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Free and Forced Vibrations by Finite Element Method

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Abstract

In this article, the free and forced vibrations of mass-spring is analyzed with finite element method. The time dependency is considered using Gauss Quadrature method of two points. It helps to study the dynamic responses for the undamped and damped system. The results are compared with analytical solution. Discretization error is computed using $H_{1,}$ $L_{1,}$ and L_2 norms. And its convergence is discussed.

Keywords: Ordinary Differential Equation, Finite Element Method, Galerkin Method, Discretization, Gauss Quadrature two points Method, Error.

1. INTRODUCTION

The vibrations analysis for the system is an essential field in computational mechanics. In this article, the mechanical problem is expressed as an ordinary differential equation with Dirichlet condition. Azadi *et al*. [1] analyzed the free and forced vibration behaviors of an off-road vehicle. They proposed an approach for the free vibrations with natural frequencies of vehicle and concluded that the forced vibration react on displacement and acceleration of the driver. Rahmani and Mirtaheri [2] searched out the effect of masonry panels on the vibrations response of infilled steel frame building. Wang *et al*. [3] showed the additional damping by eddy current in the seawater environment to be same as the air environment.

In this type of problem sometimes it is difficult to find analytical solution and if the analytical solution is known, it may contain complicated terms. So, some study is required to develop the solution in easy and compact form. In order to reduce this type of complication, numerical technique based on weak formulation has been proposed using Finite Element Method (FEM) [4-6]. In proposed scheme solution lie in some infinite dimensional space like Hilbert Sobolev space so differential equation can be solved by using computers. There are many numerical methods to solve differential equation like Finite Difference Method, Finite Element method, Boundary Integral, Spectral Method etc. but all the methods have some limitations, for example FDM is not able to solve differential equation with irregular boundaries and computational solution is obtained only on grid points so for remaining points some interpolation methods are required. Spectral method requires a global basis set expansion of the wave function whereas FEM does not require such expansion [7]. In FEM there is a reduction in the differentiation and is a more desirable property for the numerical computational. Then by discretization of the domain, construct a finite dimensional space V_h where h is the discretization parameter with property that the basis functions have small supports in domain. The FEM can be used to solve a large number of engineering problems. Here, we are using Galerkin approach for free and forced vibrations of one dimensional second order mass-spring system. These problems are little difficult to formulate and solve because vibrations problem involves various independent constraints. To overcome this type of situation, we propose FEM. Because of resonance, vibration is a major issue and for that reason system fails. For that weak formulation is worked out using weighted residual method. However, it is difficult to implement boundary conditions because of shape function. Here, we have discussed the gauss quadrature of two points method and describe the analysis of mass-spring with and without damping condition. The error analysis is also carried out to show the superiority of the proposed scheme. Using MATLAB 7.0, the validity and efficiency is validated for the proposed scheme.

In section 2, we discuss the mathematical modelling of free and forced vibrations of mass-spring. After that we proposed our numerical scheme based on FEM for the posed problem in section 3. In section 4, proposed problem is validated.

2. MATHEMATICAL MODEL

A string is attached with mass m. If we pull the mass in downward direction and release it, a vertical motion of string is observed. The motions without external force is free motions gives a homogeneous differential equation. Forced motions obtained when external force act on the body and is the result of non-homogeneous differential equation. Hence, the free and forced vibrations of mass spring is governed by ordinary differential equation

$$my'' + cy' + ky = f(t) \tag{1}$$

When equation (1) is homogeneous, it gives rise to free motion. Forced vibration is the result of non-homogeneous ordinary differential equation (1). In the proposed discussion, we will consider $f(t)$ as a sinusoidal force, and given by $H\cos(\omega t)$, where my'' denotes the force of inertia, cy' gives damping force, and ky is the spring force.

For free vibration eq. (1) results to homogeneous form as

$$my'' + cy' + ky = 0 \tag{2}$$

Eq. (2) has complementary function, which is dependent upon following cases:

