On Edge Regular Fuzzy Line Graphs

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Abstract

In this paper, degree of an edge in fuzzy line graph is obtained and some properties of edge regular fuzzy line graphs are studied. Fuzzy line graph of an edge regular fuzzy graph need not be edge regular. Conditions under which it is edge regular are provided.

Keywords: Strong fuzzy graph, complete fuzzy graph, edge regular fuzzy graph, totally edge regular fuzzy graph, fuzzy line graph.

2010 Mathematics Subject Classification: 03E72, 05C72

1. INTRODUCTION

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975 [11]. Though it is very young, it has been growing fast and has numerous applications in various fields. During the same time Yeh and Bang have also introduced various connectedness concepts in fuzzy graphs [12]. J.N.Mordeson (1993) introduced the concept of fuzzy line graph [2]. K.Radha and N.Kumaravel (2014) introduced the concept of edge regular fuzzy graphs [9]. In this paper, we study about edge regular property of fuzzy line graphs.

First we go through some basic definitions in the next section from [1] – [5], [8] and [9].

2. BASIC CONCEPTS

Let \( V \) be a non-empty finite set and \( E \subseteq V \times V \). A fuzzy graph \( G: (\sigma, \mu) \) is a pair of functions \( \sigma: V \rightarrow [0, 1] \) and \( \mu: E \rightarrow [0, 1] \) such that \( \mu(x, y) \leq \sigma(x) \wedge \sigma(y) \) for all
Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. The degree of a vertex $x$ is 
\[d_G(x) = \sum_{xy \in E} \mu(xy).\]
If each vertex in $G$ has same degree $k$, then $G$ is said to be a regular fuzzy graph or $k$-regular fuzzy graph.

The degree of an edge $e=uv \in E$ is defined by 
\[d_G^*(e) = d_G^*(u) + d_G^*(v) - 2.\]
If each edge in $G^*$ has same degree, then $G^*$ is said to be edge regular. The degree of an edge $xy \in E$ is 
\[d_G(xy) = \sum_{x \in V} \mu(xz) + \mu(zy) - 2\mu(xy).\]
If each edge in fuzzy graph $G$ has same degree $k$, then $G$ is said to be an edge regular fuzzy graph or $k$-edge regular fuzzy graph.

Given a graph $G^*:(V,E)$, its line graph $L(G^*):(Z,W)$ is a graph such that 
\[Z = \{S_x = \{x\} \cup \{u,v \mid x = u,v \in V\} \mid x \in E, x = u,v \in V\},\]
\[W = \{S_x \cap S_y \neq \emptyset \mid x,y \in E, x \neq y\}.\]
The number of edges in $L(G^*):(Z,W)$ is half of the sum of the degree of edges in $G^*:(V,E)$. $G^*:(V,E)$ is a cycle if and only if $L(G^*):(Z,W)$ is also a cycle.

The fuzzy line graph of $G:(\sigma,\mu)$ is $L(G):(\omega,\lambda)$ with underlying graph $L(G^*):(Z,W)$ where 
\[Z = \{S_x = \{x\} \cup \{u,v \mid x = u,v \in V\} \mid x \in E, x = u,v \in V\},\]
\[W = \{S_x \cap S_y \neq \emptyset \mid x,y \in E, x \neq y\}, \quad \omega(x) = \mu(x), \forall S_x \in Z \quad \text{and} \quad \lambda(S_x,S_y) = \mu(x) \land \mu(y) \text{ for every } S_x,S_y \in W \text{ [2].}\]
For the sake of simplicity, the vertices of $L(G)$ may be denoted by $x$ instead of $S_x$ and the edges by $xy$ instead of $S_x \cap S_y$.

A homomorphism of fuzzy graphs $h:G \rightarrow G'$ is a map $h:V \rightarrow V'$ which satisfies $\sigma(x) \leq \sigma'(h(x))$ for $x \in V$ and $\mu(x,y) \leq \mu'(h(x),h(y))$ for $x,y \in V$. A weak isomorphism $h:G \rightarrow G'$ is a map $h:V \rightarrow V'$ which is a bijective homomorphism that satisfies $\sigma(x) = \sigma'(h(x))$ for $x \in V$. A co-weak isomorphism $h:G \rightarrow G'$ is a map $h:V \rightarrow V'$ which is a bijective homomorphism that satisfies $\mu(x,y) = \mu'(h(x),h(y))$ for $x,y \in V$. An isomorphism $h:G \rightarrow G'$ is a map $h:V \rightarrow V'$ which is a bijective homomorphism that satisfies $\sigma(x) = \sigma'(h(x))$ for $x \in V$ and $\mu(x,y) = \mu'(h(x),h(y))$ for $x,y \in V$. We denote it as $G \cong G'$. 

