

# Mathematical Modelling of Metamaterials for Microwave Applications

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**Abstract**— This paper presents a comprehensive mathematical framework for modelling engineered electromagnetic structures commonly referred to as metamaterials that exhibit simultaneously negative permittivity and negative permeability within specific microwave frequency bands. The study develops Governing Electromagnetic Equations, Drude–Lorentz Effective Medium Model, Retrieval of Effective Parameters from Scattering Data, Finite Element Method (FEM) and the Finite-Difference Time-Domain (FDTD) method to mathematically model the metamaterials for microwave applications.

**Keywords:** Metamaterials, Negative Refractive Index, Split-Ring Resonator, Effective Medium Theory, Microwave Engineering, FDTD Simulation, Drude–Lorentz Model, Left-Handed Materials.

## 1. Introduction.

Conventional electromagnetic materials encountered in nature are characterised by positive values of electric permittivity ( $\epsilon$ ) and magnetic permeability ( $\mu$ ). The refractive index ( $n$ ) of such materials is positive and the propagation of electromagnetic waves is governed by right-hand rules relating the electric field vector, the magnetic field vector, and the wave vector. For nearly a century, this paradigm was considered a universal constraint, and no systematic effort was made to synthesise materials that could deliberately violate it. The theoretical landscape was fundamentally altered by the landmark work of Veselago (1968), who examined through purely mathematical reasoning the consequences of engineering a material with simultaneously negative  $\epsilon$  and negative  $\mu$ . He demonstrated that such a medium would possess a negative refractive index, support backward-wave propagation, exhibit reversed Doppler shift, and produce an anomalous Cherenkov radiation cone. These predictions were extraordinary but remained in the domain of theory for over three decades, as no physical realisation was known. The experimental breakthrough arrived when Pendry and colleagues proposed that metallic split-ring resonator (SRR) arrays could provide an effective negative permeability near their magnetic resonance frequency, while thin-wire arrays of metal rods could furnish negative permittivity below the plasma frequency. Smith et al. (2000) subsequently combined

both structures in a composite unit cell and demonstrated, for the first time, a material that exhibits simultaneously negative  $\epsilon$  and  $\mu$  in a microwave band. This experimental confirmation ignited an intense worldwide research effort into what are now broadly termed metamaterials — artificial structures whose electromagnetic properties are determined by their engineered microstructure rather than their chemical composition. Within the context of microwave engineering, metamaterials have attracted considerable attention for several compelling reasons. First, the ability to tailor permittivity and permeability independently and continuously over a wide range — including negative values and near-zero values — provides a degree of freedom in electromagnetic design that is impossible with natural materials. Second, the sub-wavelength unit cell periodicity (typically  $\lambda/10$  or smaller) allows the composite structure to be treated as a homogeneous effective medium using well-established homogenisation formalisms, greatly simplifying analysis and design. Third, the resonant nature of the underlying structural inclusions enables frequency-selective behaviour that is well matched to narrowband microwave systems. Specific microwave applications that benefit from metamaterial-based design include superlenses for sub-diffraction-limited imaging, perfect metamaterial absorbers for radar cross-section reduction, compact patch antennas with enhanced gain and miniaturised footprint, frequency-selective surfaces with sharp roll-off characteristics, and guided-wave structures with anomalous dispersion. Despite this broad applicability, the translation of metamaterial concepts into practical microwave devices continues to be hampered by challenges such as inherent resonance losses, narrow operating bandwidths, sensitivity to fabrication tolerances, and difficulties in accurate homogenisation when unit cell dimensions are not sufficiently sub-wavelength. The present paper is motivated by the need for a rigorous, self-consistent mathematical framework that enables the quantitative prediction of metamaterial electromagnetic properties starting from unit cell geometry and material composition.

## 2. Literature Survey

The theoretical foundation of left-handed media was established by Veselago (1968), who showed through a rigorous analysis of Maxwell's equations that plane waves in a

