Frobenius Algebras and Comultiplication

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Abstract
In this article the characterize Frobenius algebras $A$ as algebras having a comultiplication which is a map of $A$-modules. This characterization allows a simple demonstration of the compatibility of Frobenius algebra structure with direct multiplication. The relationship between two dimensional topological quantum field theories and Frobenius algebras is then formulated as an equivalence of categories. Let $k$ be a field. All vector spaces will be $k$-vector spaces and all algebras will be $k$-algebras; all tensor products are over $k$. When we write $\text{Hom}(V,V')$ it is always mean the space of $k$-linear maps.

1. Introduction
A Frobenius algebra is a finite-dimensional algebra equipped with a nondegenerate bilinear form compatible with the multiplication. Examples are matrix rings, group rings, the ring of characters of a representation and artinian Gorenstein rings. In algebra and representation theory such algebras have been studied for a century. In order to relate this to Frobenius algebras. The principal result of cobordisms and Topological quantum field theories establishes an alternative characterization of Frobenius algebras which goes back at least to Lawvere namely as algebras with multiplication denoted which are simultaneously coalgebras with this compatibility condition between subject to a certain compatibility condition is exactly the right-hand relation drawn. In fact, the basic relations valid in $2\text{Cob}$ correspond precisely to the axioms of a commutative Frobenius algebra Lowell Abrams. This comparison leads to the main theorem. There is an equivalence of categories $2\text{TQFT}_{\mathbb{K}} \simeq A_{\mathbb{K}}$, given by sending a $\text{TQFT}$ to its value on the circle.

A Frobenius algebra is vector space that is both an algebra and a coalgebra in a compatible way. This sort of compatibility is different from that involved in a bialgebra and Hopf algebra. More generally, Frobenius algebras can be defined in any monoidal category and even in any polycategory in which case they are sometimes called Frobenius monoids Stephen Sawin. Frobenius structure. During the past decade Frobenius algebras have shown up in a variety of topological contexts, in theoretical physics and in computer science. In physics, the main scenery for Frobenius algebras is that of topological quantum field theory Edward Witten, which in its axiomatisation amounts to a precise mathematical theory. In computer science, Frobenius algebras arise in the study of flowcharts, proof nets circuit diagrams. In any case, the reason Frobenius algebras show up is that it is essentially a topological structure: it turns out the axioms for a Frobenius algebra can be given completely in terms of graphs or as we shall do, in terms of topological surfaces Lowell Abrams.

### Notations:

<table>
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<tr>
<th>Principal</th>
<th>2D cobordisms</th>
<th>Algebraic operation (in a $k$-algebra $A$)</th>
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<td>$\mu$</td>
<td>$\otimes$ $A \rightarrow A$</td>
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<td>Creation</td>
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2. Algebras, Modules, and pairings:
2.1 Definition and basic properties of Frobenius algebras:
2.1.1 Definition: k-algebras
A $k$-algebra is a $k$-vector space $A$ together with two $k$-linear maps $\mu: A \otimes A \rightarrow A$, $\eta: k \rightarrow A$ Satisfying the Associativity law and the unit law

\[
(\mu \otimes \text{id}_A) \mu = (\mu \otimes \text{id}_A) \mu = (\text{id}_A \otimes \mu) \mu,
\]

In other words, a $k$-algebra is precisely a monoid in the monoidal category $(\text{Vect}_k, \otimes, k)$.

2.1.1.1 Definition: Frobenius algebra
A Frobenius algebra is a $k$-algebra $A$ of finite dimension, equipped with a linear functional $\varepsilon: A \rightarrow k$ whose nullspace contains no nontrivial left ideals. The functional $\varepsilon \in A^*$ is called a Frobenius form.
2.1.1.2 Definition
A quickly leads to a couple of other characterizations. A nondegenerate pairing \( A \otimes A \rightarrow k \), induces two isomorphisms \( A \cong A^* \), they do not in general coincide.

2.1.1.3 Definition:
Frobenius algebra is a finite dimensional \( k \)-algebra \( A \) equipped with a left \( A \)-isomorphism to its dual. Alternatively \( A \) is equipped with a right \( A \)-isomorphism to its dual.

