On Discrete Distributions Generated through Mittag-Leffler Function and their Properties

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Abstract—The Poisson distribution is a popular model for count data. However, its use is restricted by the equality of its mean and variance (equi-dispersion). Many models with the ability to represent under, equi and over dispersed data have been proposed in the literature to overcome this restriction. One of this is the hyper Poisson(HP) distribution of Bardwell and Crow(1964). Chakraborty and Ong(2017) introduced a generalization of HP distribution using Mittag-Leffler function. The hyper Poisson, displaced Poisson, Poisson and geometric distribution are particular cases of this distribution. This Mittag-Leffler function distribution(MLFD) belongs to the generalized hyper geometric and generalized power series family and also arises as weighted Poisson distributions. MLFD is a flexible distribution with varying shape and can be modeled for under, equi or over dispersed data. Chakraborty and Ong(2017) studied various distributional properties of MLFD such as recurrence relations for pmf, generating functions, formulae for different types of moments, reliability properties and stochastic ordering. They showed that the MLFD is suitable in empirical modeling of differently dispersed count data when compared with hyper Poisson distribution. Pillai(1990) introduced the Mittag-Leffler distribution in terms of the Mittag-Leffler function. The Mittag-Leffler function was introduced by Swedish Mathematician Gosta Mittag-Leffler in 1903 (Mittag-Leffler, 1903) in connection with his method of summations of some divergent series. The function $E_\alpha(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(1+\alpha k)}$, $u \in (0, \infty)$ is known as Mittag-Leffler function. It arises as the solution of certain boundary value problems involving fractional differential equation. During various developments of fractional calculus in the last two decades, this function has gained importance and popularity on account of its vast applications in the field of Science and Technology (see, Mathai (2010) and Pillai(1990)).

Feller(1971) showed that the Laplace transform of $E_\alpha(-x^\alpha)$ for $0 \leq \alpha \leq 1$ is $\frac{\lambda^{\alpha-1}}{1+\lambda^\alpha}$, $\lambda \geq 0$. But $E_\alpha(-x^\alpha)$ is not a probability distribution. Pillai(1990) showed that $F_\alpha(x) = 1- E_\alpha(-x^\alpha)$ is a distribution function. Hence he named $F_\alpha(x)$ as Mittag-Leffler distribution. We have $F_\alpha(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{\alpha k}}{\Gamma(1+\alpha k)}$, $0 < \alpha \leq 1$, $x \geq 0$ and the corresponding density function is $f_\alpha(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{\alpha k-1}}{\Gamma(\alpha k)}$. Huillet(2016) discussed various properties of Mittag-Leffler distribution and related stochastic processes. Jayakumar and Pillai(1996) characterized Mittag-Leffler distribution using spectrum function corresponding to an infinitely divisible distribution.
Pillai and Jayakumar(1995) proposed a class of discrete Mittag-Leffler distribution(DML) having probability generating function(pgf) \( P(z)=\frac{1}{1+(c(1-z)^{\alpha})} \), as a discrete analogue of the Mittag-Leffler distribution. Chakraborty and Ong(2017) introduced a generalization of hyper Poisson distribution using Mittag-Leffler function called Mittag-Leffler function distribution.

This paper is organised as follows. Section 2 deals with discrete Mittag-Leffler distribution –its mathematical origin and some properties are discussed. In section 3, we study Mittag-Leffler function distribution. The applications of Mittag-Leffler distribution in various areas are discussed in section 4.

II. DISCRETE MITTAG-LEFFLER DISTRIBUTION

A discrete version of the Mittag–Leffler distribution was introduced by Pillai and Jayakumar(1995). The discrete Mittag-Leffler (DML) distribution is a generalisation of geometric distribution.

A.Mathematical origin of DML distribution

Consider a sequence of independent Bernoulli trials in which the kth trial has probability of success \( \alpha/k \), \( 0 < \alpha<1 \), k=1,2,3… . Let N be the trial number in which the first success occurs.