$x, y \in V$. The order and size of a fuzzy graph $G:(\sigma,\mu)$ are defined by \[O(G) = \sum_{x \in V} \sigma(x) \text{ and } S(G) = \sum_{xy \in E} \mu(xy).\] A fuzzy graph $G:(\sigma,\mu)$ is strong, if $\mu(xy) = \sigma(x) \land \sigma(y)$ for all $xy \in E$. A fuzzy graph $G:(\sigma,\mu)$ is complete, if $\mu(xy) = \sigma(x) \land \sigma(y)$ for all $x,y \in V$. The underlying crisp graph is denoted by $G^*:(V,E)$. 
2.1. Theorem [9]:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on a \( k \) – regular graph \( G^* : (V, E) \). Then \( \mu \) is a constant function if and only if \( G \) is both regular and edge regular.

2.2. Theorem [5]:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on an odd cycle \( G^* : (V, E) \). Then \( G \) is regular iff \( \mu \) is a constant function.

2.3. Theorem [5]:
Let \( G : (\sigma, \mu) \) be a fuzzy graph where \( G^* : (V, E) \) is an even cycle. Then \( G \) is regular iff either \( \mu \) is a constant function or alternate edges have same membership values.

2.4. Theorem [8]:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on \( G^* : (V, E) \). If \( \mu \) is a constant function, then \( G \) is edge regular if and only if \( G^* \) is edge regular.

2.5. Theorem [9]:
Let \( \mu \) be a constant function in \( G : (\sigma, \mu) \) on \( G^* : (V, E) \). If \( G \) is regular, then \( G \) is edge regular.

2.6. Theorem [10]:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on an odd cycle \( G^* : (V, E) \). Then \( G \) is edge regular if and only if \( \mu \) is a constant function.

3. DEGREE OF AN EDGE IN FUZZY LINE GRAPH
For any \( ab \in W \), \( d_{L(G)}(ab) = \sum_{ac \in W, c \neq b} \lambda(ac) + \sum_{cb \in W, c \neq a} \lambda(cb), \forall ab \in W \).

\[
\therefore \quad d_{L(G)}(ab) = \sum_{c \in E, c \neq b} \mu(a) \land \mu(c) + \sum_{c \in E, c \neq a} \mu(b) \land \mu(c), \forall ab \in W. \quad (3.1)
\]

3.1. Theorem:
Let \( G : (\sigma, \mu) \) be a fuzzy graph such that \( \mu \) is a constant function. Then \( d_{L(G)}(ab) = c \left[ d_{G^*}(a) + d_{G^*}(b) - 2 \right], \forall a, b \in E \).

Proof:
Let \( \mu(a) = c, \forall a \in E \).
From (3.1), \( d_{L(G)}(ab) = \sum_{c \in E, c \neq b} \mu(a) \land \mu(c) + \sum_{c \in E, c \neq a} \mu(b) \land \mu(c), \forall ab \in W. \)
Therefore, for all \( ab \in W \), \( d_{L(G)}(ab) = \sum_{c \in E} c \land c + \sum_{c \in E} c \land c \)
\[ = c \left[ d_{G'}(a) - 1 \right] + c \left[ d_{G'}(b) - 1 \right]. \]
\[ = c \left[ d_{G'}(a) + d_{G'}(b) - 2 \right]. \]

4. EDGE REGULAR PROPERTY OF FUZZY LINE GRAPH OF FUZZY GRAPH

4.1. Remark:
If \( G : (\sigma, \mu) \) is an edge regular fuzzy graph, then \( L(G) : (\omega, \lambda) \) need not be edge regular fuzzy graph.
For example, in figure 4.1 \( G : (\sigma, \mu) \) is 1.8 - edge regular fuzzy graph, but \( L(G) : (\omega, \lambda) \) is not an edge regular fuzzy graph.

Fig.4.1.

