Set Valued Homeomorphisms using Ideal of a Ring and Rough Approximations

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Abstract
In this paper, using an ideal of an algebraic structure ring \((R)\) we generate a set valued homemorphism on that algebraic structure \(R\). We define rough approximations in \(R\) using this set valued homemorphism. We find approximations of subring and ideal of \(R\) using this set valued homemorphism. We find rough approximations of different ideals of the ring \(R\). We find properties of this homemorphism with for different nonempty subsets of \(R\).

Keywords: Approximation, Ring, Ideal, Homomorphism

INTRODUCTION

In this paper \(R\) represents a ring.

Definition 2.1. (Anderson, Fuller[1]) Let + and \(\cdot\) are two binary operations defined on a nonempty set \(R\). Then \(R\) is called a ring if if for all \(\alpha, \beta, \delta \in R\)

1. \((R, +)\) is a group.
2. \(\alpha + \beta = \beta + \alpha \) (addition is commutative)
3. \((\alpha \cdot \beta) \cdot \delta = \alpha \cdot (\beta \cdot \delta) \) (multiplication is associative)
4. (i) \((\alpha + \beta) \cdot \delta = \alpha \cdot \delta + \beta \cdot \delta\),

(ii) \(\alpha \cdot (\beta + \delta) = \alpha \cdot \beta + \alpha \cdot \delta, \) (left and right distributive law)

Definition 2.2. (Anderson, Fuller[1]) Let \(I \subseteq R\) be a subset of \(R\) such that \(I \neq \emptyset\). If \(I\) satisfies following conditions for all \(\alpha, \beta, \gamma \in R\) (i) \(\alpha, \beta \in I, \gamma \in R\) (i) \(\alpha + \beta \in I\) (ii) \(\alpha \gamma \in I\) and \(\gamma \alpha \in I\) Then \(I\) is called an ideal of \(R\).

Definition 2.3. (Pawlak[11]) Let \(F\) be a set having finite elements and \(\eta\) be a relation on \(F\) which is reflexive, Symmetric and transitive. The set \(F\) is called universe and \(\eta\) is called indiscernibility relation. A pair \((F, \eta)\) is called a space of approximation.

Definition 2.4. (Pawlak[11]) Let \((F, \eta)\) be the space of approximation and \(X \subseteq F\). The element \(x \in X\) under the relation \(\eta\) forms a set denoted by \(\eta(x)\) called equivalence class.
The set of all elements which can be definitely classified as members of \( X \) under the relation \( \eta \) called \( \eta \)-lower approximation of a set \( X \) with respect to \( \eta \) and denoted by \( \eta_-(X) \), that is \( \eta_-(X) = \{ x : \eta(x) \subseteq \{ \} \} \).

The set of all elements which are identified as the possible members of \( X \) under the relation \( \eta \) is called \( \eta \)-upper approximation of the nonempty set \( X \) with respect to \( \eta \) and denoted by \( \eta_+(X) \), that is \( \eta_+(X) = \{ x : \eta_+(X) \cap X \neq \{ \} \} \).

The set of all elements which are not elements of \( X \) or as elements of \( -X \) with respect to the relation \( \eta \) is called boundary region of \( X \) with respect to \( \eta \) and denoted by \( \Delta(X) \), that is \( \Delta(X) = \eta_+(X) - \eta_-(X) \).

**Definition 3.2.** (Pawlak[11]) A set \( X \) with elements is called as ordinary set if there are no elements in the boundary region of \( X \). If boundary region of \( X \) contains some elements then it is called rough set.

**Proposition 3.2.** Let \( p, q \in R \). Then we get \( p \leq q \).

(i) the binary operation addition is preserved under \( g \).

(ii) \(-g(p)\) is a subset of \( g(-p)\).

(iii) the binary operation multiplication is preserved under \( g \).

3. SET VALUED HOMOMORPHISMS USING IDEAL OF A RING

In the next part of this paper \( R, R_1, R_2 \) represents rings and \( I \) is an ideal of the ring \( R \).

**Definition 3.1.** Let \( x \) be an element of \( R \) and the set \( \eta \) of all subsets of \( R \) is denoted by \( P (R) \). Then we define \( \rho : R \rightarrow P (R) \) by \( f(m) = \{ p \in R | m - p \text{ is an element of } I \} \).

