A Fuzzy Approach To Complete Upper Semilattice And Complete Lower Semilattice

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Abstract

This study is an extension of the work “A Study of Green’s Relations on Fuzzy Semigroups” [1]. Using the definition of fuzzy compatibility [1], a new concept for fuzzy partial congruences is also introduced. On the basis of the described notions and certain conditions in [1], a complete lower fuzzy semilattice and a complete upper fuzzy semilattice is found out and defined from the set of all fuzzy partial congruences on a semigroup. The purpose of this paper is to develop the concept of a complete upper semilattice and a complete lower semilattice [5], on fuzzy views, using the percept of partially ordered and totally ordered set.

Keywords: Fuzzy compatibility, Fuzzy congruences, Fuzzy partial congruences, Dot composition of fuzzy relations, Cross composition of fuzzy relations, Complete upper fuzzy semilattices, Complete lower fuzzy semilattices.

1. INTRODUCTION:

The study introduces fuzzy partial congruences by recalling the ideas of partially ordered and totally ordered set. It is a follow up of the work [1]. Using fuzzy congruences [3, 7] and well known definitions [2], the notion of partial congruences is described.

The study concludes that the algebraic structure \(<P_f(S), \circ >\) is a complete upper fuzzy semilattice, where “\(\circ\)”denotes the dot composition of fuzzy relations [1] and \(<P_f(S), \bullet >\) is a complete lower fuzzy semilattice, where “\(\bullet\)” denotes the cross composition of fuzzy relations [1] and \(P_f(S)\) denotes the set of all fuzzy partial congruences on a semigroup.
2. BASIC NOTIONS.

Definition 2.1. Fuzzy relations: Fuzzy relation indicates the strength or association between the elements of n-tuple. When n=2 the fuzzy relation is called a fuzzy binary relation.

That is a function \( \alpha \) defined from \( SX \times S \) to \([0,1]\) is called a fuzzy binary relation on \( S \), where \( S \) is a semigroup [1].

Composition of fuzzy relations

Definition 2.2. Dot composition: Let \( \alpha \) and \( \beta \) be two fuzzy binary relations on a semigroup \( S \). A dot composition of \( \alpha \) and \( \beta \), denoted by \( \alpha \circ \beta \), is defined as

\[
\alpha \circ \beta (a,b) = \max_{z \in S} \min \{ \alpha(a,z), \beta(z,b) \} \quad \text{for all } a, b \in S \quad [1].
\]

Definition 2.3. Cross composition: A cross composition of \( \alpha \) and \( \beta \), denoted by \( \alpha \odot \beta \), is defined as

\[
\alpha \odot \beta (a,b) = \min_{z \in S} \min \{ \alpha(a,z), \beta(z,b) \} \quad \text{for all } a, b \in S \quad [1].
\]

Proposition 2.3. Dot and Cross Composition of fuzzy relation on a semigroup is associative.

Proof: Let \( \alpha, \beta, \gamma \) be fuzzy relations on \( S \). By definition, ‘\( \circ \)’ is binary on \( S \)

\[
[a \circ (\beta \circ \gamma)](a,b) = \max_{z \in S} \min \{ \alpha(a,z), \beta(z,b) \}
\]

\[
= \max_{z \in S} \min \{ \alpha(a,z), \max_{w \in S} \min \{ \beta(z,w), \gamma(w,b) \} \}
\]

Similarly, we get,

\[
[(\alpha \circ \beta) \circ \gamma](a,b) = \max_{z \in S} \min \{ \alpha(a,z), \beta(z,w), \gamma(w,b) \}
\]

From (1) and (2), \( \circ \) is associative.

By definition \( \odot \) is binary on \( S \)

\[
[\alpha (\odot \beta) (\odot \gamma)](a,d)
\]

\[
= \min_{b \in S} \min \{ \alpha(a,b), \beta \odot \gamma(b,d) \}
\]

\[
= \min_{b \in S} \min \{ \alpha(a,b), \min_{c \in S} \min \{ \beta(b,c), \gamma(c,d) \} \}
\]

Similarly, we get,

\[
[(\alpha \odot \beta) \odot \gamma](a,d)
\]

\[
= \min_{b \in S} \min \{ \alpha(a,b), \beta(b,c), \gamma(c,d) \}
\]