4.2. Remark:
If \( L(G) : (\omega, \lambda) \) is an edge regular fuzzy graph, then \( G : (\sigma, \mu) \) need not be edge regular fuzzy graph. For example, in figure 4.2 \( L(G) : (\omega, \lambda) \) is 0.7 - edge regular fuzzy graph. But \( G : (\sigma, \mu) \) is not an edge regular fuzzy graph.

Fig.4.2.
4.3. **Theorem:**
Let $G:(\sigma,\mu)$ be a fuzzy graph such that $\mu$ is a constant function. If $G:(\sigma,\mu)$ is an edge regular fuzzy graph, then $L(G):(\omega,\lambda)$ is an edge regular fuzzy graph.

**Proof:**
Since $G:(\sigma,\mu)$ is an edge regular fuzzy graph and $\mu$ is a constant function, by theorem 2.4, $G^*$ is edge regular.

Let $G^*$ be $m$–edge regular. Then by theorem 3.1, for any $ab \in W$,

$$d_{L(G)}(ab) = c \left[ d_G(a) + d_G(b) - 2 \right] = c \left[ m + m - 2 \right] = 2c \left[ m - 1 \right].$$

Hence $L(G):(\omega,\lambda)$ is an edge regular fuzzy graph.

4.4. **Theorem:**
Let $G:(\sigma,\mu)$ be a fuzzy graph such that $\mu$ is a constant function. If $G:(\sigma,\mu)$ is an edge regular fuzzy graph, then $L^n(G)$ is an edge regular fuzzy graph, where $n$ is a positive integer.

**Proof:**
By theorem 4.3, $L(G):(\omega,\lambda)$ is an edge regular fuzzy graph. Since $\mu$ is a constant function, $\lambda$ is a constant function.

Again by theorem 4.3, $L(L(G))$ is an edge regular fuzzy graph. Proceeding in this same way, we get $L^n(G)$ is an edge regular fuzzy graph, for every positive integer $n$.

4.5. **Theorem:**
Let $G:(\sigma,\mu)$ be a strong fuzzy graph such that $\sigma$ is a constant function. If $G:(\sigma,\mu)$ is an edge regular fuzzy graph, then $L(G):(\omega,\lambda)$ is an edge regular fuzzy graph.

**Proof:**
By the hypothesis of this theorem, $\mu$ is a constant function. Therefore the result follows from theorem 4.3.

4.6. **Theorem:**
Let $G:(\sigma,\mu)$ be a strong fuzzy graph such that $\sigma$ is a constant function. If $G:(\sigma,\mu)$ is an edge regular fuzzy graph, then $L^n(G):(\omega,\lambda)$ is an edge regular fuzzy graph.

**Proof:**
By the hypothesis of this theorem, $\mu$ is a constant function. Therefore the result follows from theorem 4.4.
4.7. Theorem:
Let $G: (\sigma, \mu)$ be a fuzzy graph such that $\mu$ is a constant function. If $G: (\sigma, \mu)$ is a regular fuzzy graph, then $L(G): (\omega, \lambda)$ is an edge regular fuzzy graph.

Proof:
If $G: (\sigma, \mu)$ is a regular fuzzy graph such that $\mu$ is a constant function, then by theorem 2.5, $G: (\sigma, \mu)$ is edge regular.

Then by theorem 4.3, $L(G): (\omega, \lambda)$ is an edge regular fuzzy graph.

4.8. Remark:
The converse of above theorem need not be true.
From the following figures, $L(G): (\omega, \lambda)$ is 1.6–edge regular fuzzy graph and $\mu$ is a constant function. But, $G: (\sigma, \mu)$ is not a regular fuzzy graph.

4.9. Theorem:
Let $G: (\sigma, \mu)$ be a fuzzy graph such that $\mu$ is a constant function. If $G: (\sigma, \mu)$ is a regular fuzzy graph, then $L'(G): (\omega, \lambda)$ is an edge regular fuzzy graph.

Proof:
Proof is similar to proof of the theorem 4.4.

4.10. Theorem:
If $G^*: (V, E)$ is a $k$–regular graph, then $L^m(G^*)$ is $2(2^m k - 2^{m+1} + 1)$–edge regular graph.

Proof:
Let us prove this theorem by mathematical induction.
Since $G^*: (V, E)$ is a $k$–regular graph, $G^*$ is $2(k - 1)$–edge regular. Therefore $L(G^*)$ is $2(k - 1)$–regular.
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\[ \therefore \text{For any edge } ab \text{ in } L(G^*), \quad d_{L(G^*)}(ab) = d_{L(G^*)}(a) + d_{L(G^*)}(b) - 2 \]
\[ = 2(k-1) + 2(k-1) - 2 = 2(2k-3). \]

Therefore the result is true for \( n = 1. \)

Assume that \( L^n(G^*) \) is an \( 2(2^n k - 2^{n+1} + 1) \)–edge regular graph.

