Miscellaneous Properties of Line Mycielskian Graph of a Graph.

Keerthi G. Mirajkar*1, Veena N. Mathad2, and Pooja B3

Department of Mathematics
1,3Karnatak University’s Karnatak Arts College, Dharwad - 580 001, Karnataka, India.
2University of Mysore, Manasagangotri, Mysore-06, India.

Abstract
The line Mycielskian graph [12] of a graph G, denoted by Lµ(G), is the graph obtained from L(G) by adding q + 1 new vertices E′ = {e′i : 1 ≤ i ≤ q} and e, then for 1 ≤ i ≤ q, joining e′i to the neighbours of ei and to e. The vertex e is called the root of Lµ(G). In this paper, we study the connectedness, connectivity, covering invariants, chromatic number, domination number of line Mycielskian graph of a graph.

Keywords: Line graph, Line Mycielskian graph, connectedness, connectivity, covering invariants.

2010 Mathematics Subject Classification: 05C76, 05C40.

1 Introduction
In this paper, we consider only simple finite and undirected graphs. For a graph G = (V,E), let V(G), E(G) denote the vertex set and edge set. If two distinct edges x and y are incident with a common vertex then they are called adjacent edges. The degree of a vertex vi in G is the number of edges incident to vi and is denoted by di = deg(vi). The minimum degree among the vertices of G is denoted by δ(G) = δ. The open–neighbourhood N(ei) of an edge ei in E(G) is the set of edges adjacent to ei. N(ei) = {ej / ei, ej are adjacent in G}. Let x be any number, then [x] denote the least integer greater than or equal to x and ⌊x⌋ denote the greatest integer less than or equal to x. For undefined terms and notations refer[6]. The line graph L(G) of a graph G is the graph with vertex set as the edge set of G and two vertices of L(G) are adjacent whenever the corresponding edges in G have a vertex in common [6]. Mycielski [13] used a fascinating construction in order to create a triangle free graphs with arbitrarily large chromatic numbers. Mycielski [2] introduced the graph–transformation as follows:

Let G be a graph with vertex set V = {vi : 1 ≤ i ≤ p}. The Mycielski graph of G, denoted by µ(G), is the graph obtained from G by adding p + 1 new vertices V′ = {v′i : 1 ≤ i ≤ p} and u, then for 1 ≤ i ≤ p, joining v′i to the neighbours of vi and to u. vi and v′i are known as twin–vertices, and V and V′ are known as twin–sets in µ(G). The vertex u is called the root of µ(G). Clearly, V[µ(G)] = V ∪ V′ ∪ {u}. The beauty of Mycielski graph µ(G) is that it transforms the triangle–free graph G into a triangle–free graph µ(G), and it produces three new triangles for every triangle of G.
Motivated by this concept on the same lines the concept of line Mycielskian graph of a graph is introduced in [12] and is defined as follows:

Let \( G \) be a graph with edge set \( E = \{e_i : 1 \leq i \leq q\} \). The line Mycielskian graph of a graph \( G \), denoted by \( L(\mu)(G) \), is the graph obtained from \( L(G) \) by adding \( q + 1 \) new vertices \( E' = \{e'_i : 1 \leq i \leq q\} \) and \( e \), then for \( 1 \leq i \leq q \), joining \( e'_i \) to the neighbours of \( e_i \) and to \( e \). The vertex \( e \) is called the root of \( L(\mu)(G) \). Clearly, \( V[L(\mu)(G)] = E \cup E' \cup \{e\} \). The line graph \( L(G) \) is an induced subgraph of \( L(\mu)(G) \).

Recently, there has been an increasing interest in the study of Mycielskians graph, especially, in the study of their circular chromatic numbers [2, 7, 8, 9, 10] and which are also studied in [1, 3, 4, 11, 13, 14]. In this paper, we study the connectedness, connectivity, covering invariants, chromatic number, domination number of line Mycielskian graph of a graph.

