THE MIXED TREE DOMINATION POLYNOMIAL OF SOME PATH RELATED GRAPHS

RAFIA YOOSUF
Assistant Professor, Department of Mathematics,
M E S Mampad College, Malappuram, Kerala, India

PREETHI KUTTIPULACKAL
Associate Professor, Department of Mathematics,
University of Calicut, Kerala, India

Abstract
The mixed tree domination polynomial of a connected graph G of order n is the polynomial

\[ P(G, x) = \sum_{i=\gamma_{mt}(G)}^{2n-1} p(i)x^i \]

where \( p(i) \) is the number of mixed tree dominating sets of G of cardinality \( i \) and \( \gamma_{mt}(G) \) is the mixed tree domination number of G. In this paper the mtd- polynomial of some corona graphs of the path \( P_n \) are studied

Keywords:
Domination, Mixed tree domination, domination polynomial, Mixed tree domination polynomial (mtd - polynomial)

Introduction
The domination polynomial of a graph is introduced by Saeid Alikhani and Yee-hock Peng in [5]. Preethi and Raji introduced the concept of mixed tree dominating set in connected graph [2,3,4]. While extending the concept of domination polynomial in view of mixed tree dominating set, we came across with many interesting relations with the coefficients of the polynomial and the graph parameters. Also, the coefficients of the polynomial of some important class of graphs have attractive patterns and the roots of the polynomial have interesting nature. This paper includes the mtd-polynomial of some corona graph of \( P_n \).

As the elements of a mtd-set form a tree, the maximum cardinality of an mtd-set is \( 2n-1 \), and the minimum cardinality is \( \gamma_{mt}(G) \).

Definition 1.
Let \( G = (V, E) \) be a simple graph. For any vertex \( v \in V \), the open neighborhood of \( v \) is the set \( N[v] = \{u \in V : u \neq v \} \), and the closed neighborhood of \( v \) is the set \( N[v] = N[v] \cup \{v\} \). For a set \( S \subseteq V \), the open neighborhood of \( S \) is \( N[S] = \bigcup_{v \in S} N[v] \) and the closed neighborhood of \( S \) is \( N[S] = N(S) \cup S \).

A set \( S \subseteq V \) is a dominating set of \( G \); if \( \gamma \) is the minimum cardinality of a dominating set in \( G \). A dominating set with cardinality \( \gamma \) is called a \( \gamma \) - set. For the basic concepts in graph theory we refer mainly Bondy and Murthy [1] and the concepts in domination theory and mixed tree domination are followed from Preethi [3]. Saeid Alikhani and Yee-hock Peng introduced the concept of domination polynomial of a graph as the polynomial

\[ D(G, x) = \sum_{i=\gamma(G)}^{d} d(G, i)x^i \]

where \( d(G, i) \) denotes the number of dominating sets of cardinality \( i \). A mixed dominating set of \( G \) is a subset \( K \) of \( V \cup E \), such that every element in \( \{V \cup E\} \) is either adjacent or incident to an element of \( K \). By the graph formed by a subset \( A \) of \( V \cup E \), we mean the subgraph whose edge set is \( A \cap E \) and the vertex set consists the vertices in \( A \) together with the ends of the edges in \( A \). A mixed dominating set \( S \subseteq V \cup E \) of a connected graph \( G (V, E) \) is a mixed tree dominating set (mtd - set), if the graph formed by \( S \) is a tree.

The mixed tree domination number \( \gamma_{mt}(G) \) is the minimum cardinality of a mixed tree dominating set in \( G \):

Considering the polynomial idea of Alikhani et.al., we introduced the mtd – polynomial [6] of a connected graph and study the information about the graph that we can obtain from the polynomial. The graphs considered here are all connected and simple of order \( n \).

As the elements of a mtd-set form a tree, the maximum cardinality of an mtd-set is \( 2n-1 \), and the minimum cardinality is \( \gamma_{mt}(G) \).

Example 1.
Consider the path \( P_3 = v_1v_2v_3 \). It has only one \( \gamma_{mt} \) set.
The mixed tree dominating sets are

$\{v_2\}, \{v_1, v_2, v_2, v_3\}, \{v_1, v_2, v_2\}, \{v_1, v_2, v_2, v_3\}, \{v_1, v_2, v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_3, \}$

So that the polynomial is

$x + 3x^2 + 5x^3 + 3x^4 + x^5$

Observations [6]

The following results are immediate consequences of the definition.

Theorem 1.

(1) $x^{min}(G)$ is a factor of $P(G; x)$ i.e.; '0' is a root of multiplicity $y_{min}(G)$

(2) mtd - polynomial of any graph is an odd degree polynomial.

(3) The constant term is zero. The coefficient of $x$ is 1 if and only if $G$ is a star $K_{1,t}$ : $t > 1$.

Theorem 2. [6]

(1) The leading coefficient of $P(G, x)$ is $\tau(G)$ - the number of spanning trees of $G$.

(2) The leading coefficient is 1 i.e. $P(G; x)$ is monic if and only if $G$ is tree.