Then \( L^{n+1}(G^*) \) is \( 2(2^n k - 2^{n+1} + 1) \)–regular.

\[ \therefore \text{For any edge } xy \text{ in } L(G^*), \quad d_{L^{n+1}(G^*)}(xy) = d_{L^{n+1}(G^*)}(x) + d_{L^{n+1}(G^*)}(y) - 2 \]
\[ = 2(2^n k - 2^{n+1} + 1) + 2(2^n k - 2^{n+1} + 1) - 2 \]
\[ = 2(2^{n+1} k - 2^{n+2} + 1). \]

\[ \therefore L^{n+1}(G^*) \text{ is } 2(2^{n+1} k - 2^{n+2} + 1) \text{–edge regular.} \]

By the principle of mathematical induction, we have \( L^n(G^*) \) is an \( 2(2^n k - 2^{n+1} + 1) \)–edge regular, for every positive integer \( n. \)

4.11. Theorem:

Let \( G : (\sigma, \mu) \) be a fuzzy graph such that \( \mu(e) = c, \forall e \in E \), where \( c \) is a constant. If \( G^* : (V, E) \) is a \( k \)–regular graph, then \( L^n(G) : (\omega, \lambda) \) is an \( 2c(2^n k - 2^{n+1} + 1) \)–edge regular fuzzy graph, where \( n \) is a positive integer.

Proof:

By the definition of fuzzy line graph, all the edge membership values is equal to \( c \) in \( L(G), L^2(G), \ldots, L^n(G) \), since \( \mu(e) = c, \forall e \in E \), where \( c \) is a constant. ......... (4.1)

To prove \( L^n(G) : (\omega, \lambda) \) is an \( 2c(2^n k - 2^{n+1} + 1) \)–edge regular fuzzy graph.

By (4.1) and theorem 3.1, \( d_{L(G)}(ab) = c \left[ d_{L^1(G)}(a) + d_{L^1(G)}(b) - 2 \right], \forall ab \in W \)

Using the theorem 4.10,
\[ d_{L(G)}(ab) = c \left[ 2(2^{n-1} k - 2^n + 1) + 2(2^{n-1} k - 2^n + 1) - 2 \right], \forall ab \in W. \]
\[ = c \left[ 2^n k - 2^{n+1} + 2 + 2^n k - 2^{n+1} + 2 - 2 \right]. \]
\[ \Rightarrow d_{L(G)}(ab) = 2c \left[ 2^n k - 2^{n+1} + 1 \right], \forall ab \in W. \]

Therefore \( L^n(G) : (\omega, \lambda) \) is an \( 2c \left[ 2^n k - 2^{n+1} + 1 \right] \)–edge regular fuzzy graph, for all positive integers \( n. \)

4.12. Theorem:

Let \( G : (\sigma, \mu) \) be a strong fuzzy graph such that \( \sigma \) is a constant function and let \( G^* : (V, E) \) be a regular graph. Then \( L(G) : (\omega, \lambda) \) is an edge regular fuzzy graph.
Proof:
Proof is similar to proof of theorem 4.11.

4.13. Theorem:
Let $G^*: (V,E)$ be an edge regular graph. If membership value of at least one of the two edges of any pair of adjacent edges is equal to $\wedge\{\mu(e)|e \in E\}$, the lowest edge membership value, then $L(G): (\omega, \lambda)$ is an edge regular fuzzy graph.

Proof:
Let $G^*$ be $k$ - edge regular and let $\wedge\{\mu(e)|e \in E\} = c$. Then by hypothesis, $\mu(x) \wedge \mu(y) = c$ for any pair of adjacent edges $x$ and $y$.

From (3.1), for all $xy \in W$, $d_{L(G)}(xy) = \sum_{z \in E, z \neq x, y} \mu(x) \wedge \mu(z) + \sum_{z \in E, z \neq x, y} \mu(y) \wedge \mu(z) = \sum_{z \in E, z \neq x, y} c + \sum_{z \in E, z \neq x, y} c$

$= c[d_{G^*}(x) - 1] + c[d_{G^*}(y) - 1] = c[k - 1] + c[k - 1] = 2c[k - 1]$.