**Remark 1.1.** [12] \( L(G) \) is an induced subgraph of \( L(\mu)(G) \).

### 2 Connectedness of Line Mycielskian graph of a graph

In the following theorem, we study when line Mycielskian graph \( L(\mu)(G) \) is connected and disconnected.

**Theorem 2.1.** The line Mycielskian graph \( L(\mu)(G) \) of a graph \( G \) is disconnected iff \( G = K_2 \).

**Proof.** Suppose \( G = K_2 \), then \( L(G) \) has exactly one vertex. By definition, in \( L(\mu)(K_2) \)

\[
\begin{align*}
K_2: & \quad e_1 \\
L(\mu)(K_2): & \quad e_1 \quad e
\end{align*}
\]

for each edge \( e_1 \) of \( G \), a new vertex \( e'_1 \) is taken and a new vertex \( e \) is introduced which is adjacent to new point \( e'_1 \). Further \( e_1 \) is not adjacent to either \( e'_1 \) nor \( e \) and results in a disconnected graph of order 3. By Remark 1.1, it is clear that, \( L(\mu)(G) = L(G) \cup K_2 \). Hence \( L(\mu)(G) \) is disconnected. Conversely, assume \( L(\mu)(G) \) is disconnected \( L(\mu)(G) = L(G) \cup K_2 \).

Suppose \( G \) is not \( K_2 \) and assume that \( G \) has at least two edges. Then \( L(G) \cup K_2 \) has \( q + 2 \) vertices where as \( L(\mu)(G) \) has \( 2q + 1 \) vertices, \( q > 1 \). Hence \( L(G) \cup K_2 \) has less number of vertices than \( L(\mu)(G) \). Clearly \( L(\mu)(G) \neq L(G) \cup K_2 \), a contradiction. Hence \( G \) must be \( K_2 \).

**Theorem 2.2.** For any connected \((p, q)\) graph \( G \) with \( p \geq 3 \), \( L(\mu)(G) \) is connected.

**Proof.** Let \( G \) be a connected graph with \( p \geq 3 \) vertices. Let \( V[L(\mu)(G)] = E_1 \cup E_2 \cup \{e\} \) where \( (E_1) = L(G) \) and \( E_2 \) is the set of newly introduced vertices such that \( e_i \) implies \( e'_i \) is a bijective map from \( E_1 \) onto \( E_2 \) satisfying \( N(e'_i) = N(e_i) \cap E_1 \), for all \( e_i \in E_1 \) and the vertex \( e \) is called the root of \( L(\mu)(G) \). Let \( a, b \in V(L(\mu)(G)) \).

We consider the following cases:

**Case 1.** \( a, b \in E_1 \). Since \( G \) is a connected graph with \( p \geq 3 \), \( L(G) \) is a nontrivial connected graph. Since by Remark 1.1, there exists an \( a - b \) path in \( L(\mu)(G) \).

**Case 2.** \( a \in E_1 \) and \( b \in E_2 \). Let \( e \in E_1 \) be such that \( N(b) = N(e) \cap E_1 \). Choose \( w \in N(b) \). Since \( a \) and \( w \in E_1 \), as in Case 1, \( a \) and \( w \) are joined by a path in \( L(\mu)(G) \). Hence \( a \) and \( b \) are connected by a path in \( L(\mu)(G) \).

**Case 3.** \( a, b \in E_2 \). As in Case 2, there exists \( c \) and \( d \) in \( E_1 \) such that \( c \in N(a) \) and \( d \in N(b) \). Consequently, \( ca, db \in E[L(\mu)(G)] \). Also \( c \) and \( d \) are joined by a path in \( L(\mu)(G) \). Hence \( a \) and \( b \) are connected by a path in \( L(\mu)(G) \).