(3) If $G$ is Hamiltonian, then the leading coefficient is greater than or equal to $n$. But the converse is not true.

Theorem 3. The mixed tree domination polynomial of the graph $P_n \circ K_1$ is

$$2n \sum_{k=0}^{2n} [(nC_0)(0C_G) + (nC_1)(2C_G) + (nC_2)(4C_G) + ... + (nC_n)(2nC_G)] x^{2n-1+k}$$

Proof.

Let $P_n = v_1 v_2 v_3 ... v_n$ and let $u_i$ be the vertex in $P_n \circ K_1$ adjacent to $v_i$, $i = 1, 2, ... n$. Every mtd - set must contain the set consisting of the $n - 1$ edges,

$X = \{v_2 v_3, v_2 v_4, ..., v_n v_1\}$ of $P_n$ --- (1)

And $y_{min}(P_n \circ K_1) = 2n - 1$.

Let us consider the mtd - sets of cardinality $2n - 1$. Following are the possible sets.

(1) $n - 1$ edges and $n$ vertices: There is only one set in this category - the set consisting of all vertices and edges of the $P_n$.

(2) $n$ edges and $n - 1$ vertices: Because of (1), there are $n$ choices for taking $n$ edges i.e., any one of the edge $u_i v_i$ together with the edges in $X$. To dominate the remaining vertices all, the remaining $v_i$'s must be included. Therefore, $nC_1$ mtd-sets in this category.

(3) In general, for an mtd - set with $(n - 1) + r$, $0 \leq r \leq n$ edges and $n - r$ vertices: There are $nC_i$ choices for edges and then the remaining vertices $v_i$'s which are not the ends of the $r$ edges chosen must be included; so that the number of mtd-sets with this specification is $nC_i$.

The total number of mtd - sets of cardinality

$2n - 1 = (nC_0) + (nC_1) + (nC_2) + ... + (nC_n) = 2^n$.

Now consider the mtd - sets of cardinality $2n$.

Following are the possible sets.

Case: $n - 1$ edges and $n + 1$ vertices: Not possible since the elements must form a tree.

Case: $n$ edges and $n$ vertices: For the $n$th edge $nC_1$ choices are there. If we choose the $n$th edge as $u_y v_i$, then the $n$- 1 vertices $\{v_1, ... v_{i-1}, v_{i+1}, ... v_n\}$ must be included in the sets. The $n$th vertex can be one of $u_i$ or $v_i$, so there are $2C_1$ choices for $n$th vertex. So in this case $(nC_i)(2C_1)$ choices for the set.

For an mtd-set with $(n+1) + r$, $1 \leq r \leq n$ edges and $n - r + 1$ vertices:

There are $nC_i$ choices for edges and then the remaining vertices $v$'s which are not the ends of the $r$ edges chosen must be included; so that the number of mtd - sets with this specification is $(nC_r)((2r) C_1)$.

The total number of mtd - sets of cardinality

$2n + 1 = (nC_0)(2C_1) + (nC_1)(4C_1) + (nC_2)(6C_1) + ... + (nC_n)(2nC_1)$.

For the general case, we consider, mtd - sets of cardinality $2n - 1 + i$, $0 \leq i \leq 2n$.

Case: $n - 1$ edges and $n+i$ vertices: Here $i = 0$ is the only possibility and $i = nC_0$ such mtd- sets exist.

Case: $n$ edges and $n+(i-1)$ vertices: $i = 0$; $i = 1$; and $i = 2$ are possible. By (1) only $nC_1$ choices for the $n$th edge.

When $i=0$, only one choice for the $n$-1 vertices; so there are $(nC_i)$ such mtd - sets in this category.

When $i=1$, the $n$th vertex can be selected from the two ends of the $n$th edge selected. So there are $2C_1$ choices for $n$th vertex.

So in this case $(nC_i)(2C_1)$ choices for the set.

When $i = 2$: We must include both the ends of the $n$th edge selected. So in this case $(nC_i)(2C_1)$ choices for the set.

Case: $(n+1)$ edges and $n+(i-2)$ vertices: In this case there are $nC_i$ choices for the additional two edges. Here $i$ can have values $0,1,2,3,4$.

When $i=0$, only one choice for $n$-2 vertices; so there are $nC_2$ such mtd - sets.

When $i=1$, the additional vertex must be the end of one of the two additional edges selected, so that $4C_1$ choices for vertices, and hence $(nC_i)(4C_1)$ choices for the set.

When $i=2$, the additional two vertices must be the ends of the
two additional edges selected, that yields $4C_2$ choices for vertices, and hence $(nC_2)$ $(4C_2)$ mtd- sets. When $i=3$, there are $4C_1$ choices for vertices and so in this case $(nC_2)$ $(4C_2)$ mtd - sets.

When $i=4$, there are $(nC_2)$ $(4C_2)$ mtd-sets. We consider the general case as follows: $(n-1) + k$ edges and $n-i-k$ vertices, $0 \leq k \leq n$. Here $i$ can have values $0, 1, 2, ..., 2k$.