From (2) and (3), \( \alpha (\odot \beta) (\odot \gamma) = (\alpha (\odot \beta) \odot \gamma) \)

Proposition 2.4. The set of all fuzzy binary relations \( \beta_{\mu}(S) \), with respect to the operation

\[
\circ
\]

is a semigroup.
is a semigroup

**Proof:** Result is obvious by proposition 2.3.

**Types of Fuzzy binary relations on a non-empty set X**

**Definition 2.5.** A fuzzy binary relation $R$ defined on a non-empty set $X$ is reflexive if $R(x, x) = 1$ for all $x \in X$. If $R(x, x) = 1$ for some and not for all $x \in X$, then $R$ is fuzzy irreflexive, and when $R(x, x) \neq 1$ for all $x \in X$, then $R$ is said to be anti-fuzzy reflexive. Since $R$ is a fuzzy relation it can take any value between 0 and 1. So we can define different grades of reflexivity, called $\varepsilon$-reflexivity where $0 \leq \varepsilon < 1$, then the fuzzy relation is said to be $\varepsilon$- reflexive. If $R(x, x) = \varepsilon$ for all $x \in X$.

**Definition 2.6.** Fuzzy symmetric: A fuzzy relation $R$ on a non-empty set $X$ is fuzzy symmetric if $R(x, y) = R(y, x)$ for all $x, y \in X$, $R$ is fuzzy asymmetric if $R(x, y) = R(y, x)$ for some and not for all $x, y \in X$, and is fuzzy anti-symmetric if $R(x, y) \neq R(y, x)$ for all $x, y \in X$ where $x \neq y$. Hence $R$ is fuzzy anti-symmetric if $R(x, y) \neq R(y, x)$ implies $x \neq y$.

**Definition 2.7.** Fuzzy transitive: A fuzzy relation defined on a non-empty set $X$ is fuzzy transitive if $R(x, z) \geq \max_{y \in X} \min\{R(x, y), R(y, z)\}$ for all $x, z \in X$ or $R \circ R \leq R$, where $\circ$ denotes the ordinary composition of fuzzy relations. A fuzzy relation is fuzzy non transitive if $R(x, z) < \max_{y \in X} \min\{R(x, y), R(y, z)\}$ for some and not for all $x, z \in X$.

A fuzzy relation defined on a non-empty set $X$ is fuzzy anti-transitive if $R(x, y) < \max_{y \in X} \min\{R(x, y), R(y, z)\}$ for all $x, z \in X$.

**Remark 2.8:** When the operation of fuzzy relation is the cross composition, fuzzy relation $\alpha$ is transitive if and only if $\alpha \circ \alpha \geq \alpha$.

**Definition 2.9.** Similarity relation: A fuzzy binary relation $R$ on a non-empty set $X$ is said to be a similarity relation if it is

1) fuzzy reflexive
2) fuzzy symmetric
3) fuzzy transitive

**Definition 2.10.** Fuzzy partial order relation: A fuzzy binary relation $\omega$ which is fuzzy reflexive, fuzzy anti-symmetric and fuzzy transitive is called a fuzzy partial order relation. In a non-empty set $X$, if $x$ and $y$ are $\omega$ related, we usually write $(x, y) \in \omega$. Here we write $x \leq_{\omega} y$ rather than $(x, y) \in \omega$. If in a non-empty set $X$, $x \leq_{\omega} y$ or
We call the partial order as, total order. Then \((X, \preceq)\) denotes a totally ordered set.

When the greatest lower bound of \(x\) and \(y\) denoted by \(x \wedge y\) exists for every \(x, y \in X\), then \((X, \preceq)\) is called a complete lower fuzzy semilattice.\(^2\)

The totally ordered set \((X, \preceq)\), called a complete upper fuzzy semilattice if the least upperbound of \(x\) and \(y\) denoted by \(x \vee y\) exists for every \(x, y \in X\) \(^2\).

3. FUZZY COMPATIBLE RELATIONS.

Ju Pil Kim, , D.R. Bae, have defined fuzzy compatibility \(^3\). Here the same definition is described more precisely in \(^1\). Using the newly defined fuzzy compatibility \(^1\), we establish some important results and properties.