Hence $L(G): (\omega, \lambda)$ is an edge regular fuzzy graph.

4.14. Remark:
The converse of above theorem need not be true.
From the figure 4.2, $L(G): (\omega, \lambda)$ is 0.7 - edge regular fuzzy graph. But, membership value of at least one of the two edges of any pair of adjacent edges is not equal to $\wedge\{\mu(e)|e \in E\}$, the lowest edge membership value.

5. EDGE REGULAR PROPERTY OF FUZZY LINE GRAPH OF FUZZY GRAPH ON A CYCLE

5.1. Theorem:
Let $G: (\sigma, \mu)$ be a fuzzy graph and let $L(G): (\omega, \lambda)$ be a fuzzy line graph of $G: (\sigma, \mu)$. Then $G: (\sigma, \mu)$ is a fuzzy graph on a cycle $G^*: (V,E)$ if and only if $L(G): (\omega, \lambda)$ is a fuzzy graph on the cycle $L(G^*): (Z,W)$.

Proof:
Since $G^*: (V,E)$ is a cycle if and only if $L(G^*): (Z,W)$ is also a cycle, $G: (\sigma, \mu)$ is a fuzzy graph on a cycle $G^*: (V,E)$ if and only if $L(G): (\omega, \lambda)$ is a fuzzy graph on a cycle $L(G^*): (Z,W)$.

5.2. Theorem:
If $G: (\sigma, \mu)$ is a strong fuzzy graph on a cycle $G^*: (V,E)$ such that $\sigma$ is a constant function, then $G$ is isomorphic to $L(G)$.
Proof:
Let \( \sigma(u) = c \), for all \( u \in V \), where \( c \) is a constant.
Since \( G \) is a strong fuzzy graph, \( \mu(e) = c, \forall e \in E \).
By the definition of fuzzy line graph, \( \omega \) and \( \lambda \) are also constant functions of same constant value \( c \).

Let \( G \) be a fuzzy graph on a cycle \( v_1 e_1 v_2 e_2 ... v_n e_n v_{n+1}, \forall v_i \in V \& e_i \in E \), where \( v_{n+1} = v_1 \). Then \( L(G) \) is a fuzzy line graph on the cycle \( e_1 x_1 e_2 x_2 ... e_n v_i e_{n+1}, \forall e_i \in Z \& x_i \in W \), where \( e_{n+1} = e_1 \).

Now, define a mapping \( h: V \rightarrow Z \) by \( h(v_i) = e_i, \forall v_i \in V \).
Then \( \omega(h(v_i)) = \omega(e_i) = c = \sigma(v_i), \forall v_i \in V \) and
\[
\lambda(h(v_i)h(v_{i+1})) = \mu(h(v_i)) \land \mu(h(v_{i+1})) = \mu(e_i) \land \mu(e_{i+1}) = c = \mu(v_i v_{i+1}), \forall e_i \in E .
\]
Therefore \( G \) is isomorphic to \( L(G) \).

5.3. Theorem:
If \( G: (\sigma, \mu) \) is a fuzzy graph on a cycle \( G^*: (V, E) \), then \( L(G) \) is homomorphic to \( G \).

Proof:
Let \( G \) be a fuzzy graph on a cycle \( v_1 e_1 v_2 e_2 ... v_n e_n v_{n+1}, \forall v_i \in V \& e_i \in E \), where \( v_{n+1} = v_1 \). Then \( L(G) \) is a fuzzy line graph on the cycle \( e_1 x_1 e_2 x_2 ... e_n v_i e_{n+1}, \forall e_i \in Z \& x_i \in W \), where \( e_{n+1} = e_1 \).

Now, define a mapping \( h: Z \rightarrow V \) between \( L(G) \) and \( G \) by \( h(e_i) = v_i, \forall e_i \in Z \).
Therefore \( \sigma(h(e_i)) = \sigma(v_i) \geq \mu(e_i) = \omega(e_i), \forall e_i \in Z \) and
\[
\mu(h(e_i)h(e_{i+1})) = \mu(v_i v_{i+1}) = \mu(e_i) = \omega(e_i) \geq \lambda(e_i e_{i+1}), \forall e_i, e_{i+1} \in Z .
\]
Therefore \( L(G) \) is homomorphic to \( G \).

