Restrained step domination number for some derived and grid related graphs

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Abstract

G. Mahadevan et.al., introduced the concept of restrained step domination number of a graph recently in [5]. A set \(S \subseteq V\) of a graph \(G\) is said to be restrained step dominating set, if \(<S>\) is the restrained dominating set and \(<V-S>\) is a perfect matching. The minimum cardinality taken over all the restrained step dominating sets is called the restrained step domination number of \(G\) and is denoted by \(Y_{rsd}(G)\). In this paper we investigate this restrained step domination number for some standard derived graphs like splitting graph, square graph and grid related graphs like triangular grid.

Keywords: Restricted domination number, restrained step domination number

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Introduction

The concept of complementary perfect domination was introduced by Paulraj Joseph et.al., [4]. A set is called a complementary perfect dominating set if \(S\) is a dominating set of \(G\) and the induced subgraph \(<V-S>\) has a perfect matching. The minimum cardinality taken over all complementary perfect dominating sets is called the complementary perfect domination number and is denoted by \(Y_{cp}(G)\). The concept of restrained domination number was introduced by Gayla. S et.al., [1]. A dominating set is said to be restrained dominating set if every vertex in \(<V-S>\) is adjacent to at least one vertex in \(S\) as well as in \(V-S\). The minimum cardinality taken over all restrained dominating sets in \(G\) is restrained domination number and denoted by \(Y_p(G)\).

Motivated by the above, G. Mahadevan, et. al., [5] introduced the concept of restrained step domination number of a graph. A set \(S \subseteq V\) of a graph \(G\) is said to be restrained step dominating set, if \(<S>\) is the restrained dominating set and \(<V-S>\) is a perfect matching. The minimum cardinality taken over all the restrained step dominating set is called the restrained step domination number of \(G\) and is denoted by \(Y_{rsd}(G)\).

The splitting graph of \(G\), \(S(G)\) is obtained from \(G\) by adding for each vertex \(v\) of \(G\) a new vertex \(v'\) so that \(v'\) is adjacent to every vertex that is adjacent to \(v\). Note that if \(G\) is a \((p,q)\) graph then \(S(G)\) is \((2p,3q)\) graph. The square of a graph \(G\) has \(V(G^2)=V(G)\) with \(u,v\) is adjacent in \(G^2\) whenever \(d(u,v) \leq 2\) in \(G\). The graph triangular grid \(T_p\) is obtained by attaching \(p\) triangles to the vertices of \(T_{p-1}\) name as \(\{a_1^p,a_2^p,...,a_{p-1}^p\}\). In \(T_1\) there is only one triangle. Let the vertices of \(T_1\) be \(\{a_1^1,a_1^2,a_2^1\}\). In \(T_2\) attach 2 triangles to the vertices \(\{a_1^1,a_2^1\}\). Let the vertices of \(T_2\) be \(\{a_1^1,a_2^1,a_3^1,a_4^1,a_5^1\}\) and the vertices of the \(T_3\) be \(\{a_1^1,a_1^2,a_2^1,a_3^2,a_4^2,a_1^3,a_2^3,a_3^3,a_4^3,a_5^3\}\).

Restrained step domination number for splitting graphs of path and cycle

In this section we have found rsd-number for splitting graph of path and cycle along with examples.

Theorem 2.1

For a path \(P_p\), \(Y_{rsd}(S(P_p))\)

\[
\begin{align*}
\frac{6p}{5} & \quad \text{if } p \equiv 0 \pmod{5} \\
\frac{6(p-1)}{5} + 2 & \quad \text{if } p \equiv 1 \pmod{5} \\
\frac{6(p-2)}{5} + 2 & \quad \text{if } p \equiv 2 \pmod{5} \\
\frac{6(p+2)}{5} - 2 & \quad \text{if } p \equiv 3 \pmod{5} \\
\frac{6(p+1)}{5} - 2 & \quad \text{if } p \equiv 4 \pmod{5}.
\end{align*}
\]

Proof

Let \(V(P_p) = \{v_1,v_2,...,v_p\}\). Hence \(V(S(P_p)) = \{v_1,v_2,v_3,...,v_p,v_1',v_2',...,v_p'\}\). Let \(S_1 = \{v_i : i \equiv 2 \pmod{5}\}\), \(S_2 = \{v_i : i \equiv 3 \pmod{5}\}\), \(S_3 = \{v_i : i \equiv 0 \pmod{5}\}\).
$S_4=\{v_i : i \equiv 1 \pmod{5}\}$,  
$S_5=\{v_i : i \equiv 4 \pmod{5}\}$,  
$S_n, \{v_i : i \equiv 0 \pmod{5}\}$

Case i \(p \equiv 0 \pmod{5}\)

$S=S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$ is the restrained step dominating set whose cardinality is $\frac{6p}{5}$.

