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A New Experiment with the Convergence and Stability of Logistic Map via SP Orbit

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Abstract

In the theory of chaos, fractals and discrete dynamics the logistic map $x_{n+1} = rx_n(1-x_n)$, n = 0, 1, 2, … where $0 < x_0 < 1$ *and* $0 < r \le 4$ plays an important role. The purpose of this paper is to extend the stability of logistic map under SP orbit. This stability has been obtained by running computer program in Mathematica 9.0 using time series representation. We observe that in SP orbit the range of convergence and stability of logistic map is extendable for a higher value of r i.e. for $r \le 32.10$ than that of obtained via Picard orbit (2004) and Noor orbit (2012). This experimental study enables us to control the chaotic behaviour of one dimensional dynamical system (logistic map) for a larger value of population growth parameter *r*.

Keywords: Dynamical system, logistic map, stability, time series analysis, SP orbit.

2010 Mathematics Subject Classification: 37F45, 37M10.

1. INTRODUCTION

The dynamical system is associated to how the situation evolves from one time period to next time period. It is the study of iteration of maps over a time period [9]. Non linear dynamical systems have been shown to exhibit surprising and complex effects that would never be anticipated by any linear techniques.

The logistic map is a non linear dynamical equation. This equation is a common model of population growth, originally due to P.F. Verhulst in 1945 [22]. Verhulst logistic map is given by a quadratic polynomial $x_{n+1} = rx_n(1-x_n)$, n = 0, 1, 2, …. Here x represents the population at any time *n* and *r* is the rate of population growth [9]. The logistic map is widely used into the consideration of chaos concept as it is relatively simplest dynamical system. A chaotic system has the property to show sensitive dependence on initial conditions i.e. once a system becomes chaotic there is no regularity and stability anymore.

The modern chaos theory is based on Verhulst logistic map. Ausloos et al. [1] studied its occurrence in the large financial crashed. Logistic map used in cryptography [6,11]. To know various applications of logistic map, one may refer Devany [7], Peitgen et al. [16,17,24], Beardon [2], Bunde and Havlin [3], A. Celletti [4], Z. Sardar and I. Abrams [23] and M.S. E. I. Naschie[12-15]. Many Techniques such as neural method, fuzzy system, OGY method and Pyragas method [8] have been used to control the chaos.

In all the literature of logistic map, Picard iteration was used for solving non linear equations. This iteration method is slow and less complex but convergence rate is also slow and for lower values of *r* i.e. for $0 < r \le 2.75$ [9,18]. In 2002, Rani and Kumar [22] studied logistic map in Mann orbit. They proved that convergence range of logistic map is extendable to a large value of r. In 2005, Kumar and Rani [10] compared the study of logistic map in Picard orbit and in Norlund orbit. Further, Rani and Agarwal [19] studied the stability of logistic map by running computer programs.

In 2011, Rani and Goel [20] enhanced the capabilities of logistic map via I-Superior orbit. They showed that logistic map is stable for $r < 17$. In 2012, Chugh at el. [5] studied the logistic map in Noor orbit. They showed computationally that logistic map is convergent in Noor orbit for a larger value of *r* i.e. for $r < 27$ than that of Picard orbit.

The SP-iteration was proposed by W. Phuengrattana and S. Suantai [18] for approximating a fixed point of continuous functions on an arbitrary interval. They proved a necessary and sufficient result for the rate of convergence of the SP-iteration of continuous functions on an arbitrary interval. They compared the convergence speed of Mann, Ishikawa, Noor and SP-iterations using some numerical examples and proved that the SP-iteration is equivalent to and converges faster than the other iterations.

In this paper, our main aim is to derive the range of convergence and stability of logistic map using SP orbit. Here, we have obtained the result that logistic map remains stable and converges to a fixed point for higher values of *r* i.e. for $r \le 32.10$ which is greater than that of using Picard and Noor orbit. This experimental study enables us to control the chaotic behaviour of one dimensional dynamical system (logistic map) for a larger value of population growth parameter *r*. This experimental study has been made by running program in Mathematica 9.0. In section 2, we give some basic definitions which are useful in our further study. Section 3 deals with the description of our experimental approach and results obtained. Finally, we conclude the paper in section 4.