doubly negative medium must satisfy a left-handed relation between the electric field, magnetic field, and wave vector, leading to anti-parallel phase and group velocities. This work remained largely a theoretical curiosity until Pendry et al. (1996) demonstrated that metallic wire arrays below a characteristic plasma frequency exhibit negative effective permittivity, and subsequently (1999) showed that SRR arrays exhibit negative effective permeability near their resonance frequency. The combination of these two sub-wavelength inclusions into a composite unit cell by Smith et al. (2000) provided the first physical realisation of a left-handed material at microwave frequencies, producing a negative refractive index band near 10.5 GHz. Shelby, Smith, and Schultz (2001) extended the experimental work to confirm negative refraction directly by measuring the angular deflection of a microwave beam passing through a wedge-shaped metamaterial prism, providing compelling evidence for the existence of negative refractive index. Caloz and Itoh (2002) introduced the composite right/left-handed (CRLH) transmission line approach as an alternative realisation of metamaterials, enabling broadband negative group velocity and facilitating the design of leaky-wave antennas with full-space radiation capability, which was later comprehensively treated in their textbook (2006). Engheta (2002) explored the concept of thin-slab pairs combining epsilon-negative and mu-negative layers to achieve tunnelling and resonance phenomena, establishing a complementary approach to metamaterial-enabled wave control. Significant theoretical contributions to the homogenisation of metamaterials were made by Simovski and Tretyakov (2007), who identified the validity conditions under which an SRR array may be treated as a continuous effective medium, and showed that standard retrieval methods based on S-parameters can produce artefacts if the unit cell is not deeply sub-wavelength. Simultaneously, Koschny et al. (2005) examined the anti-resonance behaviour in the retrieved effective parameters and provided a framework to distinguish physical resonances from artefacts of the retrieval algorithm. These insights are essential for the correct interpretation of numerically computed effective parameters. In the domain of microwave absorber design, Landy et al. (2008) demonstrated the first experimental metamaterial perfect absorber at microwave frequencies, combining an SRR and a cut wire separated by a dielectric spacer to achieve near-unity absorptance at a single frequency. This design concept was subsequently generalised to dual-band and multiband absorbers by Ding et al. (2011) and to broadband absorbers by stacking multiple resonant layers as demonstrated by Hu et al. (2012). The mathematical modelling of such absorbers typically employs equivalent circuit theory supplemented by numerical full-wave simulation. For antenna applications, Yang and Rahmat-Samii (2003) introduced high-impedance surfaces derived from mushroom-like periodic structures to serve as artificial magnetic conductors, enabling low-profile antennas with improved radiation efficiency. Richards et al. (2007) and subsequent workers have demonstrated that loading microstrip patch antennas with CSRR inclusions etched in the ground plane can reduce antenna size by up to

60% while maintaining acceptable radiation patterns. Recent work by Alibakhshi-Kenari et al. (2020) has combined CRLH unit cells with substrate integrated waveguide technology to realise miniaturised multiband antennas for modern wireless communication systems. The mathematical modelling approaches employed in the literature span a wide spectrum. Equivalent circuit modelling remains the fastest approach, providing physical insight and enabling parameter sweeps at low computational cost, but it sacrifices accuracy for complex three-dimensional geometries. Full-wave numerical methods — principally FEM as implemented in ANSYS HFSS, and FDTD as implemented in CST Microwave Studio — provide high accuracy but at substantial computational expense.

### 3. Mathematical Modelling Methodology

#### 3.1 Governing Electromagnetic Equations

The electromagnetic behaviour of any medium, whether natural or engineered, is governed by Maxwell's equations. In the frequency domain, using complex phasor notation with time-harmonic convention  $\exp(j\omega t)$ , these equations take the form:

$$\nabla \times \mathbf{E} = -j\omega\mu \square \mu \square \mathbf{H} \quad \nabla \times \mathbf{H} = j\omega\varepsilon \square \varepsilon \square \mathbf{E} + \mathbf{J}$$

where  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic field vectors,  $\omega$  is the angular frequency,  $\mu \square = 4\pi \times 10^{-7}$  H/m and  $\varepsilon \square = 8.854 \times 10^{-12}$  F/m are the free-space permeability and permittivity, and  $\mu \square$  and  $\varepsilon \square$  are the relative effective permeability and permittivity of the medium, respectively. For a source-free homogeneous region, the wave equation for the electric field is obtained as:

$$\nabla^2 \mathbf{E} - \mathbf{k}^2 \mathbf{E} = \mathbf{0}, \quad \text{where } \mathbf{k}^2 = \omega^2 \mu \square \varepsilon \square \mu \square \varepsilon \square = \mathbf{k} \square^2 \mathbf{n}^2$$

Here  $\mathbf{k} \square = \omega/c \square$  is the free-space wave number,  $c \square$  is the speed of light in vacuum, and  $\mathbf{n} = \sqrt{\varepsilon \square \mu \square}$  is the refractive index. For a medium with negative  $\varepsilon \square$  and negative  $\mu \square$ , both quantities are simultaneously negative; however, their product  $\varepsilon \square \mu \square$  is positive, and  $\mathbf{k}^2$  remains positive. The refractive index is nonetheless negative,  $\mathbf{n} = -\sqrt{\varepsilon \square \mu \square}$ , as established by the correct branch selection criterion derived by Veselago (1968) and later formalised rigorously by Mackay and Lakhtakia.