Case-i: Overdamping

$$y_c = c_1 e^{(-\alpha+\beta)t} + c_2 e^{(-\alpha-\beta)t}$$

Case-ii: Critical Damping

$$y_c = (c_1 + c_2 t)e^{-\alpha t}$$

Case-iii: Under Damping

$$y_c = e^{-\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$$

$$= ce^{-\alpha t}\cos(\beta t - \delta)$$

where $c^2 = c_1{}^2 + c_2{}^2$, $\tan\delta = \dfrac{c_2}{c_1}$

Here, we are using method of undetermined coefficient for finding particular integral

$$y_p(t) = a\cos\omega t + b\sin\omega t\,, \tag{3}$$

$$\text{where } a = H\,\frac{m(\omega_0{}^2 - \omega^2)}{m^2(\omega_0{}^2 - \omega^2)^2 + \omega^2 c^2} \tag{4}$$

$$b = H\,\frac{\omega c}{\left(k - m\omega^2\right)^2 + \omega^2 c^2} \tag{5}$$

3. NUMERICAL SCHEME

In this section, we discuss the finite element approximation which is one of the discretization method, using the Galerkin approach for free and forced vibrations of mass-spring. Here, discretization method reduces the continuous system to a simple discrete system. For the proposed scheme initially we derive a weak form of mass-spring.

Free and Forced vibrations of mass-spring problem is of the form:

$$my'' + cy' + ky = f(t), \qquad 0 < t < L;$$

$$y(0) = y(L) = 0. \tag{6}$$

The above form of the equation is referred as the strong form.

Let $T_h : 0 = x_0 < x_1 < ... < x_M < x_{M+1} = L$ be a partition of the interval $(0, L)$ into the subinterval $I_j = \left(x_{j-1}, x_j\right)$, with length $\left|I_j\right| = h_j = x_j - x_{j-1}$, $j = 1, 2, ..., M$.

Define finite dimensional space

$$V_h^0 = \left\{ \begin{array}{r} v \in C(0,1) : v \text{ is piecewise quadratic function} \\ \text{on } \mathrm{T_h} \text{ and } v(0) = v(1) = 0 \end{array} \right\},$$

with the basis function $\left\{\phi_j\right\}_{j=1}^{M}$.

Since y is known at the boundary points 0 and L, it is not necessary to apply basis function corresponding to $x_0 = 0$ and $x_{M+1} = L$. The orthogonality condition of the residual $R(Y) = mY'' + cY' + kY - f(t)$ to the test function space V_h^0; i.e. $R(Y) \perp V_h^0$ is defined as follow:

$$\int_0^L \left(mY'' + cY' + kY - f(t)\right) v(t)dt = 0, \quad \forall v(t) \in V_h^0 \tag{7}$$

Integrating by parts gives

$$\int_0^L \left(\begin{array}{l} mY'(t)v'(t) + cY(t)v'(t) \\ \quad + kY(t)v(t) - f(t)v(t) \end{array} \right) dt = 0, \quad \sin ce\ v(0) = v(L) = 0 \tag{8}$$

Determining $\xi_j = Y(x_j)$ the approximate value at the node x_j.

Choose $Y(t) = \sum_{j=1}^{M} \xi_j \phi_j(t)$ implies $Y'(t) = \sum_{j=1}^{M} \xi_j \phi_j'(t)$.

Then, Eq. (8) can be written as

$$\sum_{j=1}^{M} \xi_j m \int_0^L \phi_j' v'(t)dt + \sum_{j=1}^{M} \xi_j c \int_0^L \phi_j v'(t)dt + \sum_{j=1}^{M} \xi_j k \int_0^L \phi_j v(t)dt = \int_0^L f(t)v(t),$$

$$\forall v(t) \in V_h^0 \tag{9}$$

Since every $v(t) \in V_h^0$ is a linear combination of basis function $\phi_j(t)$, we can write $v(t) = \phi_i(t)$, for $i = 1, 2, ..., M$.

Matrix form of the equation is

$$A\xi = B \tag{10}$$

where, A is the stiffness matrix and B is the load vector.

4. NUMERICAL VALIDATION

Here we are taking $m = 1,\ c = 3,\ k = 2\ and\ f(t) = 20\cos(2t)$ for proposed differential equation.