When $i=0$ as $k$ runs from 0 to $n$, we get all the mtd- sets of cardinality $2n-1$, which is $(nC_0) + (nC_1) (2C_0) + (nC_2) (4C_0) + ... + (nC_k)(2C_k) + ... + (nC_n)(2nC_0)$. Similarly, $i=1$ counts the mtd sets of cardinality $2n$, that is $(nC_1)(2C_1) + (nC_2)(4C_1) + ... + (nC_n)(2nC_1)$. From the above argument we conclude that mtd - sets of cardinality $(2n-1) + k$, $0 \leq k \leq 2n$ is given by

$$ (nC_0)((2k) C_0) + (nC_1)((2k) C_1) + ... + (nC_n)((2k) C_n) $$

Where $0C_i = 0$, if $m < k$.

**Observations**

Coefficients of mtd - polynomial of $P_{n,0}K_2$ shows some properties. Some of them are given below.

1. Coefficient of $x^{ymtd}$ is $2^n$
2. Coefficient of $x^{ymtd+1}$ is $(2^n)n$
3. Coefficient of $x^{ymtd-1}$ is $1$
4. Coefficient of $x^{3n-1}$ is $2n$
5. Coefficient of $x^{3n-2}$ is $2n$

**Theorem 4.**

The mixed tree domination polynomial of the graph $P_{n,0}K_2$ is

$$ \sum_{k=0}^{2n} \left\{ (2C_1)^{n-2} (nC_0)((2k) C_0) + (2C_1)^{n-3} (3C_2)((n + 2)C_k) (2)^{-n-2} \right\} \sum_{j=0}^{(n-1) + k} x^{3n-1-k} $$

**Proof.**

Let $P_n = v_1v_2v_3...v_n$ and let $u_i, w_i$, $i = 1, 2, ..., n$ be the vertices in $P_{n,0}K_2$ adjacent to $v_i$. Every mtd - set must contain the $n-1$ edges $v_1v_2, v_2v_3, ..., v_{n-1}v_n$ and one edge from each pair \{\{v_i, u_i, w_i\}\}. Hence the $m_{\text{td}}$- polynomial of $P_{n,0}K_2$ is $3n - 1$.

For convenience the blocks $\{v_i, u_i, w_i\}$, $i = 1, 2, ..., n$ is called the 3-blocks.

Let us consider the mtd - sets of cardinality $3n+i$, $0 \leq i \leq 3n$. The following are the possible sets.

1. **2n-1 edges and n vertices:**
   - For 2n-1 edges we have $(2C_1)^n$ choices, by the above arguments. Let the 2n-1 edges be $v_1v_2, v_2v_3, ..., v_{n-1}v_n, u_1v_1, u_2v_2, ..., u_nv_n, w_1v_1, w_2v_2, ..., w_nv_n$, where $v_i \in \{u_i, w_i\}$. Then for the $n$ vertices, $v_i$ or $w_i$ can be chosen from each block; so that there are $(2C_1)^n 2^n$ mtd - sets in this category.

2. **2n edges and n-1 vertices:**
   - For 2n edges we have $(2C_1)^{n-1}(3C_2)$ choices; because, as we cannot include all the three edges of a block, we must choose the 2n-1 edges as above and the additional edge can be any one of $v_1w_1$ or $u_1w_1$ where $v_1 \in \{u_1, w_1\}$.

3. **In general, for an mtd set with 2n-1+r, 0 \leq r \leq n edges and n-r vertices:**
   - There are $(2C_1)^{n-1}(nC_1)^{r}(n+2r)C_0$ mtd - sets exist.

Case: $i=0$; we consider the mtd - sets of cardinality $3n-1$. Following are the possible sets.

1. **2n-1 edges and n+1 vertices:**
   - As in the above case, there are $(2C_1)^n$ choices for 2n-1 edges, $2^n$ choices for n vertices, which are compulsory and nC1 choices for the additional vertex. So that $(2C_1)^n(nC1)$ mtd - sets in this category.
3n = n + 2n.

the number of mtd sets of cardinality 3n is

\[ k \binom{3n}{r} \]

(2)

\[ \binom{2C}{k} \]

Case: i=k, 0 + 2n) \binom{2C}{k} + \binom{2C}{n-2} \binom{2C}{n-1+k} + \binom{2C}{n-2} \binom{2C}{n-1} + \binom{2C}{n-2} \binom{2C}{n-1} \binom{2C}{n} ((n + 2) C_i).

Now, for n-r+1 vertices, we must choose one end of the edge vi from the n-r 3-blocks from which only one edge is included; that gives 2n \cdot n choices. The remaining vertex has 3r+n-r = n+2r choices; that is (n + 2r) C_i (See the graph given below). So that, the number of mtd -sets in this category is

\[(2)_n \cdot (2C)_n \cdot (2C)_i \cdot (n + 2r) C_i \]

\[ \binom{3n-1}{r} \binom{3n-1+k}{r} \]

\[ k \]

\[ n \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]