Then Probability that \( N=r \) is given by

\[
P_r = (1- \alpha)(1- \alpha/2)\ldots (1- \alpha/(r-1)) \alpha /r = (-1)^{r-1} \alpha(\alpha-1)(\alpha-2)\ldots (\alpha-r+1)/r!.
\]

The probability generating function(pgf) of \( N \) is given by \( G(z)=1-(1-z)^{\alpha} \).

Let \( X_1, X_2, X_3, \ldots, X_n \) be independent and identically distributed as \( N \).

Let \( M \) be geometric with parameter \( p \). That is, \( P(M=k) = q^k p, \ k=0,1,2,\ldots, \ 0<q<1, \ q=1-p \). Then \( X_1+X_2+X_3+\ldots+X_M \) has the pgf \( P(z)=p/\{1-q[1-(1-z)^{\alpha}]\} = 1/c(1-z)^{\alpha} \) with \( p=1/(1+c) \).

The distribution with above said pgf is called DML distribution with parameter \( \alpha \).

A random variable \( X \) on \( \{0,1,2,\ldots\} \) is said to follow DML distribution if its pgf is \( P(z)=1/(1+(c(1-z)^{\alpha})^{\alpha}) \) with \( 0<\alpha<1, \ c >0 \).

The DML distribution can be viewed as the distribution of geometric sum of independent and identically distributed Sibuya random variables.

The Sibuya distribution was introduced first by Sibuya(1979) while studying the generalized hypergeometric, digamma and trigamma distributions. Devroye(1993) studied some properties of Sibuya distribution. Sibuya distribution plays an important role in the mathematical origin of DML random variables.

DEFINITION 2.2

In a sequence of independent Bernoulli trials let \( \alpha/k \) be the probability of success in \( k^{th} \) trial, \( 0<\alpha<1 \). Then the number of trials required to obtain the first success has Sibuya distribution.

Note that the pgf of Sibuya distribution is \( G(z)=1-(1-z)^{\alpha} \).

B. Some Properties of discrete Mittag-Leffler distribution

(a) DML \( (\alpha) \) is geometrically infinitely divisible and hence infinitely divisible.
(b) DML \( (\alpha) \) is normally attracted to stable(\( \alpha \)).
(c) ML(\( \alpha \)) is obtained as a sequence of DML(\( \alpha \)).
(d) For \( \alpha=1 \), the distribution is geometric.
(e) DML(\( \alpha \)) is in discrete class L.
(f) DML(\( \alpha \)) is directly attracted to discrete stable(\( \alpha \)).

III. MITTAG-LEFFLER FUNCTION DISTRIBUTION

Chakraborty and Ong(2017) proposed a new generalisation of the HP distribution using the Mittag-Leffler function called Mittag-Leffler function distribution (MLFD).

The Poisson distribution is a popular model for count data. However, its use is restricted by the equality of its mean and variance (equi-dispersion). Many models with the ability to represent under, equi and over dispersed data have been proposed in the
literature to overcome this restriction. One of this is the HP distribution of Bardwell and Crow (1964).

The HP distribution proposed by Bardwell and Crow (1964) can handle both over and under dispersion. The probability mass function of the HP distribution is given by

$$P(X = k) = \frac{\Gamma(\beta)}{\Gamma(k + \beta)} \frac{\lambda^k}{\phi(1, \beta, \lambda)}, k = 0, 1, 2, \ldots$$

where

$$\phi(1, \beta, \lambda) = \sum_{i=0}^{\infty} \frac{(1)^i}{(\beta)_i k!} = \beta(\beta + 1)(\beta + 2)\ldots(\beta + k - 1)$$

is the confluent hypergeometric function.

A new generalization of the HP distribution, which is a continuous bridge between geometric and HP, is derived by Chakraborty and Ong (2017) using the generalized Mittag-Leffler function. The generalization of the HP distribution is done by replacing $\Gamma(k + \beta)$ in (1) with $\Gamma(ak + \beta)$, $a > 0$ and the normalization constant becomes $E_{a, \beta}(\lambda)$ which is the generalized Mittag-Leffler function defined by

$$E_{a, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + \beta)}.$$  

**DEFINITION 3.1**

A discrete random variable $X$ is said to follow MFLD if its pmf is given by

$$P(X = k) = \frac{\lambda^k}{\Gamma(ak + \beta)E_{a, \beta}(\lambda)},$$

$$k = 0, 1, 2, \ldots; \lambda, a, \beta > 0,$$

where $E_{a, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + \beta)}$ is the generalized Mittag-Leffler function. The distribution with the above pmf is denoted as MLFD$(\lambda, a, \beta)$.

