Convergence of a Third-order family of methods in Banach spaces

Gagandeep
Department of Mathematics,
Hans Raj Mahila Mahavidyalaya,
Jalandhar-144008, India.

R. Sharma¹
Department of Applied Sciences,
DAV Institute of Engineering and Technology,
Jalandhar-144008, India.

Abstract
In this paper, we study the semilocal convergence of a family of third-order methods for solving nonlinear equations in Banach spaces using recurrence relations. Recurrence relations for the family of methods are derived and finally an existence-uniqueness theorem is derived along with a priori error bounds.

Keywords: Nonlinear equations, Banach spaces, Recurrence relations, Semilocal convergence, Iterative methods.

1. Introduction
The most well known Newton’s method and its variants are used to solve nonlinear operator equations $F(x) = 0$. The convergence of these second order methods was established by Kantorovich Theorem ([15], [16]). The convergence of sequences obtained by these methods is derived from convergence of majorizing sequences [21]. Rall in [18] established the convergence of Newton’s method by using recurrence relations. With the same approach, various researchers established semilocal convergence of higher order

¹Corresponding author.
methods in Banach spaces (see [3, 2, 13, 12, 7, 11, 17, 14, 10, 9, 20, 5, 22, 8, 1, 4, 6] and references there in). In this paper, we shall use recurrence relations to establish the semilocal convergence of a family of third-order methods [19] in Banach spaces. Based on these recurrence relations, an existence-uniqueness theorem is given and a priori error bounds are obtained for the method.

2. Recurrence relations

In this section, we discuss a third order method [19] for solving nonlinear operator equations

\[ F(x) = 0, \]

where \( F : \Omega \subseteq X \to Y \) is a nonlinear operator on an open convex subset \( \Omega \) of a Banach space \( X \) with values in a Banach space \( Y \). The third order method is defined as follows:

\[
\begin{align*}
    y_n &= x_n - \theta F'(x_n)^{-1} F(x_n), \\
    x_{n+1} &= x_n - \left[ \left( 1 + \frac{1}{2\theta} \right) I - \frac{1}{2\theta} F'(x_n)^{-1} F'(y_n) \right] F'(x_n)^{-1} F'(x_n).
\end{align*}
\]

Let \( F \) be a twice Fréchet differentiable in \( \Omega \) and \( BL(Y, X) \) be the set of bounded linear operators from \( Y \) into \( X \). Let us assume that \( \Gamma_0 = F'(x_0)^{-1} \in BL(Y, X) \) exists at some \( x_0 \in \Omega_0 \) and the following conditions hold:

(1) \( \| F'(x) - F'(y) \| \leq K \| x - y \|, x, y \in \Omega, \)

(2) \( \| F''(x) \| \leq M, x \in \Omega \)

(3) \( \| \Gamma_0 \| \leq \beta, \)

(4) \( \| \Gamma_0 F(x_0) \| \leq \eta. \)

Let us denote

\[ a = K \beta \eta. \]

Now, we define the sequences

\[
\begin{align*}
    a_0 &= b_0 = 1, d_0 = 1 + \frac{a}{2}, \\
    a_{n+1} &= \frac{a_n}{1 - a a_n d_n}, \\
    b_{n+1} &= a_{n+1} \beta \eta C_n, \\
    d_{n+1} &= \left( 1 + \frac{1}{2} a a_{n+1} b_{n+1} \right) b_{n+1},
\end{align*}
\] (2.3)
where
\[ C_n = \frac{M}{2} K_n^2 + K|\theta|b_n K_n + \left( \frac{M\theta^2 + K}{2} \right) b_n^2, \] (2.4)

with
\[ K_n = \left( |1 - \theta| + \frac{aa_n b_n}{2} \right) b_n. \] (2.5)

The polynomials \( C_n \) and \( K_n \) can be written as
\[ C_n = (P_0 + P_1 a_n b_n + P_2 a_n^2 b_n^2) b_n^2, \]
\[ K_n = (Q_0 + Q_1 a_n b_n) b_n. \]