Case ii \(p \equiv 1 \pmod{5}\)

$S=S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup \{v_p\}$ is the restrained step dominating set whose cardinality is $\frac{6(p-1)}{5}+2$.

Case iii \(p \equiv 2 \pmod{5}\)

$S=S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup \{v_p\}$ is the restrained step dominating set whose cardinality is $\frac{6(p-2)}{5}+2$.

Case iv \(p \equiv 3 \pmod{5}\)

$S=S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup \{v_p\}$ is the restrained step dominating set whose cardinality is $\frac{6(p+1)}{5}+2$.

In all the cases \(S\) is the minimum restrained step dominating set.

Example.

![Figure 2.1](image1)

![Figure 2.2](image2)

![Figure 2.3](image3)

![Figure 2.4](image4)

![Figure 2.5](image5)

Here the darkened vertices are restrained step dominating set. In the figure 2.1, $S=\{v_2, v_3, v_5, v_1', v_4', v_3'\}$ is the restrained step dominating set, $|S|=6$.

Also, \(p \equiv 0 \pmod{5}\) implies $\gamma_{\text{rsd}}(S(P_5))=\frac{6p}{5}$

Hence, $\gamma_{\text{rsd}}(S(P_5))=\frac{6p}{5}=6$.

In the figure 2.2, $S=\{v_2, v_3, v_6, v_1', v_4', v_5', v_6'\}$ is the restrained step dominating set, $|S|=8$.

Also, \(p \equiv 1 \pmod{5}\) implies $\gamma_{\text{rsd}}(S(P_5))=\frac{6(p-1)}{5}+2$

Hence, $\gamma_{\text{rsd}}(S(P_5))=\frac{6(p-1)}{5}+2=8$.

In the figure 2.3, $S=\{v_2, v_3, v_5, v_1', v_4', v_3'\}$ is the restrained step dominating set, $|S|=8$.

Also, \(p \equiv 2 \pmod{5}\) implies $\gamma_{\text{rsd}}(S(P_5))=\frac{6(p-2)}{5}+2$

Hence, $\gamma_{\text{rsd}}(S(P_5))=\frac{6(p-2)}{5}+2=10$.

In the figure 2.4, $S=\{v_2, v_3, v_5, v_2', v_3', v_4', v_5', v_6', v_8\}$ is the restrained step dominating set, $|S|=10$.

Also, \(p \equiv 3 \pmod{5}\) implies $\gamma_{\text{rsd}}(S(P_5))=\frac{6(p+1)}{5}+2$

Hence, $\gamma_{\text{rsd}}(S(P_5))=\frac{6(p+1)}{5}+2=10$.

In the figure 2.5, $S=\{v_2, v_3, v_5, v_2', v_3', v_4', v_5', v_6', v_9\}$ is the restrained step dominating set, $|S|=10$.

Also, \(p \equiv 4 \pmod{5}\) implies $\gamma_{\text{rsd}}(S(P_5))=\frac{6(p+1)}{5}+2$

Hence, $\gamma_{\text{rsd}}(S(P_5))=\frac{6(p+1)}{5}+2=10$.