2.1.2 Definition: (Alternative)
A Frobenius algebra is a \( k \)-algebra \( A \) of finite dimension, equipped with an associative nondegenerate pairing: \( A \otimes A \rightarrow k \). We call this pairing the Frobenius pairing.

2.1.6 Functionals and associative pairings on \( A \):
Every linear functional \( \epsilon: A \rightarrow k \) determines canonically a pairing \( A \otimes A \rightarrow k \), namely \( x \otimes y \mapsto (xy) \epsilon \). clearly this pairing is associative. Conversely, given an associative pairing \( A \otimes A \rightarrow k \), denoted \( x \otimes y \mapsto (\frac{x}{y}) \), a linear functional is canonically determined, namely \( A \rightarrow k, a \mapsto \langle 1_A \rangle \). This gives a one-to-one correspondence between linear functional on \( A \) and associative pairings.

2.2 Graphical language:
The maps above are all linear maps between the tensor powers of \( A \)- note that \( k \) is naturally the 0’th tensor power of \( A \). We introduce the following graphical notation for these maps,

\[
\eta \quad \mu \quad i d_A
\]

These symbols are meant to have status of formal mathematical symbols, just like the symbols \( \rightarrow \) or \( \otimes \). The symbol corresponding to each \( k \)-linear map \( \phi: A^{\otimes m} \rightarrow A^{\otimes n} \) has \( m \) boundaries on the left, one for each factor of \( a \) in the source, and ordered such that the first factor in the tensor product corresponds to the bottom input hole and the last factor corresponds to the top input hole. If \( m=0 \) we simply draw no in boundary. Similarly there are \( n \) boundaries on the right which correspond to the target \( A^{\otimes n} \), with the same convention for the ordering.

The tensor product of two maps is drawn as the union of the two symbols mimics \( \sqcup \) as monoidal operator in the category \( 2 \text{Cob} \). Composition of maps is pictured by joining the output holes of the first figure with the input holes of the second.

Now we can write down the axioms for algebra like this,

\[
\text{Associativity}
\]

\[
\text{Unit axiom}
\]

The canonical twist map \( A \otimes A \rightarrow A \otimes A, v \otimes w \mapsto w \otimes v \) is depicted , so the property of being a commutative algebra reads

\[
\text{Comm. Algebra}
\]

2.3 Pairings of vector spaces:
A bilinear pairing – or just a pairing – of two vector spaces \( V \) and \( W \) is by definition a linear map \( \beta: V \otimes W \rightarrow k \). By the universal property of the tensor product, giving a pairing \( V \otimes W \rightarrow k \) is equivalent to giving a bilinear map \( V \times W \rightarrow k \). For this reason we will allow ourselves to write like \( \beta: V \otimes W \rightarrow k, v \otimes w \mapsto (v/w) \).

2.3.1 Lemma:
Given a pairing \( \beta: V \otimes W \rightarrow k, v \otimes w \mapsto (v/w) \), between finite-dimensional vector spaces, the following are equivalent.

(i) \( \beta \) is nondegenerate.
(ii) The induced linear map \( W \rightarrow V^* \) is an isomorphism.
(iii) The induced linear map \( V \rightarrow W^* \) is an isomorphism.

If we already know for other reasons that \( V \) and \( W \) are of the same dimension, then non-degeneracy can also be characterized by each of the following a priori weaker conditions:

(ii') \( (v/w) = 0 \) \( \forall v \in V \Rightarrow w = 0 \)

(iii') \( (v/w) = 0 \) \( \forall w \in W \Rightarrow v = 0 \)

2.4 Pairings of \( A \)-modules:
Suppose now $M$ is a right $A$-module and let $N$ be a left $A$-module. A pairing $\beta: M \otimes N \rightarrow k$, $x \otimes y \mapsto \langle x/y \rangle$ is said to be associative when $\langle xa/y \rangle = \langle x/ay \rangle$ for every $x \in M, a \in a, y \in N$.

**2.4.1 Lemma:**

For a pairing $M \otimes A \rightarrow k$ as above, the following are equivalent:

(i) $M \otimes N \rightarrow k$ is associative.

(ii) $N \rightarrow M^*$ is left $A$-linear.

(iii) $M \rightarrow N^*$ is right $A$-linear.