2. PRELIMINARIES

There are generally four types of feedback procedures one-step, two-step, three-step and four-step procedures [5,16,19,20,21]. SP iteration scheme is an example of four step feedback procedure, which was due to W. Phuengrattana and S. Suantai [18]. Iteration procedures are one way to achieve the self similarity. In the literature of logistic map, iteration procedures are known as orbits.

Definition 2.1 (Orbit) [7]: Given $x_0 \in R$, then the orbit of x_0 under the mapping F is defined to be the sequence of points

$$x_0,\ x_1 = F(x_0),\ x_2 = F^2(x_0), \ldots,\ x_n = F^n(x_0),\ \ldots$$

The point x_0 is called the seed of the orbit.

In the following definitions, let X be a non-empty set and T: X→X be a self mapping.

Definition 2.2. (Picard Orbit)[16]: For $x_0 \in X$, the orbit $\{x_n\}$ defined by

$$O(T, x_0) = \{x_n : x_n = Tx_{n-1},\ n = 1, 2, \ldots\},$$

is known as Picard orbit, where the orbit $O(T, x_0)$ of T is the sequence $\{T^n x_0\}$ at the initial point x_0.

Definition 2.3: (SP Orbit) [18]: Let us consider a sequence $\{x_n\}$ of iterates for initial point $x_0 \in X$, such that

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n)z_n + \beta_n T z_n,$$
$$z_n = (1-\gamma_n)x_n + \gamma_n T x_n,\ \ n = 0,1,2,\ldots$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in [0,1]. The above sequence of iterates is called SP orbit, which is a function of five tuples $(T, z_0, \alpha_n, \beta_n, \gamma_n)$.

Remarks 2.4: The SP Orbit reduces to:

1. The **Mann orbit** [22] when $\beta_n = \gamma_n = 0$, i.e.

   $$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n;\ n = 0,1,2,\ldots$$

2. The **Picard orbit** [16] when $\beta_n = \gamma_n = 0$ *and* $\alpha_n = 1$, i.e.

   $$x_{n+1} = Tx_n;\ n = 0,1,2,\ldots$$

   For the sake of simplicity, throughout the paper we consider $\alpha_n = \alpha,\ \beta_n = \beta$ *and* $\gamma_n = \gamma$.

3. EXPERIMENTAL STUDY OF LOGISTIC MAP VIA SP ORBIT

In this paper, we try to find the maximum value of *r* for which logistic map remains convergent and stable for all $x \in [0,1]$.

This experimental study is performed by running a program in Mathematica 9.0.

3.1 Graphical presentation of SP orbit using time series method:

Using time series representation of logistic map in SP orbit, we graphically present some tables which exhibit optimum value of r for different choices of $\alpha, \beta, \gamma \in [0,1]$ against some initial values.

Table 1. For $\alpha = \beta = \gamma = 0.9$

x_0	Maximum value of **r**	
	for convergence	for stability
0.15	3.13	4.26
0.25	3.13	4.22
0.35	4.15	4.23
0.50	3.13	4.22