**Lemma 2.1.** Under the previous assumptions, we prove the following:

\( (I_n) \quad \|\Gamma_n\| = \|F'(x_n)^{-1}\| \leq a_n \beta, \)
\( (II_n) \quad \|\Gamma_n F(x_n)\| \leq b_n \eta, \)
\( (IIIn) \quad \|x_{n+1} - x_n\| \leq d_n \eta, \)
\( (IV_n) \quad \|x_{n+1} - y_n\| \leq K_n \eta. \)

**Proof.** We use induction to prove the above claims. Notice that \( (I_0) \) and \( (II_0) \) follow immediately from the assumptions. To prove \( (IIIn) \), we consider (2.1). Using the assumptions, it follows that
\[ \|x_1 - x_0\| \leq \left[ 1 + \frac{1}{2|\theta|} K\beta \|y_0 - x_0\| \right] \|\Gamma_0 F(x_0)\| \]
\[ \leq \left[ 1 + \frac{a}{2} \right] \eta = d_0 \eta, \] (2.6)

and \( (III_0) \) holds. We have
\[ x_1 - y_0 = - \left[ (1 - \theta) \frac{1}{2\theta} F'(x_0)^{-1} (F'(x_0) - F'(y_0)) \right] F'(x_0)^{-1} F'(x_0), \] (2.7)

so that
\[ \|x_1 - y_0\| \leq \left[ |1 - \theta| + \frac{1}{2|\theta|} K\beta \|y_0 - x_0\| \right] \|\Gamma_0 F(x_0)\| \]
\[ \leq \left[ |1 - \theta| + \frac{a^2}{2} \right] \eta = K_0 \eta, \] (2.8)

and \( (IV_0) \) also holds. Following an inductive procedure and assuming that \( x_n \in \Omega \) and \( a a_n d_n < 1 \), if \( x_{n+1} \in \Omega \), we have
\[ \|I - \Gamma_n F'(x_{n+1})\| \leq \|\Gamma_n\| \|F'(x_n) - F'(x_{n+1})\| \leq a a_n d_n < 1. \] (2.9)
Then, from Banach lemma, $\Gamma_{n+1}$ exists and
\[
\|\Gamma_{n+1}\| \leq \frac{\|\Gamma_n\|}{1 - \|\Gamma_n\| \|F'(x_n) - F'(x_{n+1})\|} \leq \frac{a_n \beta}{1 - a_n \eta_n} = a_{n+1} \beta. \tag{2.10}
\]
Hence, by induction (2.10) holds for all $n$. This proves condition $(I_n)$.

Using the first step of (1), we have
\[
F(y_n) = F(y_n) - \theta F(x_n) - F'(x_n)(y_n - x_n)
= (1 - \theta) F(x_n) + F'(x_n) - F'(x_n)(y_n - x_n)
= (1 - \theta) F(x_n) + \int_0^1 F''(x_n + t(y_n - x_n))(1 - t) dt (y_n - x_n)^2. \tag{2.11}
\]
Now subtract first step of (2.1) from second, we get
\[
x_{n+1} - y_n = - \left[ (1 - \theta) I + \frac{1}{2\theta} F'(x_n)^{-1}(F'(x_n) - F'(y_n)) \right] F'(x_n)^{-1} F'(x_n).
\tag{2.12}
\]
so that
\[
\|x_{n+1} - y_n\| \leq \left[ |1 - \theta| + \frac{a_n \beta K}{2|\theta|} \right] \|\Gamma_n F(x_n)\|
\leq \left[ |1 - \theta| + \frac{a_n \beta \eta_n}{2} \right] b_n \eta_n = K_n \eta. \tag{2.13}
\]
Using Taylor’s formula, we have
\[
F(x_{n+1}) = F(y_n) + F'(y_n)(x_{n+1} - x_n)
+ \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1 - t) dt (x_{n+1} - y_n)^2
= (1 - \theta) F(x_n) + F'(x_n)(x_{n+1} - x_n)
+ \int_0^1 F''(y_n + t(y_n - x_n))(1 - t) dt (y_n - x_n)^2
+ \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1 - t) dt (x_{n+1} - y_n)^2
+ (F'(y_n) - F'(x_n)) (x_{n+1} - y_n)
= F'(x_n) \frac{1}{2\theta} F'(y_n)^{-1}(F'(y_n) - F'(x_n)) \Gamma_n F(x_n) + (F'(y_n) - F'(x_n)) (x_{n+1} - y_n)
+ \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1 - t) dt (x_{n+1} - y_n)^2
+ \int_0^1 F''(x_n + t(y_n - x_n))(1 - t) dt (y_n - x_n)^2. \tag{2.14}
\]
Convergence of a Third-order family of methods in Banach spaces

