On Directed Pathos Line Digraph of an Arborescence

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Abstract

For an arborescence \( T \), a directed pathos line digraph \( DPL(T) \) has vertex set \( V(Q) = A(T) \cup P(T) \), where \( A(T) \) is the arc set and \( P(T) \) is a directed pathos set of \( T \). The arc set \( A(Q) \) consists of the following arcs: \( ab \) such that \( a, b \in A(T) \) and the head of \( a \) coincides with the tail of \( b \); \( Pa \) such that \( a \in A(T) \) and \( P \in P(T) \) and the arc \( a \) lies on the directed path \( P \); \( P_iP_j \) such that \( P_i, P_j \in P(T) \), and it is possible to reach the head of \( P_j \) from the tail of \( P_i \) through a common vertex. For this class of digraphs, we discuss the planarity, outer planarity, maximal outer planarity and minimally non-outer planarity properties of these digraphs.

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1. Introduction

We shall assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [3,6]. The concept of pathos of a graph \( G \) was introduced by Harary [4] as a collection of minimum number of edge disjoint open paths whose union is \( G \). The path number of a graph \( G \) is the number of paths in any pathos. The path number of a tree \( T \) equals \( k \), where \( 2k \) is the number of odd degree vertices
of $T$. Harary [5] and Stanton [8] have calculated the path number of certain classes of graphs like trees and complete graphs.

For a tree $T$ with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$, R. Chandrasekhar et al. [2] gave the following definition.

A pathos line graph of $T$, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of $T$, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of $T$ are adjacent or the edge lies on the corresponding path of the pathos. The order and size of $PL(T)$ are $(q + k)$ and $\frac{1}{2} \sum_{i=1}^{n} d_i^2$, respectively, where $q$ is the size, $k$ is the path number and $d_i$ is the degree of vertices of $T$.

In this paper, we extend the definition of a pathos line graph of a tree to an arborescence. Furthermore, some of its characterizations such as the planarity, outer planarity, etc., are discussed.

M. Aigner [1] defines the line digraph of a digraph as follows. Let $D$ be a digraph with $n$ vertices $v_1, v_2, \ldots, v_n$ and $m$ arcs and $L(D)$ its associated line digraph with $n'$ vertices and $m'$ arcs. We immediately have $n' = m$ and $m' = \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i)$.

Furthermore, the in-degree, respectively out-degree of a vertex $v = (v_i, v_j)$ in $L(D)$ are $d^-(v') = d^-(v_j)$ and $d^+(v') = d^+(v_j)$.

We need some concepts and notations on directed graphs. A directed graph (or just digraph) $D$ consists of a finite non-empty set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pair of distinct vertices called arcs. Here $V(D)$ is the vertex set and $A(D)$ is the arc set of $D$. If $(u, v)$ or $uv$ is an arc in $D$, then we say that $u$ dominates $v$ ($v$ is dominated by $u$) and denote it by $u \rightarrow v$. A digraph $D$ is semicomplete if for each pair of distinct vertices $u$ and $v$, at least one of the arcs $(u, v)$ and $(v, u)$ exists in $D$. A semicomplete digraph of order $n$ is denoted by $D_n$.

The out-degree of a vertex $v$, written $d^+(v)$, is the number of arcs going out from $v$ and the in-degree of a vertex $v$, written $d^-(v)$, is the number of arcs coming into $v$. The total degree of a vertex $v$, written $td(v)$, is the number of arcs incident with $v$. We immediately have $td(v) = d^-(v) + d^+(v)$. A vertex with an in-degree(out-degree) zero is called a source(sink). The directed path on $n \geq 2$ vertices is the digraph $\tilde{P}_n = \{V(\tilde{P}_n), E(\tilde{P}_n), \eta\}$, where $V(\tilde{P}_n) = \{u_1, u_2, \ldots, u_n\}$, $E(\tilde{P}_n) = \{e_1, e_2, \ldots, e_{n-1}\}$, and $\eta$ is given by $\eta(e_i) = (u_i, u_{i+1})$, for all $i \in \{1, 2, \ldots, (n - 1)\}$.

An arborescence $T$ is a directed graph in which, for a vertex $u$ called the root (a vertex of in-degree zero) and any other vertex $v$, there is exactly one directed path from $u$ to $v$. A root arc of $T$ is an arc which is directed out from the root of $T$, i.e., an arc whose tail is the root of $T$.

Since most of the results and definitions for undirected planar graphs are valid for planar digraphs also, the following definitions hold good for planar digraphs. If $D$ is a planar digraph, then the inner vertex number $i(D)$ of $D$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of $D$ in the plane. A digraph $D$ is outerplanar if $i(D) = 0$ and minimally non-outerplanar if
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\[ i(D) = 1 \] Finally, a fan graph \( F_{m,n} \) is defined as the graph join \( \overline{K}_m + P_n \), where \( \overline{K}_m \) is the empty graph on \( m \) vertices and \( P_n \) is the path graph on \( n \) vertices.