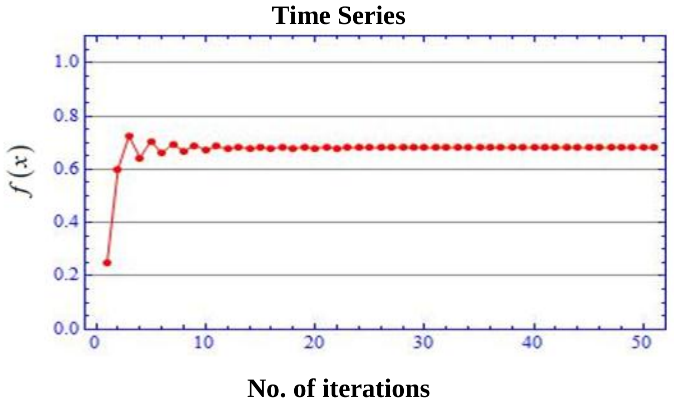

Fig. 1. $(r,\ x_0, \alpha = \beta = \gamma) = (3.13, 0.25, 0.9)$

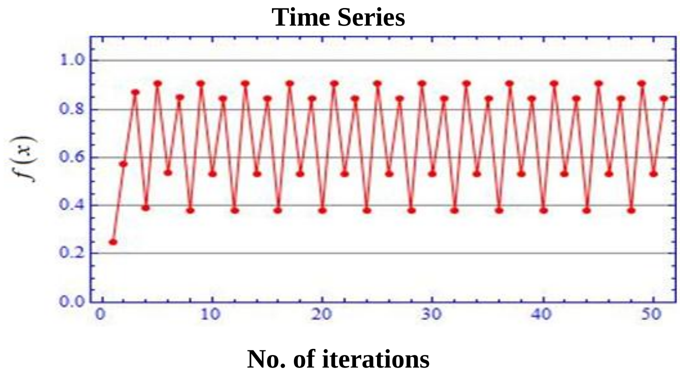

Fig. 2. $(r,\ x_0, \alpha = \beta = \gamma) = (3.81, 0.25, 0.9)$

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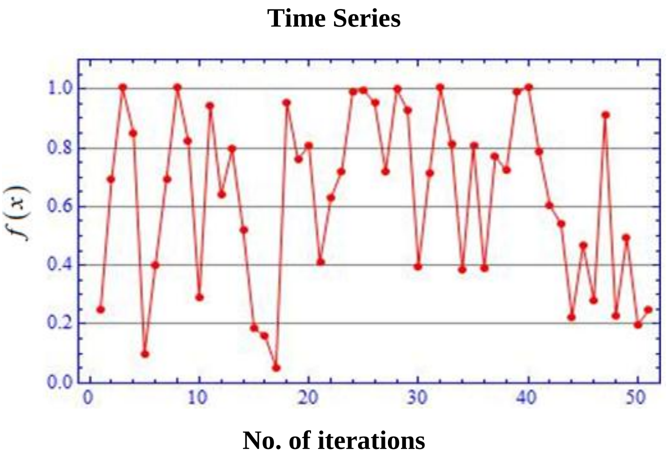

Fig. 3. $(r,\ x_0, \alpha=\beta=\gamma)=(4.28, 0.25, 0.9)$

We observe from Table 1 that for $0 < r \leq 3.13$ *and* $\forall\ x \in [0,1]$ logistic map is convergent to a fixed point. Convergent behavior of logistic map at $\alpha=\beta=\gamma=0.9$ is shown in Fig. 1. For $0 < r \leq 4.22$, logistic map shows stable behavior and it remains cyclic for $3.13 \leq r \leq 3.81$ (see Fig. 2).We notice that logistic map becomes unstable for $r \geq 4.27$. In Fig. 3, we show the unstable divergent behavior of logistic map for $r = 4.28$.

Table 2. For $\alpha=\beta=\gamma=0.6$

x_0	Maximum value of **r**	
	for convergence	for stability
0.15	4.21	5.24
0.25	4.17	5.24
0.35	4.18	5.24
0.50	4.18	5.24