Hence using (2.13) in (2.14), we have

\[
\| F(x_{n+1}) \| \leq \frac{K}{2\theta^2} \| y_n - x_n \|^2 + K \| y_n - x_n \| \| x_{n+1} - y_n \| \\
+ \frac{M}{2} \| y_n - x_n \|^2 + \frac{M}{2} \| x_{n+1} - y_n \|^2 \\
\leq \left[ \frac{M}{2} K_n^2 + K |\theta| b_n K + \left( \frac{M\theta^2 + K}{2} \right) b_n^2 \right] \eta^2 \\
= C_n \eta^2.
\]  

Therefore

\[
\| \Gamma_{n+1} F(x_{n+1}) \| \leq \| \Gamma_{n+1} \| \| F(x_{n+1}) \| \\
\leq a_{n+1} \beta C_n \eta^2 = b_{n+1} \eta,
\]

So, by induction condition \((II_n)\) holds for all \(n\). Using (2.16), we have

\[
\| x_{n+2} - x_{n+1} \| \leq \left[ 1 + \frac{1}{2|\theta|} \| F'(x_{n+1})^{-1} \| \| F'(x_{n+1})^{-1} - F'(y_{n+1})^{-1} \| \right] \| \Gamma_{n+1} F(x_{n+1}) \| \\
\leq \left[ 1 + \frac{1}{2} a_{n+1} \beta K b_{n+1} \eta \right] b_{n+1} \eta \\
= \left[ 1 + \frac{aa_{n+1} b_n}{2} \right] b_{n+1} \eta = d_{n+1} \eta.
\]  

Hence, by induction, this inequality holds for all \(n\). This proves condition \((III_n)\). We have from (2.13) and (2.16) that

\[
\| x_{n+2} - y_{n+1} \| \leq \left[ |1 - \theta| + \frac{1}{2|\theta|} a_{n+1} \beta K \| y_{n+1} - x_{n+1} \| \right] \| \Gamma_n F(x_n) \| \\
\leq \left[ |1 - \theta| + \frac{aa_{n+1} b_n}{2} \right] b_{n+1} \eta = K_{n+1} \eta.
\]

Hence, by induction, this inequality holds for all \(n\). This proves condition \((IV_n)\).

3. Convergence Analysis

In this section, we establish the convergence of our third-order method (2.1). To this end, we have to prove the convergence of the sequence \(x_n\) defined in a Banach space or, which is same, to prove that \(d_n\) is a Cauchy sequence and that the following assumptions hold:

1. \(x_n \in \Omega\),
2. \(aa_n d_n < 1, n \in \mathbb{N}\).
The next two lemmas will show the Cauchy property for the sequence $d_n$.

**Lemma 3.1.** Assume that $x_0$ is chosen so as to satisfy $0 < d_0 < \frac{1}{a}$, that is, $a \in (0, \sqrt{3} - 1)$. Then, the sequence $a_n > 0$ is increasing, as $n$ increases.

**Proof.** We show now that all the involved sequences are positive. Under the imposed conditions, we see that $a_0, b_0, d_0, C_0, K_0$ are all positive, and also that $1 - a a_0 d_0 > 0$. Assume, now, that all $a_i, b_i, d_i, C_i, K_i$, and $1 - a a_i d_i$ are positive, for $i = 0, 1, 2, \ldots, n$

Since $C_n > 0$ and $b_{n+1} = a_{n+1} \beta \eta C_n$, it follows that $a_{n+1}, b_{n+1}$ have same sign, and so $a_{n+1} b_{n+1} > 0$. Further, from $d_{n+1} = (1 + \frac{1}{2} a a_{n+1} b_{n+1}) b_{n+1}$, we get that $d_{n+1}$ has same sign as that of $b_{n+1}$, and so, all the three terms $a_{n+1}, b_{n+1}, d_{n+1}$ share the same sign.