2. Definition of \( DPL(T) \)

If a directed path \( \vec{P}_n \) starts at one vertex and ends at a different vertex, then \( \vec{P}_n \) is called an open directed path. A directed path is said to be non-empty if it has at least one arc.

The directed pathos of an arborescence \( T \) is defined as a collection of minimum number of arc disjoint open directed paths whose union is \( T \). The directed path number \( k' \) of \( T \) is the number of directed paths in any directed pathos of \( T \), and is equal to the number of sinks in \( T \), i.e., \( k' = \text{number of sinks in } T \). Finally, we assume that the direction of the directed pathos is along the direction of the arcs in \( T \).

For an arborescence \( T \), a directed pathos line digraph \( Q = DPL(T) \) has vertex set \( V(Q) = A(T) \cup P(T) \), where \( A(T) \) is the arc set and \( P(T) \) is a directed pathos set of \( T \). The arc set \( A(Q) \) consists of the following arcs: \( ab \) such that \( a, b \in A(T) \) and the head of \( a \) coincides with the tail of \( b \); \( Pa \) such that \( a \in A(T) \) and \( P \in P(T) \) and the arc \( a \) lies on the directed path \( P \); \( PiPj \) such that \( Pi, Pj \in P(T) \), and it is possible to reach the head of \( Pj \) from the tail of \( Pi \) through a common vertex.

Note that if the out-degree of the root of \( T \) is more than one, then \( DPL(T) \) is disconnected. Hence we consider the arborescences having out-degree of the root exactly one. See Fig. 1 and Fig. 2 for an example of an arborescence and its directed pathos line digraph.

![Arborescence T](image)

**Figure 1: Arborescence T.**

The following observations are easily justified:

Observation 1: The number of arcs whose end vertices are the directed pathos vertices in \( DPL(T) \) equals \( (k' - 1) \), where \( k' \) is the directed path number of \( T \).
Observation 2: A connected digraph $D$ with $n$ vertices $v_1, v_2, \ldots, v_n$ is a non-empty directed path if $\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n - 2$.

3. Basic properties of $DPL(T)$

In this section, we establish some basic relationships between an arborescence and its directed pathos line digraph.

**Proposition 3.1.** Let $T$ be an arborescence with $n$ vertices $v_1, v_2, \ldots, v_n$ and $k'$ sinks. Then the order and size of $DPL(T)$ are $(n - 1 + k')$ and $\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + n + k' - 2$, respectively.

**Proof.** Let $T$ be an arborescence with $n$ vertices $v_1, v_2, \ldots, v_n$ and $k'$ sinks. By definition, the order of $DPL(T)$ equals the sum of size and directed paths of $T$. Thus, $V(DPL(T)) = n - 1 + k'$. Now, the size of $DPL(T)$ equals the sum of sizes of $T$ and $L(T)$, and the arcs whose end vertices are the directed pathos vertices. By Observation 1, the size of $DPL(T)$ is

$$\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + n - 1 + k' - 1.$$

$$\Rightarrow \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + n + k' - 2.$$
Theorem 3.2. [6] Let $D$ be an acyclic digraph with precisely one source $x$ in $D$. Then for every $v \in V(D)$, there is an $(x, v)$-directed path in $D$.

Proposition 3.3. Every $DPL(T)$ has a source $s$ and for every vertex $v \in DPL(T)$, there is an $(s, v)$− directed path in $DPL(T)$.

Proof. Let $T$ be an arborescence with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and arc set $A(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ such that $v_1$ and $e_1$ are the root and root arc of $T$, respectively. Then the vertices $e_2, e_3, \ldots, e_{n-1}$ of $L(T)$ are reachable from $e_1$ by a unique directed path. Let $P(T) = \{P_1, P_2, \ldots, P_r\}$ be a directed pathos set of $T$ such that $P_1$ lies on the arc $e_1$ in $T$. Since the direction of the directed pathos is along the direction of the arcs in $T$, by definition, the corresponding directed pathos vertices except $P_1$ are both tail and the head in $DPL(T)$. Clearly, $DPL(T)$ is acyclic. By Theorem 3.2, for every vertex $v \in DPL(T)$, there is a $(P_1, v)$− directed path in $DPL(T)$. This completes the proof.

4. Characterization of $DPL(T)$

We will give a constructive proof to show the planarity, outer planarity and minimally non-outer planarity properties of $DPL(T)$.

Theorem 4.1. For any arborescence $T$, $DPL(T)$ is planar.

Proof. We consider the following three cases.