**Theorem 2.2** For a cycle $C_p$, $\gamma_{\text{rsd}}(S(C_p))$

\[
\begin{align*}
\begin{cases}
\frac{6p}{5} + 2 & \text{if } p \equiv 0 \pmod{5} \\
\frac{6(p-1)}{5} + 2 & \text{if } p \equiv 1 \pmod{5} \\
\frac{6(p-2)}{5} + 4 & \text{if } p \equiv 2 \pmod{5} \\
\frac{6(p+1)}{5} - 2 & \text{if } p \equiv 3 \pmod{5} \\
\frac{6(p+1)}{5} - 2 & \text{if } p \equiv 4 \pmod{5}.
\end{cases}
\end{align*}
\]

**Proof** Let $V(C_p)=\{v_1, v_2, \ldots, v_p\}$. Hence $V(S(C_p))=\{v_1, v_2, v_3, \ldots, v_p, v_1', v_2', \ldots, v_p'\}$.

For

\(p \equiv 0 \pmod{5}, \ p \equiv 1 \pmod{5}, \ p \equiv 3 \pmod{5}\) and \(p \equiv 4 \pmod{5}\) the result follows from the previous theorem 2.1.

For \(p \equiv 2 \pmod{5}\), $S=\{v_i : i \equiv 2 \pmod{5} \ \& \ i \neq p\} \cup \{v_i : i \equiv 3 \pmod{5}\}$ is the restrained step dominating set whose cardinality is $\frac{6(p-2)}{5}+4$.

In all the cases $S$ is the minimum restrained step dominating set.

**Restrained step domination number for square graph of path and cycle**

In this section we have found rsd-number for square graph of...
path and cycle along with some examples.

**Theorem 3.1** For a path, 

\[
Y_{rdsd}(P_p^2) = \begin{cases} 
\frac{p}{2} & \text{if } p \equiv 0 \pmod{4} \\
\frac{p+1}{2} & \text{if } p \equiv 1 \pmod{4} \\
\frac{p-2}{2} & \text{if } p \equiv 2 \pmod{4} \\
\frac{p-1}{2} & \text{if } p \equiv 3 \pmod{4}
\end{cases}
\]

**Proof** Let \(v_1, v_2, \ldots, v_p\) be the set of all vertices such that \(v_1 v_2 \ldots v_p\) is a walk and \(v_1 v_3, v_3 v_5, \ldots, v_{p-1} v_p, v_p v_2, v_4 v_6, \ldots, v_{p-2} v_p \in E(G)\). Let \(S = \{v_i : i \equiv 3 \pmod{4}\} \cup \{v_i : i \equiv 0 \pmod{4}\}\). If \(p \equiv 0 \pmod{4}\), then \(S\) is the restrained step dominating set implies \(Y_{rdsd}(P_p^2) = \frac{p}{2}\). If \(p \equiv 1 \pmod{4}\) then \(SU\{v_p\}\) is the restrained step dominating set implies \(Y_{rdsd}(P_p^2) = \frac{p+1}{2} + 1\). If \(p \equiv 2 \pmod{4}\) then \(S\) is the restrained step dominating set implies \(Y_{rdsd}(P_p^2) = \frac{p-2}{2}\). If \(p \equiv 3 \pmod{4}\), then \(S\) is the restrained step dominating set implies \(Y_{rdsd}(P_p^2) = \frac{p-1}{2}\). In all the cases \(S\) is the minimum restrained step dominating set.

**Example**

![Figure 3.1](image1)

![Figure 3.2](image2)

![Figure 3.3](image3)

![Figure 3.4](image4)

**Illustration**

Here the darkened vertices are restrained step dominating set.

In figure 3.1, \(S = \{v_3, v_4, v_7, v_8\}\) is the restrained step dominating set, \(|S| = 4\),

\[Y_{rdsd}(P_8^2) = \frac{8}{2} = 4\]

In figure 3.2, \(S = \{v_3, v_4, v_7, v_8\}\) is the restrained step dominating set, \(|S| = 5\),

\[Y_{rdsd}(P_8^2) = \frac{9}{2} + 1, \text{if } p \equiv 1 \pmod{4} \text{ hence } Y_{rdsd}(P_9^2) = \frac{9}{2} + 1 = 5\]

In figure 3.3, \(S = \{v_3, v_4, v_7, v_8\}\) is the restrained step dominating set, \(|S| = 4\),

\[Y_{rdsd}(P_8^2) = \frac{8}{2} - 2 = 4\]

In figure 3.4, \(S = \{v_3, v_4, v_7, v_8, v_9\}\) is the restrained step dominating set, \(|S| = 5\),

\[Y_{rdsd}(P_8^2) = \frac{9}{2} - 3 = 3\]