5.4. Theorem:
Let \( G: (\sigma, \mu) \) be a fuzzy graph on an even cycle \( G^*: (V, E) \). If either \( \mu \) is a constant function of constant value \( k \) or each edge in any one set of alternate edges has same lowest membership value \( k \) in \( G \), then \( L(G): (\omega, \lambda) \) is \( 2k \) – edge regular fuzzy graph.

Proof:
If \( \mu \) is a constant function, then there is nothing to prove.
Let \( e_1, e_2, e_3, ..., e_{2m} \) be the edges of the even cycle \( G^* \) in that order.
Suppose any one set of the alternate edges have same lowest membership value \( k \) in \( G \).
Then either
\[ \mu(e_i) = \mu(e_j) = \mu(e_k) = \ldots = \mu(e_{2m-1}) < \mu(e_l), \forall e_i \in E, i = 2, 4, \ldots, 2m \text{ (or)} \]
\[ \mu(e_i) = \mu(e_j) = \mu(e_k) = \ldots = \mu(e_{2m}) < \mu(e_l), \forall e_i \in E, i = 1, 3, 5, \ldots, 2m-1. \]
\[ \therefore \text{For any two adjacent edges } e_i \text{ and } e_{i+1}, \mu(e_i) \wedge \mu(e_{i+1}) = k. \]
\[ \therefore \lambda \text{ is a constant function with } \lambda(e_i e_{i+1}) = k, \forall i = 1, 2, \ldots, 2m, \text{ where } e_{2m+1} = e_1. \]
\[ \therefore d_{L(G)}(e_i e_{i+1}) = d(e_i) + d(e_{i+1}) - 2\lambda(e_i e_{i+1}), \forall e_i e_{i+1} \in W. \]
\[ = 2k + 2k - 2k \]
\[ = 2k. \]
Therefore \( L(G) \) is \( 2k \) – edge regular fuzzy graph.

5.5. Remark:
The converse of above theorem need not be true.
From the figure 4.2, \( L(G) : (\omega, \lambda) \) is \( 0.7 \) – edge regular fuzzy graph. But, alternate edges do not has same lowest membership values in \( G : (\sigma, \mu) \).

5.6. Theorem:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on an even cycle \( G^* : (V, E) \). If each edge in any one set of alternate edges has same lowest membership value \( k \) in \( G \), then \( L''(G) : (\omega, \lambda) \) is \( 2k \) – edge regular fuzzy graph.

Proof:
From the theorem 5.4, \( L(G) \) is \( 2k \) – edge regular fuzzy graph.
Since \( \lambda \) is a constant function, by the definition of fuzzy line graph, all edge membership values in \( L(L(G)) \) is equal to \( k \). By the theorem 2.1, \( L(L(G)) \) is \( 2k \) – edge regular fuzzy graph.
Proceeding in the same way, finally, we get \( L'(G) : (\omega, \lambda) \) is \( 2k \) – edge regular fuzzy graph.

5.7. Theorem:
Let \( G : (\sigma, \mu) \) be a fuzzy graph on an odd cycle \( G^* : (V, E) \). Then \( L(G) : (\omega, \lambda) \) is an edge regular fuzzy graph if and only if either \( \mu \) is a constant function or any two adjacent edges and all the edges that lie alternatively from them receives lowest membership value.

Proof:
As in the proof of theorem 5.4, \( L(G) \) is \( 2k \) – edge regular fuzzy graph.
Conversely, assume that \( L(G) \) is an edge regular fuzzy graph.
Since \( L(G) \) is a fuzzy graph on an odd cycle \( G^* : (V, E) \), \( \lambda \) is a constant function (using theorem 2.6).
\[ \lambda(xy) = \mu(x) \land \mu(y) = k, \text{ for all } xy \in W, \text{ where } k \text{ is a constant.} \]

\………………… (5.1)

If \( \mu(x) = \mu(y) \), for all \( xy \in W \), then \( \mu \) is a constant function.

Suppose not, then for any two adjacent edges \( x \) and \( y \) such that \( \mu(x) \neq \mu(y) \).

Without loss of generality, assume that \( \mu(x) < \mu(y) \).

Hence the result follows from (5.1).