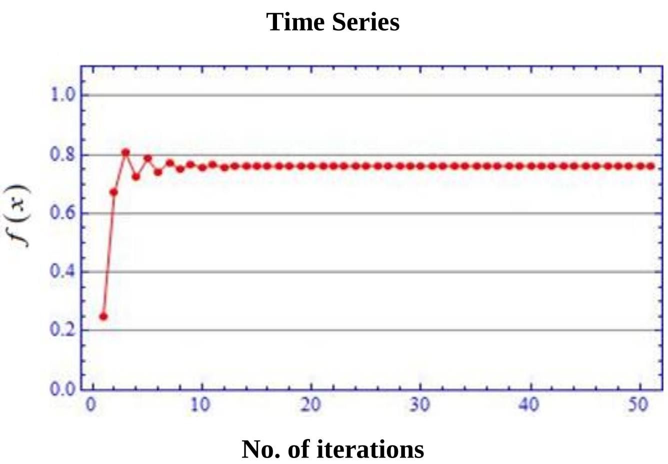

Fig. 4. $(r,\ x_0, \alpha=\beta=\gamma)=(4.17, 0.25, 0.6)$

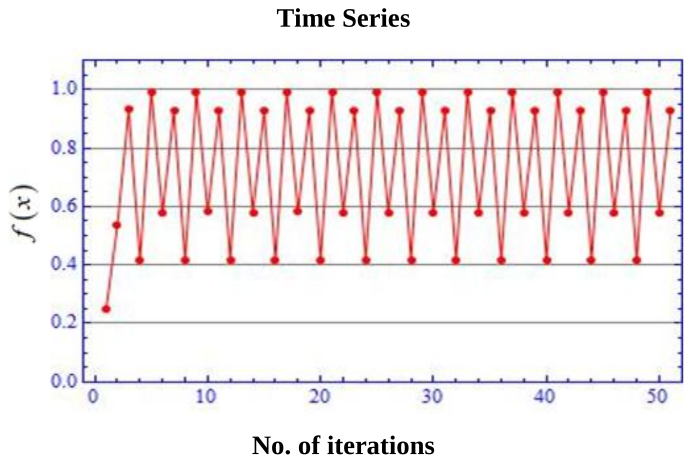

Fig. 5. $(r,\ x_0, \alpha=\beta=\gamma)=(5.21, 0.25, 0.6)$

In Table 2, we notice that logistic map is stable for $0 < r \leq 5.24$, $\forall\ x \in [0,1]$ and converges to a fixed point for $0 < r \leq 4.17$. Also, for $r \geq 5.25$ the value of x_n exceeds 1, i.e., $x_n \notin [0,1]$ which shows unstability of logistic map. Convergent and cyclic behavior of the map for $\alpha=\beta=\gamma=0.6$ is shown in Figs. 4 and 5 respectively.

Table 3. For $\alpha=\beta=\gamma=0.44$

x_0	Maximum value of **r**	
	for convergence	for stability
0.15	5.36	6.30
0.25	5.35	6.32
0.35	5.36	6.30
0.50	5.36	6.29

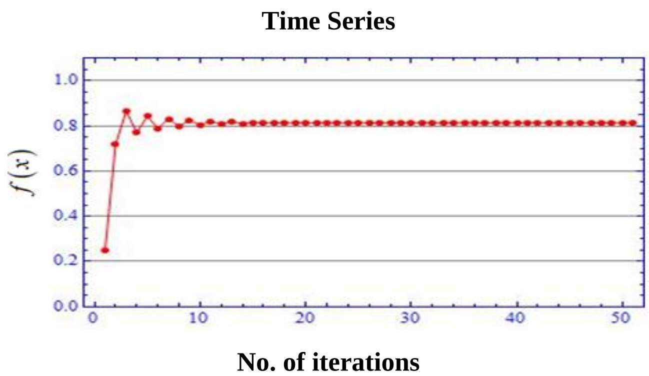

Fig. 6. $(r,\ x_0, \alpha=\beta=\gamma)=(5.35, 0.25, 0.44)$

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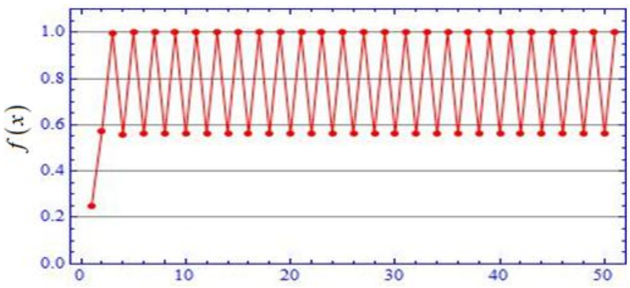
Time Series



No. of iterations

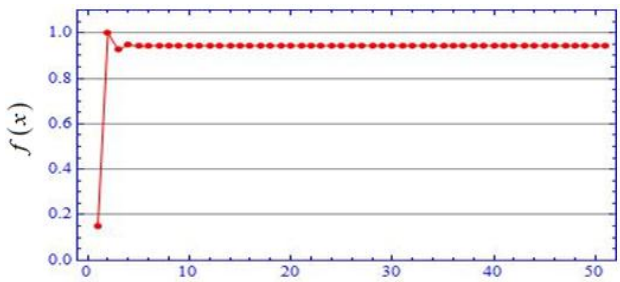
Fig. 7. $(r,\ x_0, \alpha=\beta=\gamma)=(6.33, 0.25, 0.44)$