**Case 4.** \( a \in E_2 \) and a root vertex \( \{e\} \). By definition of \( L(\mu)(G) \), all points of \( E_2 \) are joined by a root vertex \( e \). Hence there exists a path from the vertices of \( E_2 \) to the root vertex of \( L(\mu)(G) \).

In all the cases, \( a \) and \( b \) are connected by a path in \( L(\mu)(G) \). Thus \( L(\mu)(G) \) is connected.

### 3 Connectivity and Edge Connectivity of line Mycielskian graph of a graph

**Connectivity or vertex–connectivity** is a minimum number of vertices whose removal from \( G \) results into a disconnected or trivial graph and is denoted by \( \kappa(G) \). **Edge–connectivity** of a graph is the minimum number of edges whose removal from \( G \) results into a disconnected or trivial graph and is denoted by \( \lambda(G) \).

In the following theorem, we determine vertex–connectivity of line Mycielskian graph \( \kappa(L(\mu)(G)) \) and the edge–connectivity of line Mycielskian graph \( \lambda(L(\mu)(G)) \) of a graph.

**Theorem 3.1.** For a connected \((p, q)\) graph \( G \), the vertex–connectivity of line Mycielskian graph of a graph is

\[
\kappa(L(\mu)(G)) = \min \{2\kappa(L(G)) + 1, \delta(L(G)) + 1\}.
\]
Proof. From Whitney’s result, we have

$$\kappa(L_\mu(G)) \leq \lambda(L_\mu(G)) \leq \delta(L_\mu(G))$$

and

$$\kappa(L(G)) \leq \lambda(L(G)) \leq \delta(L(G)).$$

By Remark 1, we have, $$\kappa(L_\mu(G)) \geq \kappa(L(G)).$$

Case 1. If $$\kappa(L(G)) = 0$$, then obviously $$\kappa(L_\mu(G)) = 0.$$

Case 2. If $$\kappa(L(G)) = 1$$, then $$L(G) = K_2$$ or it is a connected graph with a cutvertex $$e_i.$$ We have the following subcases:

Subcase 2.1. If $$L(G) = K_2.$$ Then $$L_\mu(G) = C_5.$$ Consequently, $$\kappa(L_\mu(G)) = \delta(L_\mu(G)) + 1 = 2.$$

Subcase 2.2. $$L(G)$$ is connected with a cutvertex $$e_i.$$ If $$\delta(L(G)) = 1$$, then let $$e_j$$ be a pendant vertex of $$L(G)$$ which is adjacent to $$e_i.$$ Then $$e'_i$$ is a vertex of $$L_\mu(G)$$ such that $$\deg_{L_\mu(G)}(e'_i) = 2$$ and so, removal of $$e'_i$$ results in a disconnected graph with root vertex $$e$$ as a cutvertex. So removal of root vertex $$e$$ results in a disconnected graph. Hence $$\kappa(L_\mu(G)) = \delta(L(G)) + 1.$$ If $$\delta(L(G)) \geq 2$$, then the removal of a cutvertex $$e_i$$ of $$L(G)$$ and its corresponding vertex $$e'_i$$ and the root vertex $$e$$ from $$L_\mu(G)$$ results in a disconnected graph. Hence $$\kappa(L_\mu(G)) = 2\kappa(L(G)) + 1.$$ Now, suppose $$\kappa(L(G)) = n$$, where $$n$$ is an integer. Then $$L(G)$$ has a minimum vertex–cut $$\{e_1 : 1 \leq l \leq n\}$$ whose removal from $$L_\mu(G)$$ results in a disconnected graph.

There are two types of vertex–cuts in $$L_\mu(G)$$ depending on the structure of $$L(G).$$

1. vertex–cut containing exactly $$2n + 1$$ vertices, that is, $$\{e_1, e'_1, e : 1 \leq l \leq n\}$$ whose removal increases the number of components of $$L_\mu(G).$$

2. vertex–cut containing $$\delta(L(G)) + 1$$ vertices.