**Proposition 3.3.** By Definition 2.7 and by Proposition 3.2 we get \( \rho \) is a set valued homomorphism on \( R \).

**Definition 3.4.** Let \( f_i \) be the set valued homomorphism on the ring \( R \) and \( S \) be a subset of \( R \) and \( S \neq \{ \} \). Then we define \( L(S) = \{ p \in R | f(p) \subseteq S \} \)

and \( f(S) = \{ p \in R | f(p) \cap S \neq \{ \} \} \) Then \( f(S) \) and \( f(S) \) are called rough approximations of \( S \) with respect to \( \rho \).

In particular \( f(S) \) is called lower approximation of \( S \) and \( f(S) \) is called upper approximation of \( S \) with respect to \( \rho \).

**Proposition 3.5.** Let \( f_i : R \rightarrow P (R) \) be the set-valued homomorphism. Let \( S \)

and \( T \) be nonempty subsets of \( R \). Then

1. \( f(S \cap T) = f(S) \cap f(T) \).

2. \( f(S \setminus T) = f(S) \setminus f(T) \).

3. \( f(S \cup T) \subseteq f(S) \cup f(T) \).

4. \( f(S) \supseteq f(T) \subseteq f(S) \cup f(T) \).

**Proof.** To prove (1) let \( x \in f(S) \cap f(T) \). Then \( x \in f(S) \cap f(T) \).

Let \( m \in f(x) \cap (S \cup T) \). Then \( m \in f(x) \) and \( m \in S \cup T \) which implies \( m \in f(x) \) and \( m \in S \) or \( m \in T \). Then \( m \in f(x) \) or \( m \in S \) or \( m \in T \). Hence \( m \in f(x) \) and \( m \in S \) or \( m \in T \). Then \( m \in f(x) \) or \( m \in S \) or \( m \in T \). Then \( m \in f(x) \) or \( m \in S \) or \( m \in T \). Then \( m \in f(x) \) or \( m \in S \) or \( m \in T \). Then \( m \in f(x) \) or \( m \in S \) or \( m \in T \). Hence \( m \in f(x) \) or \( m \in S \) or \( m \in T \).

With out loss of generality assume \( f(x) \cap S \neq \{ \} \). Let \( m \in f(x) \cap S \). Then \( m \in f(x) \) and \( m \in S \). As \( S \subseteq S \cup T \) we get \( m \in S \cup T \) so that \( m \in f(x) \) and \( m \in S \cup T \). Hence \( f(x) \cap (S \cup T) \) is nonempty subset of \( S \) and \( f(x) \); \( f(x) \) is a subset of \( S \cap T \). Then \( x \in f(x) \) and \( x \in f(x) \). Hence \( x \in f(x) \). Therefore \( f(S \cap T) \subseteq f(S) \cup f(T) \).

The proof of (3) and (4) is similar to that of (1) and (ii).
Proposition 3.7. Let \( f : R \rightarrow P(\mathbb{R}) \) be a set-valued homomorphism. If \( a \in \mathbb{R} \), then \( f(a) \in P(\mathbb{R}) \) is a subset of \( \mathbb{R} \) and \( f(a) \) is a subset of \( \mathbb{R} \).

(i) \( f(a) \times f(b) = f(a \times b) \)

(ii) \( f(a) + f(b) = f(a + b) \)

(iii) \( f(a) \cdot f(b) = f(ab) \)

Proposition 3.8. Let \( f : R \rightarrow P(\mathbb{R}) \) be a set-valued homomorphism. Let \( I, J \) be two ideals of \( R \) such that \( I \subseteq J \) and \( A \) be a nonempty subset of \( R \).

(i) \( f(J(A)) \subseteq f(A) \), \( (i \alpha f(A) \subseteq f(J(A)) \).

Proof. To prove (i), let \( T \subseteq J \) be two ideals of \( R \) such that \( I \subseteq J \) and \( A \) be a nonempty subset of \( R \). Suppose \( x \in f(J(A)) \) then \( f(x) \subseteq A \). As \( I \subseteq J \) we get \( f(x) \subseteq f(J(x)) \subseteq A \). Hence \( x \in f(J(A)) \). Therefore \( f(J(A)) \subseteq f(J(A)) \).

To prove (ii), suppose \( x \in f(J(A)) \). Then \( f(x) \subseteq A \), \( A \subseteq \emptyset \). Let \( a \in f(x) \cap A \). Then \( a \in A \). We have \( a \in f(x) \) which gives \( x - a \in I \). As \( I \subseteq J \) we get \( x - a \in J \).