Case 1: Suppose that $T$ is a directed path on $n \geq 2$ vertices. For $n = 2$, $DPL(T)$ is $D_2$, which is planar. For $n \geq 3$, $DPL(T)$ is a digraph whose underlying graph is the fan graph $F_{1,n-1}$. Clearly, $DPL(T)$ is planar.

Case 2: Suppose that $T$ is an arborescence whose underlying graph is a star graph $K_{1,n}$ on $n \geq 2$ vertices. Let $A(T) = \{e_1, e_2, \ldots, e_n\}$ such that $e_1$ is the root arc of $T$. By definition, $L(T)$ is an out-star of order $n$ with the central vertex $e_1$. Now, the directed path number of $T$ is $(n - 1)$. Then $DPL(T)$ is a connected digraph in which every block is either $D_2$ or $D_3$. Finally, the arcs joining directed pathos vertices gives $DPL(T)$ such that the crossing number of $DPL(T)$ is zero. Clearly, $DPL(T)$ is planar.

Case 3: Suppose that $T$ is an arborescence whose underlying graph is not a star graph such that $td(v) \geq 1$, for every vertex $v \in T$. Let $V(T) = \{v_1, v_2, \ldots, v_n\}$ and $A(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ such that $v_1$ and $e_1 = v_1v_2$ are the root and root arc of $T$, respectively. By definition, $L(T)$ is an out-tree of order $(n - 1)$. The directed path number of $T$ is the number of sinks in $T$. Then $DPL(T)$ is a connected digraph in which every block is either $D_2$ or $D_3$ or $D_4 - e$. Finally, the arcs joining directed pathos vertices gives
A directed pathos line digraph $DPL(T)$ of an arborescence $T$ is outerplanar if and only if the underlying graph of $T$ is a star graph $K_{1,n}$ on $n \leq 3$ vertices.

Proof. Suppose $DPL(T)$ is outerplanar. Assume that $T$ is an arborescence whose underlying graph is the star graph $K_{1,4}$. Let $V(T) = \{a, b, c, d, e\}$ and $A(T) = \{(a, c), (c, b), (c, d), (c, e)\}$ such that $a$ and $ac$ are the root and root arc of $T$, respectively. Then $A(L(T)) = \{(ac, cb), (ac, cd), (ac, ce)\}$. Let $P(T) = \{P_1, P_2, P_3\}$ be a directed pathos set of $T$ such that $P_1$ lies on the arcs $(a, c)$ and $(c, b)$, $P_2$ lies on $(c, d)$ and $P_3$ lies on $(c, e)$. Then $DPL(T)$ is a connected digraph in which every block is either $D_2$ or $D_3$, which shows that $i(DPL(T)) = 0$. Finally, the arcs joining directed pathos vertices $P_1, P_2$ and $P_3$ gives $DPL(T)$ such that the inner vertex number of $DPL(T)$ becomes exactly one, i.e., $i(DPL(T)) = 1$, a contradiction.

For the sufficiency, we consider the following three cases.

Case 1: If $T$ is $K_{1,1}$, then $DPL(T)$ is $D_2$, which is outerplanar.

Case 2: If $T$ is $K_{1,2}$, then $DPL(T)$ is $D_3$, which is also outerplanar.

Case 3: If $T$ is $K_{1,3}$, then $DPL(T)$ is a connected digraph in which every block is either $D_2$ and $D_3$ and the inner vertex number of $DPL(T)$ is zero. Hence $DPL(T)$ is outerplanar.

Theorem 4.3. [3] Every maximal outerplanar graph $G$ with $n$ vertices has $(2n − 3)$ edges.

Theorem 4.4. A directed pathos line digraph $DPL(T)$ of an arborescence $T$ is maximal outerplanar if and only if $T$ is a directed path $\vec{P}_n$ on $n \geq 2$ vertices.

Proof. Suppose that $DPL(T)$ is maximal outerplanar. Then $DPL(T)$ is connected. Hence $T$ is connected. If $DPL(T)$ is $D_2$, then $T$ is also $D_2$. Let $T$ be an arborescence with $n \geq 3$ vertices and directed path number $k'$. By Proposition 3.1, the order and size of $DPL(T)$ are $(n − 1 + k')$ and $\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + n + k' - 2$, respectively. Since $DPL(T)$ is maximal outerplanar, by Theorem 4.3, the size of $DPL(T)$ is $2(n − 1 + k') − 3$. Hence $\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + n + k' - 2 = 2(n − 1 + k') − 3$. It is known that for an arborescence
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which is a directed path, the directed path number \( k' = 1 \).

\[
\Rightarrow \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + n - 1 = 2n - 3,
\]

\[
\Rightarrow \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = 2n - 3 - n + 1,
\]

\[
\Rightarrow \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n - 2.
\]

By Observation 2, \( T \) is a non-empty directed path. The necessity is thus proved.