Thus, $$\kappa(L_\mu(G)) = \begin{cases} 2n + 1, & \text{if } n \leq \frac{\delta(L(G)) + 1}{2}; \\ \delta(L(G)) + 1, & \text{otherwise}. \end{cases}$$

Hence,

$$\kappa(L_\mu(G)) = \min\{2\kappa(L(G)) + 1, \delta(L(G)) + 1\}.$$  \hfill \Box$

Theorem 3.2. If $$G$$ is a $$(p,q)$$ connected graph, then the edge–connectivity of line Mycielskian of a graph $$G$$ is given as

$$\lambda(L_\mu(G)) = \begin{cases} (\delta(L(G)) + 1) \lambda(L(G)), & \text{if } n \leq \delta(L(G)) + 1; \\ \delta(L(G)) + 1, & \text{otherwise}. \end{cases}$$

Proof. From Whitney’s result we have

$$\kappa(L_\mu(G)) \leq \lambda(L_\mu(G)) \leq \delta(L_\mu(G))$$

and by Remark 1, we have $$\lambda(L_\mu(G)) \geq \lambda(L(G)).$$

We now consider the following cases.

Case 1. If $$\lambda(L(G)) = 0,$$ then obviously $$\lambda(L_\mu(G)) = 0.$$

Case 2. If $$\lambda(L(G)) = 1,$$ then $$L(G) = K_2$$ or it is a connected graph with a bridge $$x = e_\ell e_j,$$ say.

We have the following subcases of this case.

Subcase 2.1. If $$L(G) = K_2.$$ Then $$L_\mu(G) = C_5.$$ Consequently, $$\lambda(L_\mu(G)) = \delta(L_\mu(G)) + 1 = 2.$$

Subcase 2.2. $$L(G)$$ is connected with a bridge $$e_\ell e_j.$$ If $$e_i$$ is a pendant vertex in $$L(G)$$ then $$L_\mu(G)$$ is a connected graph having a vertex $$e'_i$$ with a degree $$\deg_{L_\mu(G)}(e'_i) = 2.$$ The removal of edges incident with $$e'_i$$ disconnects $$L_\mu(G).$$ Thus, $$\lambda(L_\mu(G)) = \delta(L_\mu(G)) + 1 = 2.$$ If neither $$e_i$$ nor $$e_j$$ is a pendant vertex in $$L(G),$$ then $$\delta(L_\mu(G)) = 3$$ and let $$e_k$$ be a vertex of $$L_\mu(G)$$ with $$\deg_{L_\mu(G)}(e_k) = 3.$$ In $$L_\mu(G),$$ there are only three edges incident with $$e_k$$ and the removal of these edges disconnects $$L_\mu(G).$$ Hence, $$\lambda(L_\mu(G)) = \delta(L_\mu(G)) + 1.$$ If $$\delta(L(G)) = m \geq 3,$$ where $$m$$ is an integer. Then let $$e_k$$ be a vertex in $$L(G)$$ with $$\deg_{L(G)}(e_k) = \delta(L(G)) = m.$$ Then, $$\deg_{L_\mu(G)}(e_k) = m + 1$$ and the removal of the set $$\{e_k e'_k, e_k e''_k : 1 \leq l \leq m\}$$ results in a disconnected graph, so that

$$\lambda(L_\mu(G)) = (m + 1)\lambda(L(G)) = (\delta(L(G)) + 1) \lambda(L(G)).$$

Case 3. If $$\lambda(L(G)) = n,$$ where $$n$$ is an integer. Then $$L(G)$$ has minimum edge–cut $$\{e_1 : e_1 = u_1 v_1, 1 \leq l \leq n\}$$ whose removal makes $$L(G)$$ disconnect. As above, there are two types of edge–cuts in $$L_\mu(G)$$ depending on the structure of $$L(G).$$ Among these, one edge–cut contains exactly $$\delta(L(G)) + 1$$ $$\lambda(L(G))$$ edges whose removal increases the number of components of $$L_\mu(G)$$ and the other is $$\delta(L(G)) + 1$$ edge cuts.

