths and cycles are graceful. Xu et.al. [2008] verified that the graphs $C_{13}$ are graceful where $t \equiv 0, 1 \pmod{4}$. Sethuraman and Jesintha developed a new class of graceful lobsters.

SECTION 2: Edge-odd gracefulness of few semi-shell graphs

Definition 2.1: Graceful Graph: A function $f$ of a graph $G$ is called a graceful labeling with $m$ edges, if $f$ is an injection from the vertex set of $G$ to the set $\{0, 1, 2, \ldots, m\}$ such that when each edge $uv$ is assigned the label $|f(u) - f(v)|$ and the resulting edge labels are distinct. Then the graph $G$ is graceful.

Definition 2.2: Edge-odd graceful graph: A $(p, q)$ connected graph is edge-odd graceful graph if there exists an injective map $f$: $E(G) \rightarrow \{1, 3, \ldots, 2q-1\}$ so that
induced map $f_+: V(G) \rightarrow \{0, 1, 2, \ldots, (2k-1)\}$ defined by $f_+(x) \equiv \sum f(x, y) \pmod{2k}$, where the vertex $x$ is incident with other vertex $y$ and $k = \max \{p, q\}$ makes all the edges distinct and odd. Hence the graph $G$ is edge-odd graceful.

**Definition 2.3:** $C(n, n-4)$ is a connected $(p, q)$-graph whose vertex set is \{v_1, v_2, v_3, \ldots, v_n\} and edge set is \{v_iv_{i+1}: i = 1 \text{ to } (n-1)\} \cup \{v_nv_1\} \cup \{v_1v_j: j = 3, 4, 5, \ldots, (n-3)\}$. Here $p = n; q = 2n-5$. It is obtained from a shell graph by removing two additional chords at $v_1$. It is called a weak shell graph.

**Theorem 2.4:** The weak shell graph $C(n, n-4)$ is edge-odd graceful if $n \geq 9$.

**Proof:** One of the arbitrary labeling for vertices is as follows:

The given graph $C(n, n-1)$ is connected whose vertex set is \{v_1, v_2, v_3, \ldots, v_n\} and edge set is \{v_iv_{i+1}: i = 1 \text{ to } (n-1)\} \cup \{v_nv_1\} \cup \{v_1v_j: j = 3, 4, 5, \ldots, (n-3)\}$. Here $p = n; q = 2n-5$

Define a map $f$: $E(C(n, n-4)) \rightarrow \{1, 2, 3, \ldots, q\}$ by

- $f(v_iv_{i+1}) = 2i-1, \; i = 1 \text{ to } (n-1)$
- $f(v_nv_1) = 2n-1$
- $f(v_1v_j) = (2n-1) + 2(j-2), \; j = 3, 4, 5, \ldots, (n-3)$. 
Define $f^*: V(C(n, n-4)) \to \{0, 1, 2, \ldots, q\}$ by $f^*(U) = \sum_{V \in G} f(UV) \pmod{2q}$ where this sum runs over all edges through the vertex $U$.

Hence the map $f$ and the induced map $f^*$ provide labels as distinct odd numbers for edges and also the labelings for vertex set have distinct values in $\{0, 1, 2, \ldots, (2k - 1)\}$. Hence the graph $C(n, n-4)$ is edge-odd graceful.

**Definition 2.5:** $C_n*S_{n-4}$ is a connected graph whose vertex set is $\{v_1, v_2, \ldots, v_n\}$ and edge set is $\{v_iv_{i+1}: i=1 \text{ to } (n-1)\} \cup \{v_nv_1\} \cup \{v_1v_j: j=3, 4, 5, \ldots, \frac{n-3}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, (n-1)\}$. Here it is obtained from a shell graph by removing two chords at the middle vertex. It is called also weak shell graph.