The probability recurrence relation for MLFD pmf in (3) is

$$\Gamma(ak + a + \beta)P(X = k + 1) = \lambda \Gamma(ak + \beta)P(X = k), \quad k \geq 1$$

with

$$P(X = 0) = \frac{1}{\Gamma(\beta)E_{a, \beta}(\lambda)}.$$  

A. Related Distributions and Connection with Other Families of Distributions

The MLFD$(\lambda, a, \beta)$ includes a number of well-known distribution as particular cases:

1. When $a = 0$, MLFD$(\lambda, a, \beta)$ reduces to the Poisson distribution with parameter $\lambda$.
2. When $a = 0, \beta(\geq 0)$, MLFD$(\lambda, a, \beta)$ becomes geometric distribution with parameter $\lambda$ provided $\lambda < 1$.
3. When $a = 1, \beta(\geq 0)$, MLFD$(\lambda, a, \beta)$ reduces to the HP$(\lambda, \beta)$ distribution described in Bardwell and Crow (1964).
4. When $a = 1$ and $\beta(=t+1)$ is a positive integer, MLFD$(\lambda, a, \beta)$ reduces to displaced Poisson distribution with parameters $\lambda$ and $t$.

**REMARK 3.1**

(a) For $0 \leq a \leq 1$, the MLFD$(\lambda, a, \beta)$ can be viewed as a continuous bridge between geometric ($a = 0$) and HP ($a = 1$) distributions in the range of the parameter $a$.

(b) For $0 \leq a \leq 1$, the MLFD$(\lambda, a, 1)$ can be viewed as a continuous bridge between geometric ($a = 0$) and Poisson ($a = 1$) distributions in the range of the parameter $a$. This property is also shared by the Com-Poisson distribution.

B. MLFD as Member of Some Families of Discrete Distributions

1. MLFD$(\lambda, a, \beta)$ is a member of the generalized hypergeometric family.
(2) MLFD($\lambda, \alpha, \beta$) is a member of the generalized power series distribution when $\lambda$ is the primary parameter.

(3) For fixed value of the parameter $\alpha$ and $\beta$ the MLFD($\lambda, \alpha, \beta$) is a member of the exponential family of distribution.

Chakraborty and Ong (2017) studied various distribution properties of MLFD such as recurrence relations for pmf, generating functions, formulae for different types of moments, reliability properties and stochastic ordering.

4. APPLICATIONS

During the last two decades, the Mittag-Leffler distribution was applied in diverse areas like Physics, Reliability, Financial Modelling, Time series Analysis, Queueing Theory, etc. (see, Mariamma(2017)). Kozubowski and Rachev (1994) discussed the applications of Mittag-Leffler distribution in modeling financial data. Semi Mittag-Leffler distribution, which is a generalization of Mittag-Leffler distribution, arises as the stationary solution of a first order autoregressive equation (see, Jayakumar and Pillai, 1993). Bunge (1996) used Mittag-Leffler distribution in the study of random stability. Lin (1998) studied some distributional properties of Mittag-Leffler distribution. Kozubowski and Rachev (1999) developed certain representations of the Mittag-Leffler distribution.


The DML distribution proposed by Pillai and Jayakumar (1995) arises as a mixture of Poisson and Mittag-Leffler distribution. They have studied different properties of DML distribution and gave a probabilistic derivation and application in a first order auto regressive integer-valued process. Chakraborty and Ong (2017) showed that the MLFD is suitable in empirical modeling of differently dispersed count data when compared with HP distribution. MLFD is a flexible distribution with varying shape and can be modeled for under, equi or over dispersed data.

Acknowledgement

The author would like to thank UGC for the financial support under Minor Research Project No: MRP 14-15/KLCA033/UGC-SWRO.

References