**Definition 3.1. Fuzzy right compatible.** A fuzzy binary relation \(\alpha\) on a semigroup \(S\) is fuzzy right compatible if \(\alpha(a, b) \leq \alpha(at, b)\) and \(\alpha(a, b) \leq \alpha(a, bt)\) for all \(a, b, t \in S\).

**Example 3.2.** Consider the semigroup \(Z^+\) of all positive integers with respect to the operation ordinary multiplication. Define a fuzzy binary relation defined by

\[
\alpha(a, b) = P[x \leq a, y \leq b], a, b \in Z^+.
\]

That is \(\alpha\) is a probability measure. Clearly \(\alpha\) is always non-negative and its maximum value is 1.

Also \(\alpha(at, b) = P[x \leq at, y \leq b] \text{ where } a, b, t \in Z^+ \geq P[x \leq a, y \leq b] \geq \alpha(a, b)\)

Similarly we get

\[
\alpha(a, bt) \geq \alpha(a, b)
\]

Hence \(\alpha\) is fuzzy right compatible on \(Z^+\).

**Definition 3.3.** (Fuzzy Left Compatible). A fuzzy binary relation \(\alpha\) on a semigroup \(S\) is fuzzy left compatible if \(\alpha(a, b) \leq \alpha(a, tb)\) and \(\alpha(a, b) \leq \alpha(ta, b)\) for all \(a, b, t \in S\).

**Example 3.4** Consider \(f: R \times R \rightarrow [0, \infty]\) defined by

\[
f(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}; x, y \in R.
\]

Here \(R\) is a set of real numbers, which is a semigroup with respect to ordinary multiplication. Define a fuzzy relation \(\alpha: R \times R \rightarrow [0, 1]\) defined by

\[
\alpha(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)dudv
\]

\(f(u, v)\) is a joint probability density function of a bivariate normal distribution.

We have
\( \alpha(a, tb) = \int \int f(x, y) dx \, dy \) where \( a, b, t \in \mathbb{Z}^+ \)

\[ \geq \int \int f(x, y) dx \, dy \]

\[ \geq \alpha(a, b) \]

Again

\( \alpha(ta, b) = \int \int f(x, y) dx \, dy \) where \( a, b, t \in \mathbb{Z}^+ \)

\[ \geq \int \int f(x, y) dx \, dy \]

\[ \geq \alpha(a, b) \]

Hence the above defined fuzzy binary relation is left compatible on \( \mathbb{Z}^+ \).

**Definition 3.5 Fuzzy Compatible:** A fuzzy binary relation \( \alpha \) on a semigroup \( S \) is compatible if it is both fuzzy left compatible and fuzzy right compatible.

**Example 3.6.** Consider the semigroup of positive integers \( \mathbb{Z}^+ \), with respect to the operation of ordinary multiplication. Define a relation \( \alpha \) on \( \mathbb{Z}^+ \) defined by

\[ \alpha(x, y) = \int \int f(u, v) du \, dv \]

Where \( f(u, v) = \frac{1}{2\pi} e^{-\frac{1}{2}(u^2 + v^2)}, u, v \in \mathbb{Z}^+ \)

Clearly \( \alpha \) is both fuzzy left compatible and fuzzy right compatible. Hence \( \alpha \) is compatible.

**Proposition 3.7** If \( \alpha \) is a fuzzy right compatible relation on a semigroup \( S \), then \( \alpha(a, b) \leq \alpha(ac, bd) \) for all \( a, b, c, d \in S \). The converse is true when \( S \) is a monoid.

**Proof:** Given \( \alpha \) is fuzzy right compatible. Then for any \( a, b, c \in S \),

\[ \alpha(a, b) \leq \alpha(ac, b) \] ………………… (1)

Since \( ac, b \in S \) and \( \alpha \) is fuzzy right compatible

\[ \alpha(ac, b) \leq \alpha(ac, bd) \forall a, b, c, d \in S \] ………………… (2)

From (1) and (2), \( \alpha(a, b) \leq \alpha(ac, bd) \) for all \( a, b, c, d \in S \)

Conversely assume \( S \) is a monoid and

\[ \alpha(a, b) \leq \alpha(ac, bd) \] for all \( a, b, c, d \in S \) ………………… (3)
Since $S$ is a monoid, the identity $e \in S$. Therefore from (3),
\[ \alpha(a, b) \leq \alpha(ae, bd) \text{ for all } a, e, b, d \in S \]
That is $\alpha(a, b) \leq \alpha(a, bd) \forall a, b, d \in S$
That is $\alpha(a, b) \leq \alpha(a, bt) \forall a, b, t \in S$
Similarly we get $\alpha(a, b) \leq \alpha(at, b) \forall a, b, t \in S$. Hence $\alpha$ is fuzzy right compatible.