Below is the numerical values for the undamped system for nodes n=50

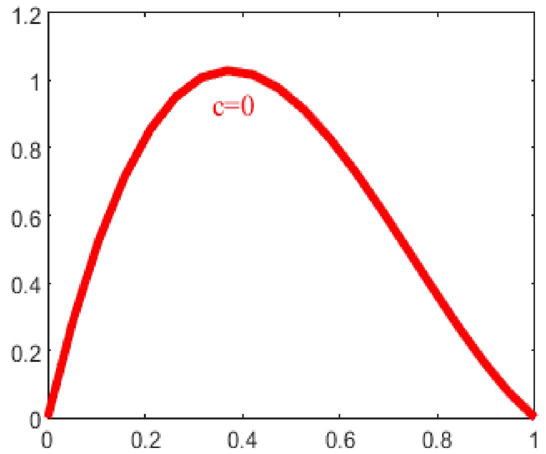

Figure 1: Undamped system for 50-nodes

0.0000	1.0207	0.9866	0.4142
0.1248	1.0499	0.9541	0.3655
0.2415	1.0730	0.9187	0.3177
0.3500	1.0902	0.8805	0.2709
0.4505	1.1017	0.8400	0.2256
0.5432	1.1077	0.7973	0.1820
0.6282	1.1084	0.7528	0.1404
0.7057	1.1041	0.7067	0.1011
0.7757	1.0950	0.6594	0.0644
0.8385	1.0814	0.6111	0.0306
0.8942	1.0635	0.5621	0.000
0.9430	1.0415	0.5128	
0.9851	1.0158	0.4634	

In the presence of damping constant, for $c = 3$ and rest of the parameters are the same as above

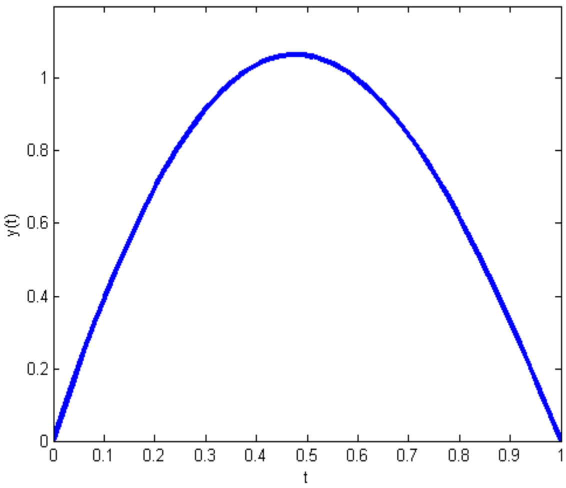

Figure 2: Damped system

-0.0000	0.8514	1.0526	0.6261
0.0859	0.8908	1.0404	0.5722
0.1688	0.9263	1.0246	0.5160
0.2485	0.9578	1.0050	0.4576
0.3249	0.9853	0.9819	0.3970
0.3979	1.0087	0.9552	0.3346
0.4674	1.0282	0.9251	0.2704
0.5334	1.0436	0.8916	0.2047
0.5958	1.0550	0.8548	0.1376
0.6545	1.0624	0.8149	0.0693
0.7095	1.0658	0.7719	0.0000
0.7607	1.0653	0.7261	
0.8080	1.0608	0.6774	

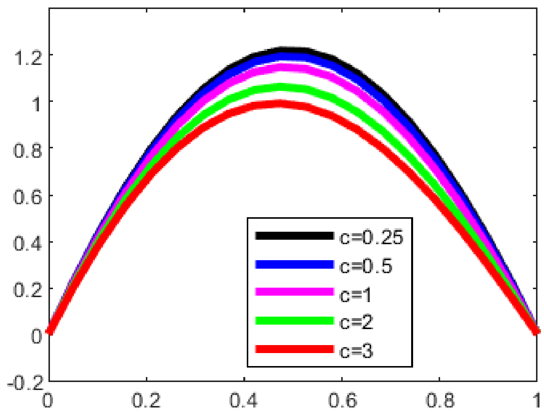

Figure 3: effect of damping constant

Damping in vibrations is shown in figure 3 that vibrations decreases with increase in damping.

5. ERROR ANALYSIS

Accuracy and error estimation is a crucial stage in establishing the acceptability of any numerical scheme, so here we derived the observed error estimates for the proposed problem in H_1, L_1 and L_2 norm which are given by

$$H_1 = \sqrt{\int_0^L (y'(x) - v'(x))^2 \, dx}$$

$$L_1 = \sum_{i=1}^{N} | y(i) - exact(x(i)) |$$

and

$$L_2 = \int_0^L \left(y(x) - exact(x) \right)^2 \, dx$$

After substituting numerical data we get, errors in terms of H_1, L_1 and L_2 norm as 0.0061, 0.3139 and 0.0051, respectively.

6. CONCLUSION

The proposed numerical approach based on FEM provides better accuracy for complicated solution of ordinary differential equations. The numerical results obtained using FEM converges to the exact solution as the number of nodes are increased and it improves the accuracy of approximate solution, moreover error terms obtained in the results are significant. Gauss Quadrature two points method is performed for the numerical integration. Main purpose of this effort is to develop a powerful and practical scheme for numerical solution of FEM which ensure the better output.

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