Proposition 3.9. Let \( f : R \rightarrow P(\mathbb{R}) \) and \( f_j : R \rightarrow P(\mathbb{R}) \) be set-valued homomorphisms where \( I, J \) are two ideals of \( R \).

Then for a subring \( A \) of \( R \). Then \( f(A) \subseteq f(J(A)) \).

Proof. Suppose \( x \in f(J(A)) \). Then \( f(x) \subseteq A \). Hence \( f(x) \subseteq f(J(A)) \).

Proposition 3.10. Let \( f : R \rightarrow P(\mathbb{R}) \) and \( f_j : R \rightarrow P(\mathbb{R}) \) be the set-valued homomorphisms where \( I, J \) be two ideals of \( R \).

Let \( A \) be any subset of \( R \). Then \( f(A) \subseteq f(J(A)) \).

Proof. Suppose \( x \in f(A) \subseteq f(J(A)) \).

Proposition 3.11. Let \( f_i : R \rightarrow P(\mathbb{R}) \) be the set-valued homomorphism. If \( A, B \) be two subsets of \( R \) such that \( A \subseteq B \), then \( f(A) \subseteq f(B) \).

Proof. Suppose \( x \in f(A) \). Then \( f(B(x)) \subseteq A \), which implies \( f(B(A)) \subseteq B \). So we get \( x \in f(B(A)) \), which gives \( f(A) \subseteq f(B) \).

Proposition 3.12. Let \( R \) be a ring and \( I = \{0\} \) and \( A \) be a nonempty subset of \( R \).

Then \( f(A) = A \neq f(I(A)) \).

4. Generalized lower and upper approximations with respect to an ideal of a ring

Definition 4.1. Let \( I, J \) be ideals of \( R \) and \( X \) be a nonempty subset of \( R \). Let \( \rho : R \rightarrow P(\mathbb{R}) \) be the set valued homomorphism. Then the sets \( \rho(I(X)) = \{ x \in R | (\rho(x) + J) \cap X = \emptyset \} \) and \( \rho(J(X)) = \{ x \in R | (\rho(x) + J) \subseteq X \} \) are called generalized upper and lower approximations of \( X \) with respect to ideal \( J \).

Proposition 4.2. Let \( I, J, K \) be ideals of \( R \) such that \( J \subseteq K \) and \( A \) be a nonempty subset of \( R \). Let \( \rho : R \rightarrow P(\mathbb{R}) \) be the set valued homomorphism. Then

(i) \( \rho(A) \subseteq \rho(A) \) (ii) \( \rho(J(A)) \subseteq \rho(J(A)) \).

Proof. To prove (i), let \( x \in \rho(A) \). Then \( \rho(x) \subseteq A \). As \( J \subseteq K \) we get

(ii) \( \rho(J(A)) \subseteq \rho(J(A)) \).

Proposition 4.3. Let \( I, J, K \) be ideals of \( R \) and \( A \) be a nonempty subset of \( R \). Let \( \rho : R \rightarrow P(\mathbb{R}) \) be the set valued homomorphism. Then

(i) \( \rho(A) \subseteq \rho(A) \) (ii) \( \rho(J(A)) \subseteq \rho(J(A)) \).

Proof. To prove (i), let \( x \in \rho(A) \). Then \( \rho(x) \subseteq A \). As \( J \subseteq K \) we get

(ii) \( \rho(J(A)) \subseteq \rho(J(A)) \).

Proposition 4.4. Let \( \rho : R \rightarrow P(\mathbb{R}) \) be the set valued homomorphism and \( J \) be an ideal of \( R \). If \( A, B \) be two nonempty subset of \( R \) then

(i) \( \rho(A) \subseteq \rho(A) \) (ii) \( \rho(J(A)) \subseteq \rho(J(A)) \).
Proposition 4.6. Let $\rho : R \rightarrow P(R)$ be the set-valued homomorphism. Let $I, J, K$ be ideals of $R$ and $A$ be a subring of $R$. Then

(i) $\rho^I(A) \ast \rho^K(A) \subseteq \rho^{I+K}(A)$

(ii) $\rho^I(A) + \rho^K(A) \subseteq \rho^{I+K}(A)$

REFERENCES