**2.5 Graphical expression of the Frobenius structure:**

According to our principles we draw like this,

\[ \begin{array}{c}
e \\
\text{Frobenius form} \\
\beta \\
\text{Frobenius pairing}
\end{array} \]

The relations $\langle x/y \rangle = (xy)e$ and $(1_N/x) = xe
\sum_i a_{ii}$ 2.1.6 then get this graphical expression.

\[ \begin{array}{c}
e \\
\text{Frobenius form} \\
\beta \\
\text{Frobenius pairing}
\end{array} \]

It is trickier to express the axioms which $\varepsilon$ and $\beta$ must satisfy in order to be a Frobenius form and a Frobenius pairing, respectively. The axiom for a Frobenius form $\varepsilon: A \rightarrow k$, is not expressible in our graphical language because we have no way to represent an ideal. In contrast, it is easy to write down the two axioms for the Frobenius pairing. The associativity condition reads

\[ \begin{array}{c}
e \\
\text{Frobenius form} \\
\beta \\
\text{Frobenius pairing}
\end{array} \]

And the nondegeneracy condition is this,

\[ \begin{array}{c}
e \\
\text{Frobenius form} \\
\beta \\
\text{Frobenius pairing}
\end{array} \]

There exists $\delta$ such that

\[ \begin{array}{c}
e \\
\text{Frobenius form} \\
\beta \\
\text{Frobenius pairing}
\end{array} \]

This is really the crucial property – we will henceforth refer to this as the snake relation.

**2.5.1 Examples**

In each example, $A$ is assumed to be a $k$-algebra of finite dimension over $k$.

**2.5.1.2 Algebraic field extensions:**

Let $A$ be a finite field extension of $k$. Since fields have no nontrivial ideals, any nonzero $k$-linear map $A \rightarrow k$ will do as Frobenius form.

**2.5.1.3 Division rings:**

Let $A$ be a division ring. Since just like a field a division ring has no nontrivial left ideals, any nonzero linear from $A \rightarrow k$ will make $A$ into a Frobenius algebra over $k$.

**2.5.1.4 Matrix rings:**

The ring $\text{Mat}_n(k)$ of all $n$-by-$n$ matrices over $k$ is a Frobenius algebra with the usual trace map as Frobenius form.

\[ \text{Tr}: \text{Mat}_n (k) \rightarrow k, \quad (a_{ij}) \mapsto \sum_i a_{ii} . \]

**2.5.1.5 Group algebras:**

Let $G = \{t_0, \ldots, \ldots, t_n\}$ be a finite group written multiplicatively, and with $t_0 = 1$. The group algebra $kG$ is defined as the set of formal linear combinations $\sum_i c_i t_i$ with multiplication given by the multiplication in $G$. It can be made into a Frobenius algebra by taking the Frobenius form to be the functional $\varepsilon: kG \rightarrow k$, $t_0 \mapsto 1, t_i \mapsto 0$ for $i \neq 0$. Indeed, the corresponding pairing $g \otimes h \mapsto (gh)e$ is nondegenerate since $g \otimes h \mapsto 1$ if and only if $h = g^{-1}$.

**2.5.1.6 The ring of group characters:**

Assume the ground field is $k = \mathbb{C}$. Let $G$ be a finite group of order $n$. A class function on $G$ is function $G \rightarrow \mathbb{C}$ which is constant on each conjugacy class; the class functions from a ring denoted $R(G)$. In particular, the characters are class functions, and in fact a very class function is a linear combination of characters, There is a bilinear pairing on $R(G)$ defined by

\[ \langle \phi/\psi \rangle = \frac{1}{n} \sum_{t \in G} \phi(t) \psi(t^{-1}) . \]

Now the orthogonality relations, state that the characters form an orthonormal basis of $R(G)$ with respect to this bilinear pairing, so in particular the pairing is
nondegenerate and provides a Frobenius algebra structure on $R \otimes (G)$.

2.6 Frobenius algebras and comultiplication:

2.6.1 Coalgebras:

Recall that a coalgebra over $k$ is a vector space $A$ together with two $k$-linear maps

\[ \delta: A \to A \otimes A, \quad \varepsilon: A \to k \]

Satisfying axioms dual to the algebra axioms (2.1.1). The map $\delta$ is called comultiplication and $\varepsilon: A \to k$ is called the counit.