**Proposition 3.8.** If $\alpha$ is fuzzy left compatible relation on a semigroup $S$, then $\alpha(a, b) \leq \alpha(ca, db) \forall a, b, c, d \in S$. The converse is true when $S$ is a monoid.

**Proof:** Result follows from proposition 3.7 by applying the definition of fuzzy left compatibility.

**Proposition 3.9.** $\max\{\alpha(a, b), \alpha(c, d)\} = \alpha(ac, bd)$. Converse is true when $S$ is a monoid.

**Proof:** Given $\alpha$ is fuzzy compatible. That is $\alpha$ is fuzzy left compatible and fuzzy right compatible. By proposition 3.7 and 3.8 we have
\[ \alpha(a, b) \leq \alpha(ac, bd) \forall a, b, c, d \in S \] \hspace{1cm} (4)
\[ \alpha(c, d) \leq \alpha(ac, bd) \forall a, b, c, d \in S \] \hspace{1cm} (5)
From (1) and (2), $\max\{\alpha(a, b), \alpha(c, d)\} \leq \alpha(ac, bd) \forall a, b, c, d \in S$
Conversely assume, $\max\{\alpha(a, b), \alpha(c, d)\} \leq \alpha(ac, bd) \forall a, b, c, d \in S$.
That implies $\alpha(a, b) \leq \alpha(c, d) \forall a, b, c, d \in S$ and $\alpha(c, d) \leq \alpha(ac, bd) \forall a, b, c, d \in S$.
Then by proposition 3.7 and 3.8 (converse part) $\alpha$ is fuzzy right compatible and $\alpha$ is fuzzy left compatible, when $S$ is a monoid. That is $\alpha$ is fuzzy compatible.

**Proposition 3.10.** If $\alpha$ is fuzzy reflexive compatible fuzzy relation on a semi group $S$, then $\alpha(ac, bc) = \alpha(ca, cb) \forall a, b, c \in S$.

**Proof:**
Given $\alpha$ is reflexive and compatible fuzzy relation on $S$. That is
\[ \alpha(x, x) = 1 \forall x \in S \] \hspace{1cm} (6)
And by proposition 3.9
\[ \max\{\alpha(a, b), \alpha(c, d)\} \leq \alpha(ac, bd) \forall a, b, c, d \in S \] \hspace{1cm} (7)
That is $\max\{\alpha(a, b), 1\} \leq \alpha(ac, bc)$. That is $1 \leq \alpha(ac, bc)$.
Since maximum value of a fuzzy relation is one, $\alpha(ac, bc) = 1$.
Similarly we get $\alpha(ca, cb) = 1$. Hence the result.

**Proposition 3.11.** A fuzzy left compatible relation on a semigroup $S$ is fuzzy right
compatible (hence compatible), if S is commutative.

**Proof:** Let $\alpha$ be fuzzy left compatible relation on a commutative semigroup S. then
\[ \alpha(a, b) \leq \alpha(ta, tb) \ orall a, b, t \in S, \text{ .......... (8)} \]
And
\[ \alpha(a, b) \leq \alpha(ta, b) \ orall a, b, t \in S, \text{ .......... (9)} \]
Since S is commutative for $t, a, b \in S; ta = tb \text{ and } tb = bt$.
So (8) and (9) can be written as
\[ \alpha(a, b) \leq \alpha(a, bt) \text{ and } \alpha(a, b) \leq \alpha(at, b) \ orall a, b, t \in S. \]
Hence $\alpha$ is right compatible. That is $\alpha$ is fuzzy left compatible and fuzzy right compatible. So $\alpha$ is compatible.

**Proposition 3.12.** A fuzzy right compatible relation on a semigroup S is fuzzy left compatible (hence compatible) when S is commutative.