In Table 3, for $0<r\le 5.35$ and for all $x\in[0,1]$ logistic map is convergent to a fixed point. This fact is shown in Fig. 6 for $\alpha=\beta=\gamma=0.44$ at $x_0=0.25$. The map is stable for $0<r\le 6.29$. Also, we show the cyclic but unstable behavior of the map for $r=8.18$ in Fig. 7.

Table 4. For $\alpha=\beta=\gamma=0.1$

x_0	Maximum value of **r**	
	for convergence	for stability
0.15	17.44	17.44
0.25	20.09	20.16
0.35	19.75	19.75
0.50	20.22	21.12

Time Series



No. of iterations

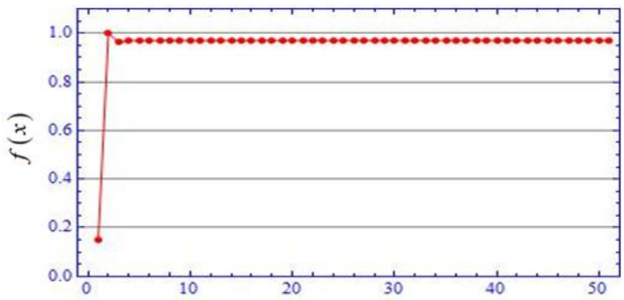
Fig. 8. $(r,\ x_0, \alpha=\beta=\gamma)=(17.44, 0.15, 0.1)$

In Table 4, the behavior of logistic map is depicted for $\alpha=\beta=\gamma=0.1$. We see that the map remains stable leading to a fixed point for $0<r\le 17.44,\ \forall\ x\in[0,1]$ (see Fig. 8).

Table 5. For $\alpha=\beta=\gamma=0.05$

x_0	Maximum value of **r**	
	for convergence	for stability
0.15	32.10	32.10
0.25	38.06	38.06
0.35	36.33	36.33
0.50	40.01	41.11

Time Series



No. of iterations

Fig. 9. $(r,\ x_0, \alpha=\beta=\gamma)=(32.10, 0.15, 0.05)$

For $0<r\le 17.44$ and for all x belonging to [0,1], we see in Table 5 that logistic map converges to a fixed point and shows stable behavior at $\alpha=\beta=\gamma=0.05$. This behavior is shown in Fig. 9 using time series representation. For $r\ge 41.11$, logistic map becomes unstable.

We observe that in SP orbit, logistic map is convergent and remains stable for larger value of r than that of Picard orbit [16] and Noor orbit [5]. Comparative table of logistic map in these orbits is as follows:

Table 6. Comparative Table

Convergence in PO	Convergence in NO	Convergence in SP orbit
$\alpha=1, \beta=\gamma=0,\ 0\le r\le 2.75$	$\alpha=\beta=\gamma=0.9,\ 0\le r\le 3.11$	$\alpha=\beta=\gamma=0.9,\ 0\le r\le 3.13$
	$\alpha=\beta=\gamma=0.44,\ 0\le r\le 4.67$	$\alpha=\beta=\gamma=0.44,\ 0\le r\le 5.35$
	$\alpha=\beta=\gamma=0.1,\ 0\le r\le 15.00$	$\alpha=\beta=\gamma=0.1,\ 0\le r\le 17.44$
	$\alpha=\beta=\gamma=0.05,\ 0\le r\le 27.52$	$\alpha=\beta=\gamma=0.05,\ 0\le r\le 32.10$

4. CONCLUSION

By an experimental approach of logistic map in SP orbit, we obtain some interesting results as compared to that of Picard and Noor orbit. From the experimental study we derive following results via SP orbit:

1. The logistic map is convergent and stable for higher values of *r*.
2. As the values of parameters $\alpha, \beta \ and \ \gamma$ come closer to 0, logistic map exhibits its stable behavior for larger values of *r*.
3. For $r \leq 3.13$, logistic map is convergent for all $\alpha, \beta, \gamma \in [0,1]$ and for all $x \in [0,1]$.
4. In SP orbit, logistic map converges to a fixed point and remains stable for $r \leq 32.10$ which is greater than that of Picard and Noor orbit (See Tables 5, 6 and Fig. 9).