5.8. Theorem:
Let \( G:(\sigma,\mu) \) be regular fuzzy graph on an odd cycle \( G':(V,E) \). Then \( L(G):(\omega,\lambda) \) is an edge regular fuzzy graph.

Proof:
Using theorem 2.2, \( \mu \) is a constant function. Therefore \( \lambda \) is also a constant function in \( L(G) \).

Also \( L(G') \) is a cycle. Hence by theorem 2.1, \( L(G) \) is edge regular.

5.9. Remark:
The converse of above theorem need not be true.
From the following figures, \( L(G):(\omega,\lambda) \) is 0.8 – edge regular fuzzy graph. But, \( G:(\sigma,\mu) \) is not a regular fuzzy graph.

![Fig 5.1](image)

5.10. Theorem:
Let \( G:(\sigma,\mu) \) be regular fuzzy graph on an even cycle \( G^*: (V,E) \). Then \( L(G): (\omega, \lambda) \) is an edge regular fuzzy graph.

Proof:
Using theorem 2.3, \( \mu \) is a constant function or alternate edges have same membership values.
In both cases, \( \lambda \) is a constant function in \( L(G) \). Hence by theorem 2.1, \( L(G) \) is an edge regular.
5.11. Remark:
The converse of above theorem need not be true.
From the figure 4.2, \(L(G): (\omega, \lambda)\) is 0.7 – edge regular fuzzy graph. But, \(G: (\sigma, \mu)\) is not a regular fuzzy graph.

5.12. Theorem:
Let \(G: (\sigma, \mu)\) be a fuzzy graph on a cycle \(G^*: (V, E)\) such that \(\mu(e) = c, \forall e \in E\), where \(c\) is a constant. Then \(G: (\sigma, \mu)\) is an \(2c\) – edge regular fuzzy graph and \(L(G): (\omega, \lambda)\) is an \(2c\) – edge regular fuzzy graph.

Proof:
Its proof is trivial.

5.13. Remark:
The converse of above theorem need not be true.
From the following figures, \(G: (\sigma, \mu)\) is 1.0 – edge regular fuzzy graph and \(L(G): (\omega, \lambda)\) is 0.8 – edge regular fuzzy graph, but \(\mu\) is not a constant function.

![Diagram](image)

Fig.5.2.

6. PROPERTIES OF EDGE REGULAR FUZZY LINE GRAPHS

6.1. Theorem:
Let \(G: (\sigma, \mu)\) be a fuzzy graph and let \(L(G): (\omega, \lambda)\) be a fuzzy line graph of \(G: (\sigma, \mu)\).
Then (i). \(S(G) = O(L(G))\) and
(ii). \(S(L(G)) = \sum x, y \in E \mu(x) \land \mu(y)\), where \(x\) and \(y\) are adjacent.

Proof:
(i). Since \(\omega(x) = \mu(x)\), \(\forall S_x \in Z\), \(\sum x \in E \mu(x) = \sum x \in Z \omega(x) \Rightarrow S(G) = O(L(G)).\)
(ii). By the definition of line graph, \(S(L(G)) = \sum x, y \in E \mu(x) \land \mu(y)\), where \(x\) and \(y\) are adjacent.
6.2. Theorem:
Let $G:(\sigma,\mu)$ be a fuzzy graph such that $\mu = c$, where $c$ is a constant. If $G^*:(V,E)$ be a $k$ – regular graph, then the size of $L(G):(\omega,\lambda)$ is $qc(k-1)$, where $q = |E|$.

Proof:
Given $G^*:(V,E)$ is a $k$ – regular graph.
From the proof of theorem 2.4, $G^*$ is an $2(k-1)$ – edge regular, the number of edges in $L(G^*):(Z,W) = |W| = \frac{1}{2}q(2(k-1)) = q(k-1)$. Therefore the size of $L(G):(\omega,\lambda)$ is $qc(k-1)$.

6.3. Theorem:
Let $G:(\sigma,\mu)$ be a fuzzy graph such that $\mu(e) = c, \forall e \in E$, where $c$ is a constant. If $G^*:(V,E)$ be a $k$ – edge regular graph, then the size of $L(G):(\omega,\lambda)$ is $\frac{qck}{2}$, where $q = |E|$.

Proof:
Proof is similar to proof of the theorem 6.2.

6.4. Remark:
The above theorems 6.2 and 6.3 are true, when $\lambda$ is a constant function of constant value $c$.

REFERENCES