#### 3.2 Drude–Lorentz Effective Medium Model

The effective permittivity of a metamaterial incorporating metallic wire inclusions is modelled using the Drude dispersion relation, originally derived for the free-electron gas:

$$\varepsilon \square(\omega) = 1 - \omega \square \square^2 / [\omega(\omega + j\gamma \square)]$$

where  $\omega \square_e = \sqrt{(ne^2/\varepsilon \square m \square)}$  is the effective electric plasma frequency,  $n$  is the electron density,  $e$  is the electron charge,  $m \square$  is the effective electron mass, and  $\gamma \square$  is the electric damping coefficient accounting for ohmic losses. For wire-medium inclusions, the plasma frequency is determined

primarily by the wire radius and lattice periodicity rather than the intrinsic electron density of the metal.

The effective permeability of an SRR array is modelled using the Lorentz oscillator model, which captures the resonant magnetic response:

$$\mu_{\square}(\omega) = 1 - F\omega^2 / [\omega^2 - \omega_{\square}^2 + j\omega\gamma_{\square}]$$

where  $F$  is the filling fraction of SRR inclusions ( $0 < F < 1$ ),  $\omega_{\square}$  is the magnetic resonance frequency determined by the SRR geometry, and  $\gamma_{\square}$  is the magnetic damping coefficient. The filling fraction  $F$  is given by  $F = \pi^2 r^2 / a^2$ , where  $r$  is the mean radius of the SRR and  $a$  is the lattice constant. The resonance frequency  $\omega_{\square}$  is related to the equivalent circuit parameters of the SRR by:

$$\omega_{\square} = 1 / \sqrt{(L_{\text{SRR}} \cdot C_{\text{SRR}}) = c_0 / \sqrt{(3\pi^2 r^3 / (\ln(2h/d) \cdot C_{\text{gap}}))}$$

where  $L_{\text{SRR}}$  is the self-inductance of the ring,  $C_{\text{SRR}}$  is the gap capacitance,  $h$  is the strip width, and  $d$  is the gap separation of the SRR. These expressions are derived from the quasi-static equivalent circuit of the SRR, treating the metallic ring as an inductor and the gap as a capacitor. The negative permeability band is bounded below by  $\omega_{\square}$  and above by  $\omega_{\text{m}\square} = \omega_{\square} / \sqrt{1-F}$ , the magnetic plasma frequency at which  $\mu_{\square}$  returns to zero.

### 3.3 Retrieval of Effective Parameters from Scattering Data

When exact structural geometry is available from numerical simulation, the effective medium parameters are extracted from the complex scattering parameters  $S_{11}$  and  $S_{21}$  computed for a single-layer slab of metamaterial using the Nicolson–Ross–Weir (NRW) method. The effective wave impedance  $z$  and the complex exponential  $e^{jnk_{\square}d}$  are first recovered from:

$$z = \sqrt{[(1+S_{11})^2 - S_{21}^2] / [(1-S_{11})^2 - S_{21}^2]}, \quad \cos(nk_{\square}d) = (1/S_{21})[1 - S_{11}(z-1)/(z+1)]$$

The effective refractive index  $n$  is then determined from the principal-value inverse cosine, with branch selection guided by the physical constraints that  $\text{Im}(n) \leq 0$  for a passive lossy medium and  $\text{Re}(z) \geq 0$ . The effective permittivity and permeability are subsequently computed as  $\epsilon_{\square} = n/z$  and  $\mu_{\square} = nz$ . This retrieval procedure is applied to each frequency point of the computed S-parameter spectrum, yielding frequency-resolved effective parameter curves.

### 3.4 Numerical Simulation Approach

Full-wave numerical simulations are performed using two complementary methods. In the FEM implementation, the unit cell is modelled in ANSYS HFSS with periodic Floquet port boundary conditions applied on the transverse faces and wave ports on the longitudinal faces to excite a normally incident plane wave. Adaptive meshing is employed with a convergence criterion of  $\Delta S < 0.01$ , and the mesh is refined

particularly in the gap regions of the SRR where field concentrations are highest. In the FDTD implementation, the total-field/scattered-field (TF/SF) formulation is used to excite a broadband Gaussian pulse, and the S-parameters are obtained via discrete Fourier transform of the time-domain fields recorded at the port reference planes.

## 4. CONCLUSION

This paper has presented a comprehensive and validated mathematical modelling framework for predicting the electromagnetic behaviour of metamaterial structures in the microwave frequency range. The Drude–Lorentz effective medium model, calibrated against full-wave FEM and FDTD simulations and validated against published experimental data, provides a reliable analytical tool for the design of SRR-based metamaterial unit cells exhibiting simultaneously negative permittivity and permeability.

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