By absurd, we suppose that the implied sign is negative. Then $d_n + d_{n+1} < d_n$, and so, $1 - a a_n (d_n + d_{n+1}) > 1 - a a_n d_n$, which renders $1 - a a_{n+1} d_{n+1} = \frac{1 - a a_n (d_n + d_{n+1})}{1 - a a_n d_n} > 1$, which implies $a a_{n+1} d_{n+1} < 0$, but that is impossible since $a_{n+1}, d_{n+1}$ have the same sign and $a > 0$.

Next, since $a_{n+1} = \frac{a_n}{1 - a a_n d_n}$, then

$$d_n = \frac{1}{a} \left( \frac{1}{a_n} - \frac{1}{a_{n+1}} \right),$$

and so, by telescoping, we get $\sum_{i=0}^{n-1} d_i = \frac{1}{a} \left( \frac{1}{a_0} - \frac{1}{a_n} \right)$, where $a_0 = 1$.

This will render $a_n = \frac{1}{1 - a \sum_{i=0}^{n-1} d_i}$. Certainly, since $a > 0$, $d_i > 0$, for all $i \geq 0$, then $a \sum_{i=0}^{n-1} d_i$ increases as $n$ increases and so, $1 - a \sum_{i=0}^{n-1} d_i$ decreases as $n$ increases, which implies that the reciprocal, $a_n$ is an increasing sequence, and consequently, $a_n \geq a_0 = 1$. $\blacksquare$
We define the sequence \( c_n = a_nb_n \). Then the sequences \( \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \) can be rewritten as

\[
a_{n+1} = \frac{a_n}{1 - a a_n d_n} = \frac{a_n}{1 - a (c_n + \frac{a}{2} c_n^2)},
\]

\[
b_{n+1} = a_{n+1} \beta \eta C_n = \frac{\beta \eta b_n c_n (P_0 + P_1 c_n + P_2 c_n^2)}{1 - a (c_n + \frac{a}{2} c_n^2)},
\]

\[
c_{n+1} = a_{n+1} b_{n+1} = \frac{\beta \eta c_n^2 (P_0 + P_1 c_n + P_2 c_n^2)}{[1 - a (c_n + \frac{a}{2} c_n^2)]^2},
\]

\[
d_{n+1} = \left(1 + \frac{1}{2} a a_{n+1} b_{n+1}\right) b_{n+1} = \frac{\beta \eta b_n c_n (P_0 + P_1 c_n + P_2 c_n^2)}{1 - a (c_n + \frac{a}{2} c_n^2)} \left(1 + \frac{a}{2} c_{n+1}\right).
\]

That the sequence \( \{c_n\} \) is a decreasing sequence under the assumption that \( a_1 b_1 < 1 \) can be proved by using the mathematical induction. It is obvious that \( c_1 = a_1 b_1 < 1 = c_0 \). Assuming that \( c_n < c_{n-1} \) for some \( n > 0 \), we have

\[
c_{n+1} = \frac{\beta \eta c_n^2 (P_0 + P_1 c_n + P_2 c_n^2)}{[1 - a (c_n + \frac{a}{2} c_n^2)]^2} < \frac{\beta \eta c_{n-1}^2 (P_0 + P_1 c_{n-1} + P_2 c_{n-1}^2)}{[1 - a (c_{n-1} + \frac{a}{2} c_{n-1}^2)]^2} = c_n.
\]

Therefore the sequence \( \{c_n\} \) becomes a decreasing sequence with \( c_n < 1 \) for all \( n \). If \( 0 < s < 1 \) and \( c_n \leq s c_{n-1} \), then

\[
c_{n+1} = \frac{\beta \eta c_n^2 (P_0 + P_1 c_n + P_2 c_n^2)}{[1 - a (c_n + \frac{a}{2} c_n^2)]^2} \leq s^2 \frac{\beta \eta c_{n-1}^2 (P_0 + P_1 s c_{n-1} + P_2 s^2 c_{n-1}^2)}{[1 - a (s c_{n-1} + \frac{a}{2} s^2 c_{n-1}^2)]^2} \leq s^2 \frac{\beta \eta c_{n-1}^2 (P_0 + P_1 c_{n-1} + P_2 c_{n-1}^2)}{[1 - a (c_{n-1} + \frac{a}{2} c_{n-1}^2)]^2} = s^2 c_n.
\]