**Proof:** Result follows from proposition 3.11 by applying the definition of right and left compatibility.

4. **Fuzzy Partial Congruences**

We are familiar with the definition of fuzzy congruence in groups [2] and inverse semigroup [3]. In this section we introduce a new notion, “fuzzy partial congruence” in a semigroup and arrive the conclusion to describe the set of all fuzzy partial congruences $P_f(S)$ as a complete upper fuzzy semilattice and a complete lower fuzzy semilattice.

**Definition 4.1 (Fuzzy Partial Congruences).** A fuzzy partial ordering compatible relation on a semigroup is called a fuzzy partial congruence. That is a fuzzy binary relation $\alpha$ on a semigroup S is said to be a fuzzy partial congruence if
i. $\alpha$ is fuzzy left compatible.
ii. $\alpha$ is fuzzy right compatible.
iii. $\alpha$ is fuzzy reflexive.
iv. $\alpha$ is fuzzy antisymmetric.
v. $\alpha$ is fuzzy transitive.

**Note 4.2.** $P_f(S)$ denote the set of all fuzzy partial congruence on a semi group S.

**Theorem 4.3.** If $\alpha, \beta \in P_f(S)$ and $\alpha \circ \beta = \beta \circ \alpha$ then $\alpha \circ \beta \in P_f(S)$.

**Proof:** Given $\alpha, \beta \in P_f(S)$. That is $\alpha$ and $\beta$ are fuzzy left and right compatible
partial ordering on $S$.
Now to prove that $\alpha \circ \beta \in P_f(S)$ where $\alpha, \beta \in P_f(S)$, We have to prove that $\alpha$ and $\beta$ are fuzzy reflexive relation.
Then
$$\alpha \circ \beta(x, x) = \max_{y \in S} \min\{\alpha(x, y), \beta(y, x)\} \geq \min\{\alpha(x, x), \beta(x, x)\} \geq \min\{1, 1\} \geq 1 \quad \text{.................................} \quad (10)$$
Since $\alpha \circ \beta$ is a fuzzy relation, $\alpha \circ \beta \leq 1 \quad \text{.................................} \quad (11)$
From (10) and (11), $\alpha \circ \beta(x, x) = 1$. Hence $\alpha \circ \beta$ is reflexive.
Since $\alpha, \beta \in P_f(S)$, both of them are partial ordering and hence anti-fuzzy symmetric. So
For $x \neq y$, $\alpha(x, y) \neq \alpha(y, x)$ and $\beta(x, y) \neq \beta(y, x)$.
Suppose $x \neq y$. We have $\alpha \circ \beta(x, x) = \max_{y \in S} \min\{\alpha(x, y), \beta(y, x)\}$, $y \in X$. Here right hand side becomes maximum for some $z \in S$, which can be in any of the three cases $x \neq z \neq y, x = z \neq y, x \neq z = y$.
Case (1) $x \neq z \neq y$;
$$\alpha \circ \beta(x, y) = \max_{z \in S} \min\{\alpha(x, z), \beta(z, y)\}, y \in X$$
$$\neq \max_{z \in S} \min\{\alpha(x, z), \beta(y, z)\}$$
$$\neq \max_{z \in S} \min\{\beta(y, z), \alpha(z, x)\}$$
$$\neq \beta \circ \alpha(y, x)$$
$$\neq \alpha \circ \beta(y, x) \quad \text{.................................} \quad (12)$$
Case (2) $x = z \neq y$;
$$\alpha \circ \beta(x, y) = \max_{z \in S} \min\{\alpha(x, z), \beta(z, y)\}$$
$$= \max_{z \in S} \min\{\alpha(x, x), \beta(x, y)\}$$
$$= \max_{z \in S} \min\{1, \beta(x, y)\}$$
$$= \beta(y, x) \quad \text{.................................} \quad (13)$$
Again
$$\alpha \circ \beta(x, y) = \beta \circ \alpha(y, x) = \max_{z \in S} \min\{\beta(y, z), \alpha(z, x)\}$$
$$= \max_{z \in S} \min\{\beta(y, x), \alpha(x, x)\}$$
$$= \min\{\beta(y, x), 1\}$$
$$= \beta(y, x) \quad \text{.................................} \quad (14)$$
Since for $x \neq y, \beta(x, y) \neq \beta(y, x)$
From (13) and (14),
\[\alpha \circ \beta(x, y) \neq \alpha \circ \beta(y, x)\]
Case (3) \(x \neq z = y\);
\[\alpha \circ \beta(x, y) = \max_{z \in S \,, x \cdot z = y} \min\{\alpha(x, z), \beta(z, y)\}\]
\[= \max_{z \in S \,, x \cdot z = y} \min\{\beta(x, y), \beta(y, y)\}\]
\[= \min\{\alpha(x, y), 1\}\]
\[= \alpha(x, y)\]………………………………………. (15)
Again
\[\alpha \circ \beta(y, x) = \beta \circ \alpha(y, x) = \max_{z \in S \,, z \cdot x = y} \min\{\beta(y, z), \alpha(z, x)\}\]
\[= \min\{\beta(y, y), \alpha(y, x)\}\]
\[= \min\{1, \alpha(y, x)\}\]
\[= \alpha(y, x)\]…………………………………………………. (16)
For \(x \neq y\) we have \(\alpha(x, y) \neq \alpha(y, x)\) so
Now we can show that \(\alpha \circ \beta\) is transitive. We have
\[(\alpha \circ \beta) \circ (\alpha \circ \beta) = \alpha \circ (\beta \circ \alpha) \circ \beta\]
\[= \alpha \circ (\alpha \circ \beta) \circ \beta\]
\[= (\alpha \circ \alpha) \circ (\beta \circ \beta)\]
\[\leq (\alpha \circ \beta)\]
Hence \((\alpha \circ \beta)\) is transitive.
Moreover \(\alpha \circ \beta \in P_f(S)\) implies \(\alpha\) and \(\beta\) are fuzzy left compatible, and fuzzy right compatible.
\[\alpha \circ \beta(x, y) = \max_{z \in S} \min\{\alpha(x, z), \beta(z, y)\}\]
\[\leq \max_{z \in S} \min\{\alpha(tx, z), \beta(z, y)\} \text{ for } t \in S\]
\[\leq \alpha \circ \beta(tx, y) \text{ for } t \in S\]
Again
\[\alpha \circ \beta(x, y) = \max_{z \in S} \min\{\alpha(x, z), \beta(z, y)\}\]
\[\leq \max_{z \in S} \min\{\alpha(x, z), \beta(z, ty)\} \text{ for } t \in S\]
\[\leq \alpha \circ \beta(x, ty) \text{ for } t \in S\]
So \(\alpha \circ \beta\) is fuzzy left compatible.
Similarly we can show that \(\alpha \circ \beta\) is right compatible.
Hence \(\alpha \circ \beta\) is fuzzy reflexive, anti symmetric, transitive and compatible relation on \(S\). that is \(\alpha \circ \beta\) is a partial congruence on \(S\). that is \(\alpha \circ \beta \in P_f(S)\).