**Theorem 2.6:** The weak shell graph $C_n*S_{n-4}$ is edge-odd graceful where $n$ is an odd integer.

**Proof:** Let $n$ be an odd integer. The given graph has the vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set is $\{v_iv_{i+1}: i=1 \text{ to } (n-1)\} \cup \{v_nv_1\} \cup \{v_1v_j: j=3, 4, 5, \ldots, \frac{n-3}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, (n-1)\}$. 
One of the arbitrary labeling for vertices is as above:

Define a map \( f: E(C_n \ast S_{n-4}) \to (1, 3, \ldots, (2q-1)) \)

\[
f(v_i v_{i+1}) = 2i-1, \ i=1 \text{ to } (n-1), \ i \neq \frac{n-3}{2}, \frac{n-1}{2}, \frac{n+1}{2}, n, (n-2)
\]

\[
f(v_{(n-3)/2} v_{(n-1)/2}) = n
\]

\[
f(v_{(n-1)/2} v_{(n+1)/2}) = n+2
\]

\[
f(v_{(n+1)/2} v_{(n+3)/2}) = n+4
\]

\[
f(v_n v_1) = 2n-1
\]

\[
f(v_{n-2} v_{n-1}) = 2n-5
\]

\[
f(v_1 v_j) = (4n-5)-2(j-3), \ j=3, 4, 5, \ldots, \frac{n-3}{2}
\]

\[
f(v_1 v_j) = (3n-6)-2 \left\lfloor \frac{n+3}{2} \right\rfloor, \ j=\frac{n+3}{2}, \frac{n+5}{2}, \ldots, (n-1).
\]

Define \( f^*: V(C_n \ast S_{n-4}) \to \{0, 1, 2, \ldots, q\} \) by \( f^*(U) = \sum_{V \in G} f(UV) \mod 2q \) where this sum run over all edges through the vertex \( U \).

Hence the map \( f \) and the induced map \( f^* \) provide labels as distinct odd numbers for edges and also the labelings for vertex set have distinct values in \( \{0, 1, 2, \ldots, (2k-1)\} \).

Hence the graph \( C_n \ast S_{n-4} \) is edge-odd graceful.

**Definition 2.7:** \( C(n, n-4) \ast F_k \) is a connected graph whose vertex set is \( \{v_1, v_2, \ldots, v_{n+k}\} \) and edge set is \( \{v_{i} v_{i+1}: i = 1 \text{ to } (n-1)\} \cup \{v_n v_1\} \cup \{v_j v_{j+1}: j = 3, 4, 5, \ldots, (n-3)\} \cup \{v_{n+1} v_{n+i+1}, i = 1, 2, \ldots, (k-1)\} \cup \{v_{n+i} v_{n+j}, j = 1, 2, \ldots, k\} \).

**Theorem 2.8:** The connected graph \( C(n, n-4) \ast F_k \) is edge-odd graceful if \( n \) is an odd integer.

**Proof:** One of the arbitrary labeling for vertices is as follows:
Its vertex set is \( \{ v_1, v_2, \ldots, v_{n+k} \} \).

Its edge set is \( \{ v_i v_{i+1} : i = 1 \text{ to } (n-1) \} \cup \{ v_n v_1 \} \cup \{ v_1 v_{j} : j = 3, 4, 5, \ldots, (n-3) \} \)

\( \cup \{ v_{n+1} v_{n+i+1} , i = 1, 2, \ldots, (k-1) \} \cup \{ v_n v_{n+j} , j =1, 2 \ldots, k \} \).