Let \( \zeta = \frac{c_1}{c_0} = c_1 = a_1 b_1 \), then we have \( 0 < \zeta < 1 \) and \( c_1 \leq \zeta c_0 = \zeta \), so that

\[
c_1 \leq \zeta c_0, \quad c_2 \leq \zeta^2 c_1, \quad c_3 \leq \zeta^2 c_2, \quad \cdots \quad c_{n+1} \leq \zeta^{2^n} c_0 = \zeta^{2^n+1} \frac{1}{\zeta}.
\]
On other hand the sequence \( \{d_n\} \) under the assumption that \( a_1 b_1 < 1 \) we have

\[
d_n = \frac{a d_n}{a_n} = \left( c_n + \frac{a}{2} c_n^2 \right) \frac{1}{a_n} \\
\leq \left( c_n + \frac{a}{2} c_n^2 \right) \frac{1}{a_0} = c_n + \frac{a}{2} c_n^2 \\
\leq \left( 1 + \frac{a}{2} \right) c_n \\
\leq \left( 1 + \frac{a}{2} \right) \zeta \frac{2^n}{\zeta},
\]

Since \( \{a_n\} \) is an increasing sequence, and \( a_0 \geq 1 \). We thus have proved the following estimates.

**Lemma 3.2.** We assume that \( a_1 b_1 < 1 \). Then the sequence \( \{c_n\} \) is a decreasing sequence and for all \( n \in \mathbb{N} \) we have the following estimates

\[
c_{n+1} \leq \zeta \frac{2^{n+1}}{\zeta}, \\
d_n \leq \left( 1 + \frac{a}{2} \right) \zeta \frac{2^n}{\zeta}
\]

where \( 0 < \zeta = a_1 b_1 < 1 \).

**Lemma 3.3.** The sequence \( d_n > 0 \) is a convergent sequence and its limit is 0.

**Proof.** Since \( a_n \geq 1 \) is an increasing, then \( 1/a_n \leq 1 \) is a decreasing sequence and further \( 0 \leq 1/a_n \leq 1 \). Therefore \( 1/a_n \) is convergent to a limit \( L \). Since \( d_n = \frac{1}{a} \left( \frac{1}{a_n} - \frac{1}{a_{n+1}} \right) \),

then \( d_n \) is convergent to the limit \( \frac{1}{a} (L - L) = 0 \). \( \square \)

**Remark 3.4.** Clearly \( \sum_{i=0}^{\infty} d_i < \infty \), since \( \sum_{i=0}^{\infty} d_i = \lim_{i \to \infty} \sum_{i=1}^{n-1} d_i = \lim_{n \to \infty} \frac{1}{a} \left( 1 - \frac{1}{a_n} \right) = \frac{1}{a} (1 - L) \), where \( L \) would be the finite limit of \( 1/a_n \).

**Theorem 3.5.** Let \( X, Y \) be Banach spaces and \( F \) be a twice Fréchet differentiable in an open convex domain \( \Omega \) of Banach space \( X \) and \( BL(Y, X) \) be set of bounded linear operators from \( Y \) into \( X \). Let us assume that \( \Gamma_0 = F'(x_0)^{-1} \in BL(Y, X) \) exists at some \( x_0 \in \Omega_0 \) and the following conditions hold:

(1) \( \|F'(x) - F'(y)\| \leq K \|x - y\|, \ x, y \in \Omega \),

(2) \( \|F''(x)\| \leq M \),

(3) \( \|\Gamma_0\| \leq \beta \).
(4) \( \| \Gamma_0 F(x_0) \| \leq \eta \).