Our goal is to show that a Frobenius algebra $(A, \varepsilon)$ has a natural coalgebra structure for which $\varepsilon$ is the counit. The idea is to use the copairing to turn around an input hole of the multiplication.

2.6.2 Comultiplication:

Define a comultiplication map $\delta: A \to A \otimes A$ by

\[ \gamma = \delta = \delta \]

Here and in the sequel we suppress identity maps. What is meant is actually

\[ \gamma = \delta = \delta \]

That the two expressions in the definition agree will follow from the next lemma. First some notation

2.6.3 The three-point function:

$\phi: A \otimes A \otimes A \to k$ is defined by $\phi := (\mu \otimes id_A) \beta = (id_A \otimes \mu) \beta,$

\[ \phi = \phi = \phi \]

Associativity of $\beta$ says that the two expressions coincide. Conversely, using the snake relation we can express in terms of the three-point function:

2.6.3.1 Lemma:

We have

\[ \gamma = \delta = \delta \]

Proof:

\[ \gamma = \delta = \delta \]

Here the first step was to use the definition of the three-point function, then remove some identity maps and insert one; next, apply the snake relation; and finally remove an identity map.

Now it is clear that the two expression of the definition of $\gamma$ agree: using this lemma they are both seen to be equal to

2.7 Multiplication in terms of comultiplication:
Conversely, turning some holes back again, using \( \beta \), and then using the snake relation, we also get the relations dual to 2.6.2

\[ \begin{align*}
\text{2.7.1 Lemma:} \\
\text{The Frobenius form } \varepsilon \text{ is counit for } \delta: \\
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{image1.png}
\end{array}
\end{align*} \]

\textbf{Proof}: Suppressing the identity maps, write

\[ \begin{align*}
\text{Here the first step was to use the expression section 2.5 for } \includegraphics[width=0.2\textwidth]{image2.png}. \text{The next step was to use relation section 2.7. Finally we used that } \includegraphics[width=0.2\textwidth]{image3.png} \text{ is neutral element for the multiplication.}
\end{align*} \]

\textbf{2.7.2 Lemma:}

The comultiplication satisfies the following compatibility condition with respect to the multiplication, called the Frobenius relation.

\[ \begin{align*}
\text{The left-hand equation expresses left A-linearity of } \delta; \text{ the right-hand equation expresses right A-linearity.} \\
\textbf{Proof}: \text{For the right hand equation, use } \includegraphics[width=0.2\textwidth]{image4.png} \text{; then use Associativity, and finally use the relation back again}
\end{align*} \]

\[ \begin{align*}
\text{2.7.3 Lemma:} \\
\text{The comultiplication is coassociative:} \\
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{image5.png}
\end{array}
\end{align*} \]

\textbf{Proof}: Use the definition of } \delta \text{; then the Associativity and finally the definition again.

\[ \begin{align*}
\text{2.7.4 Proposition:} \\
\text{Given a Frobenius algebra } (A, \varepsilon) \text{ there exists a unique comultiplication } \delta \text{ whose counit is } \varepsilon \text{ and which satisfies the Frobenius relation and this comultiplication is coassociative.}
\end{align*} \]
Proof: We have already constructed such a comultiplication and established its coassociativeity. The uniqueness is a consequence of the fact that the copairing corresponding to a nondegenerate pairing is unique. In detail suppose that \( \omega \) is another comultiplication with counit \( \varepsilon \) and which satisfies the Frobenius relation. Putting caps on the upper input hole and the lower output hole of the Frobenius relation we see that \( \eta \omega \) satisfies the snake equation.

\[
\begin{align*}
\eta \omega &= \omega \\
\end{align*}
\]

By the unit and counit axioms. So by the uniqueness of copairing we have \( \eta \omega = \gamma \). Using this, if instead we put only the cap \( \eta \) on, then we get

\[
\begin{align*}
\omega &= \\
\end{align*}
\]

That is \( \omega \) is nothing but \( \mu \) with an input hole turned around, just like \( \delta \) was defined.

Conclusion

Hence the focus on topological quantum field theories and in particular on dimension 2. This is by far the best picture of the Frobenius structures since the topology is explicit and since there is no additional structure to complicate things. In fact, the main theorem of above notes states that there is an equivalence of categories between that two dimensional Topological quantum field theories and that of commutative Frobenius algebras.

References