Define a map \( f : E(C(n, n-4) \ast F_k) \rightarrow \{1,2,3,\ldots,q\} \) by

\[
\begin{align*}
f(v_i v_{i+1}) &= 2i-1, \quad i = 1,2,\ldots,(n-1), i \neq (n-3),(n-2) \\
f(v_n v_{n-2}) &= 2n-5 \\
f(v_{n-2} v_{n-1}) &= 2n-7 \\
f(v_n v_{n+j}) &= (2n-1) + 4(j-1), \quad j = 1,2,\ldots,(k-1) \\
f(v_{n+i} v_{n+i+1}) &= (2n+1)+4(j-1), \quad i = 1,2,\ldots,(k-2) \\
f(v_{n+k-1} v_{n+k}) &= (2n-1)+4(k-1) \\
f(v_n v_{n+k}) &= (2n+1)+4(k-2)
\end{align*}
\]
\[ f(v_n v_1) = (2n+3)+4(k-2) \]
\[ f(v_1 v_i) = (2n+3)+4(k-2)+2(j-2), \quad j=3,4,\ldots,(n-3) \]

Define \( f^+ : V(C(n, n-4) \ast F_k) \rightarrow \{0, 1, 2,\ldots,q\} \) by \( f^+(U) = \sum_{U \in G} f(UV) \pmod{2q} \) where this sum run over all edges through the vertex \( U \).

Hence the map \( f \) and the induced map \( f^+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set have distinct values in \( \{0, 1, 2,\ldots,(2k-1)\} \).

Hence the graph \( C(n, n-4) \ast F_k \) is edge-odd graceful.

**Definition 2.9:** The semi-shell graph \( C(n, \frac{n-5}{2}) \) is connected whose vertex set : \( \{v_0, v_1,\ldots,v_{2n}\} \) and edge set is \( \{v_i v_{i+1}; \ i = 0 \text{ to } (2n-1)\} \cup \{v_{2n}v_0\} \cup \{v_0 v_j; j = 2 \text{ to } (n-1)\} \)

**Theorem 2.10:** The semi-shell graph \( C(n, \frac{n-5}{2}) \) is edge-odd graceful where \( n \) is an odd integer.
Its vertex set is \( \{ v_0, v_1, \ldots, v_{2n} \} \), and edge set is \( \{ v_i v_{i+1} ; i \text{ varies from } 0 \text{ to } (2n-1) \} \) \( \cup \{ v_{2n} v_0 \} \cup \{ v_0 v_{j} ; j = 2 \text{ to } (n-1) \} \).

Define a map \( f: E(C(n, \frac{n-5}{2})) \rightarrow \{ 0, 1, 2, \ldots, q \} \) by

\[
f(v_i v_{i+1}) = 2i-1 \quad \text{if } n = 1 \text{ (mod 3)},
\]

\[
= 2i-1 \quad \text{if } n = 2 \text{ (mod 3)},
\]

\[
= 2i-1 \quad \text{if } n = 0 \text{ (mod 3)}.
\]

\[
f(v_{(3n+1)/4} v_{(3n+5)/4}) = f(v_{(3n+1)/4} v_{(3n+1)/4}) + 4.
\]

\[
f(v_i v_{i+1}) = f(v_{(3n+1)/4} v_{(3n+5)/4}) + 4 \left( i - \frac{3n+1}{4}, \quad i = \frac{3n+5}{4}, \frac{3n+13}{4}, \ldots (2n - 1) \right).
\]

\[
F(v_{2n} v_0) = \frac{3n-1}{2} + 1 = \frac{3n+1}{4}.
\]

Define \( f^*: V(C(n, \frac{n-5}{2})) \rightarrow \{ 0, 1, 2, \ldots, q \} \) by \( f^*(U) = \sum_{V \in G} f(UV) \mod 2q \) where this sum run over all edges through the vertex \( U \).

Hence the map \( f \) and the induced map \( f^* \) provide labels as distinct odd numbers for edges and also the labeling for vertex set have distinct values in \( \{ 0, 1, 2, \ldots, (2k-1) \} \).

Hence the graph \( C(n, \frac{n-5}{2}) \) is edge-odd graceful.

REFERENCES


[10] Xu, X., Yang, Y., Han, L., and Li, H., The graphs $C_{13}(t)$ are graceful for $t \equiv 0, 1 \pmod{40}$, Ars. Combin., Volume 88, (2008), 429 -435.