**Theorem 4.4.** If \(\alpha\) and \(\beta\) are fuzzy partial congruence on a semigroup, such that
\(\alpha \circ \beta = \beta \circ \alpha\) then \(\alpha \circ \beta\) is the least upper bound of \(\alpha\) and \(\beta\). That is \(\alpha \circ \beta = \alpha \vee \beta\).

**Proof:** By theorem 4.3 \(\alpha \circ \beta\) is a fuzzy partial congruence. We have

\[
\alpha \circ \beta = \max_{z \in S} \min \{\alpha(a, a), \beta(z, b)\}
\]

\[\geq \max_{z \in S} \min \{\alpha(a, b), \beta(b, b)\}\]

\[\geq \min \{\alpha(a, b), 1\}\text{ since } \beta \text{ is reflexive}\]

\[\geq \alpha(a, b) \text{ .................................................. (17)}\]

Again

\[
\alpha \circ \beta(a, b) = \max_{z \in S} \min \{\alpha(a, a), \beta(z, b)\}
\]

\[\geq \min \{\alpha(a, a), \beta(a, b)\}\]

\[\geq \min \{1, \beta(a, b)\}\]

\[\beta(a, b) \text{ .................................................. (18)}\]

From (17) and (18), \(\alpha \circ \beta\) is an upper bound of \(\alpha\) and \(\beta\).