**Theorem 3.2** For a Cycle,

\[
Y_{rdsd}(C_p^2) = \begin{cases} 
\frac{p}{2} & \text{if } p \equiv 0 \pmod{4} \\
\frac{p+1}{2} & \text{if } p \equiv 1 \pmod{4} \\
\frac{p-2}{2} & \text{if } p \equiv 2 \pmod{4} \\
\frac{p-1}{2} & \text{if } p \equiv 3 \pmod{4}
\end{cases}
\]

**Proof** Let \(v_1, v_2, \ldots, v_p\) be the set of all vertices such that \(v_1 v_2 \ldots v_p\) is a cycle and \(v_1 v_3, v_3 v_5, \ldots, v_{p-1} v_p, v_p v_2, v_4 v_6, \ldots, v_{p-2} v_p \in E(G)\). Let \(S = \{v_i : i \equiv 3 \pmod{4}\} \cup \{v_i : i \equiv 0 \pmod{4}\}\). If \(p \equiv 0 \pmod{4}\), then \(S\) is the restrained step dominating set implies \(Y_{rdsd}(C_p^2) = \frac{p}{2}\). If \(p \equiv 1 \pmod{4}\), then \(SU\{v_p\}\) is the restrained step dominating set implies \(Y_{rdsd}(C_p^2) = \frac{p+1}{2} + 1\). If \(p \equiv 2 \pmod{4}\), then \(SU\{v_{p-1}\}\) is the restrained step dominating set implies \(Y_{rdsd}(C_p^2) = \frac{p-2}{2} + 1\). If \(p \equiv 3 \pmod{4}\), then \(S\cup\{v_{p-2}, v_{p-1}\}\) is the restrained step dominating set implies \(Y_{rdsd}(C_p^2) = \frac{p-1}{2} + 2\). In all the cases \(S\) is the minimum restrained step dominating set.

4 Restrained step domination number for triangular grid graph.

In this section we have found rdsd- number of triangular grid graph when the grid is odd and even.

**Theorem 4.1** If \(p\) is odd, then \(Y_{rdsd}(T_p) = \\
\begin{cases} 
\frac{3}{2} & \text{if } p \equiv 0 \pmod{3} \\
\frac{p}{2} & \text{if } p \equiv 1 \pmod{3} \\
\frac{p+1}{2} & \text{if } p \equiv 2 \pmod{3}
\end{cases}
\]

\[
\left\{ \begin{array}{l}
\sum_{k=0}^{\frac{p}{3}} (3k+1) + \sum_{k=0}^{\frac{p}{3}} (6k+3) + \sum_{k=0}^{\frac{p}{3}} (6k+5) - 2 & \text{if } p \equiv 0 \pmod{3} \\
\sum_{k=0}^{\frac{p-1}{3}} (3k+1) + \sum_{k=0}^{\frac{p-1}{3}} (6k+3) + \sum_{k=0}^{\frac{p-1}{3}} (6k+5) + \sum_{k=0}^{\frac{p}{3}} \frac{p}{\frac{p}{3}} & \text{if } p \equiv 1 \pmod{3} \\
\sum_{k=0}^{\frac{p-2}{3}} (3k+1) + \sum_{k=0}^{\frac{p-2}{3}} (6k+3) + \sum_{k=0}^{\frac{p-2}{3}} (6k+5) & \text{if } p \equiv 2 \pmod{3}
\end{array} \right.
\]
Proof Let \( \{v_{10}, v_{11}, v_{21}, v_{12}, v_{22}, \ldots, v_{1p}, v_{2p}, \ldots, v_{p+1p-1}\} \) be the set of all vertices of the triangular grid graph. Let \( \{v_{\cdot j}^p, v_{\cdot j-1}^p\} \) be a triangle thus the obtained graph is called as triangular grid graph.