Sufficiency. Suppose that \( T \) is a directed path on \( \vec{P}_n \) on \( n \geq 2 \) vertices. We consider the following two cases.

Case 1: If \( T \) is \( D_2 \), then \( DPL(T) \) is also \( D_2 \), which is maximal outerplanar.

Case 2: Suppose that \( T \) is a non-empty directed path \( \vec{P}_n \) on \( n \geq 3 \) vertices.

We show that \( DPL(T) \) is maximal outerplanar by the induction on the number of vertices of \( T \). It is easy to observe that \( DPL(T) \) of a directed path \( \vec{P}_3 \) is \( D_3 \), which is also maximal outerplanar. As the inductive hypothesis, let the directed pathos line digraph of a non-empty directed path \( T \) with \( n \) vertices be maximal outerplanar. We now prove that the directed pathos line digraph of a directed path \( T' \) with \( (n + 1) \) vertices is maximal outerplanar. We first prove that it is outerplanar. Let the vertex and arc sequence of the directed path \( T' \) be \( v_1e_1v_2e_2 \ldots v_{n-1}e_{n-1}v_nv_{n+1} \). Without loss of generality, let \( T' - v_{n+1} = T \). By the inductive hypothesis, \( DPL(T) \) is maximal outerplanar. Now, the vertex \( v_{n+1} \) is one vertex more in \( DPL(T') \) than in \( DPL(T) \). Also, there are only two arcs \( e_{n-1}e_n \) and \( Pe_n \) more in \( DPL(T') \). Clearly, the induced subdigraph on the vertices \( P, e_{n-1} \) and \( e_n \) is not \( D_4 \). Hence \( DPL(T') \) is outerplanar.

We now prove that \( DPL(T') \) is maximal outerplanar. Since \( DPL(T) \) is maximal outerplanar, by Theorem 4.3 it has \( 2n - 3 \) arcs. The outerplanar digraph \( DPL(T') \) has \( 2n - 3 + 2 = 2(n + 1) - 3 \) arcs. By Theorem 4.3, \( DPL(T') \) is maximal outerplanar. This completes the proof.

\[\text{Theorem 4.5.} \text{ A directed pathos line digraph } DPL(T) \text{ of an arborescence } T \text{ is minimally non-outerplanar if and only if the underlying graph of } T \text{ is the star graph } K_{1,4}.\]

\[\text{Proof.} \text{ Suppose } DPL(T) \text{ is minimally non-outerplanar. Assume that } T \text{ is an arborescence whose underlying graph is the star graph } K_{1,5}. \text{ Let } V(T) = \{a, b, c, d, e, f\} \text{ and } A(T) = \{(a, c), (c, b), (c, d), (c, e), (c, f)\} \text{ such that } a \text{ and } ac \text{ are the root and root arc of } T, \text{ respectively. Then } A(L(T)) = \{(ac, cb), (ac, cd), (ac, ce), (ac, cf)\}. \text{ Let } P(T) = \{P_1, P_2, P_3, P_4\} \text{ be a directed pathos set of } T \text{ such that } P_1 \text{ lies on the arcs } (a, c)\]
and \((c, b)\), \(P_2\) lies on \((c, d)\), \(P_3\) lies on \((c, e)\) and \(P_4\) lies on \((c, f)\). Then \(DPL(T)\) is a connected digraph in which every block is either \(D_2\) or \(D_3\), which clearly shows that \(i(DPL(T)) = 0\). Finally, the arcs joining the directed pathos vertices gives \(DPL(T)\) such that \(i(DPL(T)) \geq 2\), a contradiction.

Conversely, suppose that \(T\) is an arborescence whose underlying graph is the star graph \(K_{1,4}\). By necessity part of Theorem 4.2, \(i(DPL(T)) = 1\). Hence \(DPL(T)\) is minimally non-outerplanar.

We finally conclude with the following: It is known that since the pattern of directed pathos for an arborescence \(T\) is not unique, the corresponding directed pathos line digraph \(DPL(T)\) is also not unique. But, since the directed path number \(k'\) of a directed path of order \(n (n \geq 2)\) is exactly one, by definition, the directed pathos line digraph of a directed path is unique. One can easily observe that for different pattern of directed pathos for an arborescence whose underlying graph is the star graph \(K_{1,n}\) on \(n \geq 3\) vertices, the corresponding directed pathos line digraphs are isomorphic. Therefore, the necessity part of Theorem 4.2 and Theorem 4.5 holds for any pattern of directed pathos. Also, for an arborescence \(T\) with \(td(v) \geq 1\), for every vertex \(v \in T\), for different pattern of directed pathos, we find the same conclusion as described in Case 3 of Theorem 4.1.

References