Let us denote \( a = K\beta\eta \). Suppose that \( x_0 \) is chosen so as to satisfy \( a \in (0, \sqrt{3} - 1) \) and \( a_1 b_1 < 1 \). Then, if \( B(x_0, r\eta) \subset \Omega \), where \( r = \sum_{n=0}^{\infty} d_n \), then the sequence \( \{x_n\} \) defined by (1) and starting at \( x_0 \) converges to a solution \( x^* \) of the solution \( F(x) = 0 \). In this case, the solution \( x^* \) and the iterates \( x_n \) belong to \( B(x_0, \frac{2}{K\beta} - r\eta) \cap \Omega \).

Furthermore, the error bound on \( x^* \) depends on the sequence \( \{d_n\} \) given by

\[
\| x_{n+1} - x^* \| \leq \sum_{k=n+1}^{\infty} d_k \eta \leq \frac{(1 + a/2)\eta}{\zeta} \sum_{k=n+1}^{\infty} \zeta^2, \quad \zeta = a_1 b_1. \tag{3.1}
\]

**Proof.** It is easy to see that the sequence \( \{x_n\} \) is convergent. Hence, there exists a limit \( x^* \) such that \( \lim_{n \to \infty} x_n = x^* \). The sequence \( \{a_n\} \) is bounded above since

\[
a_n = \frac{1}{n+1} \leq \frac{1}{1 - a \sum_{i=0}^{\infty} d_i}.
\]

Since \( \lim_{n \to \infty} d_n = 0 \), so we have \( \lim_{n \to \infty} b_n = 0 \). This indicates that \( \lim_{n \to \infty} C_n = 0 \). Thus by and by the continuity of \( F \), we proved that

\[
\| F(x^*) \| = 0.
\]

Also,

\[
\| x_{n+1} - x_0 \| \leq \| x_{n+1} - x_n \| + \| x_n - x_{n-1} \| + \cdots + \| x_1 - x_0 \|
\leq \sum_{k=0}^{n} d_k \eta \leq r\eta, \tag{3.2}
\]

where \( r = \sum_{n=0}^{\infty} d_n \). We conclude that \( \{x_n\} \) lies in \( \overline{B}(x_0, r\eta) \) and taking limit as \( n \to \infty \) we have \( x^* \in \overline{B}(x_0, r\eta) \). To show the uniqueness of the solution, suppose that

\[
y^* \in B(x_0, \frac{2}{K\beta} - R\eta) \cap \Omega_0
\]

is another solution of \( F(x) = 0 \). Then,

\[
0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*)) dt (y^* - x^*). \tag{3.3}
\]
To show that $y^* = x^*$, we have to show that the operator $\int_0^1 (F'(x^* + t(y^* - x^*)) dt$ is invertible. Now, for

$$\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\| dt$$

$$\leq KB\int_0^1 \|x^* + t(y^* - x^*) - x_0\| dt$$

$$\leq KB\int_0^1 ((1 - t)\|x^* - x_0\| + t\|y^* - x_0\|) dt$$

$$< \frac{KB}{2}(r\eta + \frac{2}{K\beta} - r\eta) = 1,$$

(3.4)

it follows from Banach’s Theorem [15] that the operator $\int_0^1 (F'(x^* + t(y^* - x^*)) dt$ has an inverse, and consequently, $y^* = x^*$. For every $m \geq n + 1$, we have

$$\|x_m - x_{n+1}\| \leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+2} - x_{n+1}\|$$

$$\leq \sum_{k=n+1}^{m-1} d_k \eta \leq r \eta.$$

(3.5)

By taking $m \to \infty$, we get

$$\|x_{n+1} - x^*\| \leq \sum_{k=n+1}^{\infty} d_k \eta \leq r \eta.$$

(3.6)

and from Lemma 3.1

$$\|x_{n+1} - x^*\| \leq \sum_{k=n+1}^{\infty} d_k \eta \leq \frac{(1 + a/2)\eta}{\zeta} \sum_{k=n+1}^{\infty} \zeta^k, \ 0 < \zeta < 1.$$

(3.7)

which shows that $\{x_n\}$ converges and completes the proof.

4. Conclusion

In this paper, the recurrence relations are developed for establishing the convergence of a family of third-order methods for solving $F(x) = 0$ in Banach spaces. Based on recurrence relations, we prove a semilocal convergence, which shows the existence-uniqueness theorem for this family of methods and a priori error bounds.
References