Assume any fuzzy partial congruence \(\eta\) which is an upper bound of \(\alpha\) and \(\beta\). That is \(\alpha \leq \eta\) and \(\beta \leq \eta\).

\[
\alpha \circ \beta(a, b) = \max_{z \in S} \min \{\alpha(a, a), \beta(z, b)\}
\]

\[\leq \max_{z \in S} \min \{\eta(a, a), \eta(z, b)\}\]

\[\leq \eta \circ \eta(a, b)\]

\[\leq \eta(a, b). \text{ Since } \eta \text{ is a fuzzy partial congruence.}\]

That is \(\alpha \circ \beta\) is the greatest fuzzy partial congruence containing \(\alpha\) and \(\beta\). Hence \(\alpha \circ \beta\) is the least upper bond of \(\alpha\) and \(\beta\).

That is \(\alpha \circ \beta = \alpha \vee \beta\).

**Theorem 4.5.** The set \(P_f(S)\) of all fuzzy partial congruence is a complete upper fuzzy semilattice. If \(\alpha \circ \beta = \beta \circ \alpha\) for \(\alpha, \beta \in P_f(S)\).

**Proof:** Let \(\alpha, \beta \in P_f(S)\) and \(\alpha \circ \beta = \beta \circ \alpha\). Then by theorem 4.4 \(\alpha \circ \beta = \alpha \vee \beta\) and by theorem 4.5, \(\alpha \circ \beta \in P_f(S)\). That is for any \(\alpha, \beta \in P_f(S)\), the least upper bond \(\alpha \circ \beta = \alpha \vee \beta \in P_f(S)\). Hence \((P_f(S), \leq)\) is a partial ordering. More over for any \(\alpha, \beta \in P_f(S)\) implies \(\alpha \vee \beta \in P_f(S)\). So \(P_f(S)\) is a complete upper fuzzy semilattice.

**Lemma 5.4** If \(\alpha, \beta \in P_f(S)\) and \(\alpha \circ \beta = \beta \circ \alpha\) then \(\alpha \circ \beta \in P_f(S)\), when it is reflexive.

**Proof:** Given \(\alpha, \beta \in P_f(S)\), that is \(\alpha\) and \(\beta\) are partial congruence on \(S\). That is
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\[ a \text{ and } b \text{ are fuzzy left and fuzzy right compatible partial ordering on } S. \]

We have \( a \circ b \) is reflexive. Since \( a, b \in P_f(S) \), both \( a \text{ and } b \) anti-fuzzy symmetric. That is

\[
\text{For any } a \neq b \Rightarrow a(a) \neq a(b) \text{ and } b(a) \neq b(b) \quad \text{………………… (20)}
\]

\[ a \circ b(x, y) = \min_{z \in S} \{a(x, z), b(z, y)\}. \]

Here right hand side becomes minimum for some \( z \in S \), which can be in any of the following three cases.

1) \( x \neq z \neq y \)
2) \( x = z \neq y \)
3) \( x \neq z = y \)

Case (1) \( x \neq z \neq y \)
\[ a \circ b(x, y) = \min_{z \in S} \{a(x, z), b(z, y)\}. \]
\[ = \min_{z \in S} \min \{a(z, z), b(y, z)\} \text{ from (20)} \]
\[ = \min \{b(y, z), a(z, x)\} \]
\[ = \min \{b(y, z), a(y, x)\} \]

Case (2) \( x = z \neq y \);
\[ a \circ b(x, y) = \min_{z \in S} \{a(x, z), b(z, y)\} \]
\[ = \min_{z \in S} \min \{a(x, x), b(x, y)\text{ since }x = z \}
\[ = \min_{z \in S} \min \{1, b(x, y)\}
\[ = b(x, y) \text{ ……………………………………………….. (21)}
\]
\[ a \circ b(x, y) = b \circ a(y, x) \]
\[ = \min_{z \in S} \{b(y, z), a(z, x)\} \]
\[ = \min \{b(y, x), a(x, x)\text{ since }z = x \}
\[ = \min \{b(y, x), 1\} \]
\[ = b(y, x) \text{ ………….. } \text{ (22)}
\]

Since for \( b(x, y) \neq b(y, x) \)
From (21) and (22),
\[ a \circ b(x, y) \neq a \circ b(y, x) \]

Case (3) \( x \neq z = y \);
\[ a \circ b(x, y) = \min_{z \in S} \{a(x, z), b(z, y)\}
\[ = \min_{z \in S} \{a(x, y), b(y, y)\text{ since }z = y \}
\[ = \min_{z \in S} \{a(x, y), 1\}
\[ = a(x, y) \text{ …………………….. (23)}
\]

Again \[ a \circ b(y, x) = b \circ a(y, x) \]
\[ = \min_{z \in S} \{b(x, y), a(z, x)\} \]

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From (23) and (24) $\alpha \circ \beta(x, y) \neq \alpha \circ \beta(y, x)$ since $\alpha(y, x) \neq \beta(y, x)$.