Let \( S = \{v_{\cdot j}^p, v_{\cdot j-1}^p\} \) if \( p \equiv 0 \pmod{3} \)
\[
\begin{cases}
S' = \{v_{\cdot j}^p, v_{\cdot j-1}^p\} & \text{if } p \equiv 0 \pmod{3} \\
S' \cup \{v_{i+1j}^p\} & \text{if } p \equiv 1 \pmod{3} \\
S' \cup \{v_{i+1j}^p\} & \text{if } p \equiv 2 \pmod{3}
\end{cases}
\]

Then the restrained step dominating set is
\[
S = S' \cup \{v_{i+1j}^p\} : t \equiv 0 \pmod{3}
\]

Hence \( |S| = \frac{p-3}{3} + \frac{p-3}{3} + \frac{p-3}{3} \)

Therefore \( \gamma_{rsd}(T_p) \leq |S| \)

Now, we have to prove \( \gamma_{rsd}(T_p) \geq |S| \)

Example Consider the triangular grid graph \( T_0 \)

Illustration Here the darkened vertices are restrained step dominating set.

For \( p \equiv 0 \pmod{3} \),
\[
\gamma_{rsd}(T_p) = \left( \prod_{k=0}^{\frac{p-3}{3}} (3k + 1) + \left( \prod_{k=0}^{\frac{p-3}{3}} (6k + 3) + \left( \prod_{k=0}^{\frac{p-3}{3}} (6k + 5) \right) \right) - 2 \right)
\]

Hence \( \gamma_{rsd}(T_p) = (1 + 4 + 7 + 10) + (3 + 9) + (5) - 2 = 37 \)

Example Consider the triangular grid graph \( T_7 \)

Illustration Here the darkened vertices are restrained step dominating set.

For \( p \equiv 1 \pmod{3} \),
\[
\gamma_{rsd}(T_p) = \left( \prod_{k=0}^{\frac{p-3}{3}} (3k + 1) + \left( \prod_{k=0}^{\frac{p-3}{3}} (6k + 3) + \left( \prod_{k=0}^{\frac{p-3}{3}} (6k + 5) \right) \right) \right)
\]

Hence \( \gamma_{rsd}(T_p) = (1 + 4 + 4) + (3) + 5 + 2 = 22 \)

Theorem 4.2 If \( p \) is even, then \( \gamma_{rsd}(T_p) = \left( \prod_{k=0}^{\frac{p-3}{3}} (3k + 1) + \left( \prod_{k=0}^{\frac{p-3}{3}} (6k + 3) + \left( \prod_{k=0}^{\frac{p-3}{3}} (6k + 5) \right) \right) \right) \)

Proof Let \( \{v_{i0}, v_{i1}, v_{i2}, v_{i3}, v_{i4}, \ldots, v_{ip}, v_{ip}, \ldots, v_{i+1p-1}\} \) be the set of all vertices of the triangular grid graph. Let \( \{v_{\cdot j}^i, v_{\cdot j-1}^i\} \) be a triangle thus the obtained graph is called as triangular grid graph.

Let \( S' = \{v_{i+1j}^i\} \)

For \( p \equiv 0 \pmod{3} \),
\[
|S| = \frac{p-3}{3} + \frac{p-3}{3} + \frac{p-3}{3}
\]

Therefore \( \gamma_{rsd}(T_p) \leq |S| \)
Then the restrained step dominating set =
\[ S' = \left\{ v_p^2 + v_{p-1}^1 \mid p \equiv 0 \pmod{3} \right\} \]
\[ S' \cup \left\{ v_i^1 \mid i \equiv 0 \pmod{3} \right\} \]
Hence \(|S'| = \frac{p}{3} + \frac{p}{2} \]
\[ \left\{ \left( \sum_{k=0}^{p-1} (3k+1) \right) + \sum_{k=0}^{p-1} (6k+3) + \sum_{k=0}^{p-1} (6k+5) \right\} \]
Implies that \(|S'| = \frac{p}{3} + \frac{p}{2} \]
Therefore \(\gamma_{rstd}(T_p) \leq |S|\)
If there exists a rstd- dominating set \(T\), such that \(T \subseteq S, V-T\) does not have independent edges, which contradicts the definition. Implies \(\gamma_{rstd}(T_p) \geq |S|,\) Hence \(\gamma_{rstd}(T_p) = |S|\)

Conclusion
In this paper we have found the restrained step domination number for some derived graph such as square graph and splitting graph. Also we have found the restrained step domination number for triangular grid graph.

References