REFERENCES

[1] Ausloos M, Dirickx M. The logistic map and the route to chaos: from the beginnings to modern applications. New York: Springer-Verlag; 2006.

[2] Beardon AF. Iteration of rational functions. Berlin: Springer; 1991.

[3] Bunde Armin, Havlin Shlomo, editors. Fractals in science. Berlin Heidelberg: Springer-Verlag; 1994.

[4] Cellitte A. Stability and chaos in Celestial Mechanics: Springer; 2012.

[5] Chugh R, Rani M and Ashish. On the Convergence of Logistic Map in NOOR Orbit International Journal of Computer Applications 2012;43(18): 0975 – 8887.

[6] De Oliveira Luiz PL, Sobottka Marcelo. Cryptography with chaotic mixing. Chaos, Solitons & Fractals 2006;35: 466–71.

[7] Devaney Robert L. A first course in chaotic dynamical systems: Theory and experiment. Addison-Wesley; 1992.

[8] Hassan Salarieh, Mohammad Shahrokhi. Indirect adaptive control of discrete chaotic systems. Chaos, Solitons & Fractals, 2007; 34:1188–201.

[9] Holmgren Richard A. A first course in discrete dynamical systems. Springer-Verlag; 1994.

[10] Kumar M, Rani M. An experiment with summability methods in the dynamics of the logistic model. Indian J Math 2005;47(1):77–89.

[11] Mooney Aidan, Keating John G, Heffernan Daniel M. A detailed study of the generation of optically detectable watermarks using the logistic map. Chaos, Solitons & Fractals 2006; 30: 1088–97.

[12] Naschie M S EL, On the universality class of all universality classes and E-infinity space time physics. Chaos, Solitons & Fractals 2007; 32(3): 927-936.

[13] Naschie M S EL. Nuclear Spacetime Theories, Superstrings, Monster Group and Applications. Chaos, Solitons & Fractals 1999; 10(2-3): 567-580.

[14] Naschie M S EL. Super symmetry, transfinite neural networks, hyperbolic manifolds, quantum gravity and the Higgs. Chaos, Solitons & Fractals, 2004; 22(5): 999-1006.

[15] Naschie M S EL. Topics in the mathematical physics of E-infinity theory. Chaos, Solitons & Fractals, 2006; 30(3): 656-663.

[16] Peitgen H, Jurgens H and Saupe D. Chaos and fractals. New York; Springer-Verlag; 2004.

[17] Peitgen H, Saupe D, editors. The science of fractal images. Springer- Verlag; 1988.

[18] Phuengrattana W., Suantai S., On the rate of convergence of Mann Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, Journal of Computational and Applied Mathematics,(235)(2011), 3006-3014.

[19] Rani M and Agarwal R. A new experimental approach to study the stability of logistic map. Chaos, Solitons & Fractals, 2009; 41(4): 2062-2066.

[20] Rani M and Goel S. An experimental approach to study the logistic map in I-superior orbit. Chaos and Complexity Letters 2011; 5(2): 1-7.

[21] Rani M and Goel S. I-Superior approach to study the stability of logistic map. International conf. on Mec. And Elec. Tech. (ICMT 2010); IEEE.

[22] Rani M, Kumar V. A new experiment with the logistic map. J Indian Acad Math 2005; 27(1):143–56.

[23] Sardar Z. Introducing Chaos: A Graphic Guide; London; 2008.

[24] Vazquez-Medina R, Dı ´ az-Mendez A, Del Rı ´ o-Correa JL, Lopez-Hernandez J. Design of chaotic analogue noise generators with logistic map and MOS QT circuits. Chaos, Solitons & Fractals, 2009; 40:1779–93.