Hence in the three cases $\alpha \circ \beta$ is anti-fuzzy symmetric.

Since $\alpha, \beta \in P_f(S)$, both of them are transitive. That is $\alpha \circ \alpha \geq \alpha, \beta \circ \beta \leq \beta$.

Now $(\alpha \circ \beta) \circ (\alpha \circ \beta) = \alpha \circ (\beta \circ \alpha) \circ \beta$$\leq \alpha \circ \beta$.

Hence $\alpha \circ \beta$ is transitive.

Also $\alpha \circ \beta(x, y) = \min_{z \in S} \min \{\alpha(x, z), \beta(z, y)\}$

$\leq \min_{z \in S} \min \{\alpha(x, z), \beta(z, y)\}$

$\leq \alpha \circ \beta(x, y)$.

Similarly we get $\alpha \circ \beta(x, y) \leq \alpha \circ \beta(x, y)$

Hence $\alpha \circ \beta$ is fuzzy left compatible.

That is $\alpha \circ \beta$ is fuzzy reflexive, fuzzy anti-symmetric, fuzzy transitive and fuzzy compatible relations on $S$. That is $\alpha \circ \beta$ is a fuzzy partial congruence on $S$. so $\alpha \circ \beta \in P_f(S)$.

Hence the result.

**Lemma 5.5.** If $\alpha$ and $\beta$ are fuzzy partial congruence on a semi group $S$ such that $\alpha \circ \beta = \beta \circ \alpha$, then $\alpha \circ \beta$ is the greatest lower bound of $\alpha$ and $\beta$.

**Proof:** By lemma 5.4 $\alpha \circ \beta \in P_f(S)$.

Again $\alpha \circ \beta \in (x, y) = \min_{z \in S} \min \{\alpha(x, z), \beta(x, y)\}$

$\leq \min \{\alpha(x, x), \beta(x, y)\}$

$\leq \beta(x, y)$.............................. (25)

Also $\alpha \circ \beta \in (x, y) = \min_{z \in S} \min \{\alpha(x, z), \beta(x, y)\}$

$\leq \min \{\alpha(x, y), \beta(y, y)\}$

$\leq \min \{\alpha(x, y), 1\}$

$\leq \alpha(x, y)$.............................. (26)

From (25) and (26) $\alpha \circ \beta$ is a lower bound of $\alpha$ and $\beta$. Consider any lower bound $\eta \in P_f(S)$ of $\alpha$ and $\beta$. That is $\eta \leq \alpha$ and $\eta \leq \beta$. 

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We have \( \eta(x,y) \leq \eta \circ \eta(x,y) \)
\[ \leq \min_{z \in S} \min \{ \eta(x,z) \eta(z,y) \} \]
\[ \leq \min_{z \in S} \min \{ \alpha(x,z), \beta(z,y) \} \]
\[ \leq \alpha \circ \beta(x,y). \]

Hence any lower bound \( \eta \) is less than or equal to \( \circ \beta \). That is \( \alpha \circ \beta \) is the greatest lower bound of \( \alpha \) and \( \beta \). That is, \( \alpha \circ \beta = \alpha \wedge \beta \).

**Theorem 5.6.** The set \( P_f(S) \) of all fuzzy partial congruence is a complete lower fuzzy semilattice if \( \alpha \circ \beta \) is reflexive and \( \alpha \circ \beta = \beta \circ \alpha \).

**Proof:** By Lemma 5.4 \( \alpha \circ \beta \in P_f(S) \) and by lemma 5.5 \( \alpha \circ \beta \) is the greatest lower bound of \( \alpha, \beta \in P_f(S) \). That is for any \( \alpha, \beta \in P_f(S), \alpha \circ \beta = \alpha \wedge \beta \in P_f(S) \). So \( P_f(S) \) is a complete upper semilattice.

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