Thus, we have

$$\lambda(L_\mu(G)) = \begin{cases} (\delta(L(G)) + 1) \lambda(L(G)), & \text{if } n \leq \delta(L(G)) + 1; \\ \delta(L(G)) + 1, & \text{otherwise}. \end{cases}$$

\hfill \Box

4 Covering invariants of line Mycielskian graph of a graph

A vertex and an edge are said to cover each other if they are incident. A set of vertices in a graph $$G$$ is a vertex covering set, which covers all the edges of $$G.$$ The vertex covering number $$\alpha_0(G)$$ of $$G$$ is the minimum number of vertices in a vertex covering set of $$G.$$ A set of edges in a graph $$G$$ is an edge covering set, which covers all vertices of $$G.$$ The edge covering number $$\alpha_1(G)$$ of $$G$$ is the minimum number of edges in an edge covering set of $$G.$$ A set of vertices in a
Proof. (i) By Remark 1.1, we have $q + 1 \leq \alpha_1(L_\mu(G)) \leq 2 \left[ \frac{q}{2} \right] + 1$ and
$$2 \left[ \frac{q}{2} \right] + 1 \leq \beta_1(L_\mu(G)) \leq q + 1.$$ 

(i) By Remark 1.1, we have $q + 1 \leq \alpha_1(L_\mu(G)) \leq 2 \left[ \frac{q}{2} \right] + 1$ and
$$2 \left[ \frac{q}{2} \right] + 1 \leq \beta_1(L_\mu(G)) \leq q + 1.$$ 

Proof. Let $G$ be a connected $(p, q)$ graph $G$ with $p \geq 3$ vertices and $e_1, e_2, e_3, \ldots, e_q$ edges. Let $E_1 = e_1, e_2, e_3, \ldots, e_q$ be the set of newly introduced vertices in the construction of $L_\mu(G)$. For each pair $\{e_i, e_j\}$ of adjacent edges of $G$, we have an edge $e_i e_j$ in $L(G)$. Corresponding to this edge $e_i e_j$, there are edges $e_i e_j, e_i e_j', e_j e_j$ in $L_\mu(G)$. Among these, $e_i e_j'$ and $e_j e_j'$ are independent in $L_\mu(G)$. Thus, each pair of adjacent edges of $G$ gives rise to two independent edges in $L_\mu(G)$. That is, each edge of $L_\mu(G)$ gives rise to two independent edges in $L_\mu(G)$. So, $\beta_1(L_\mu(G))$ independent edges of $L(G)$ give rise to $2 \beta_1(L_\mu(G))$ independent edges in $L_\mu(G)$. Hence $\beta_1(L_\mu(G)) \geq 2 \beta_1(L_\mu(G))$. By Theorem 4.1, it follows that $2 \left[ \frac{q}{2} \right] + 1 \leq \beta_1(L_\mu(G))$.

(ii) $L_\mu(G)$ has $q + 1$ vertices and $\alpha_1(L_\mu(G)) + \beta_0(L_\mu(G)) = 2q + 1$. By substitution of $\alpha_0(L_\mu(G))$ from (i), we have
$$\beta_0(L_\mu(G)) = \max \{2q + 1, 2\beta_0(L_\mu(G)) + 1\}.$$

Theorem 4.3. For any connected $(p, q)$ graph $G$ with $p \geq 3$, we have
$$q + 1 \leq \alpha_1(L_\mu(G)) \leq 2 \left[ \frac{q}{2} \right] + 1 \text{ and } 2 \left[ \frac{q}{2} \right] + 1 \leq \beta_1(L_\mu(G)) \leq q + 1.$$ 

Proof. Let $G$ be a connected $(p, q)$ graph $G$ with $p \geq 3$ vertices and $e_1, e_2, e_3, \ldots, e_q$ edges. Let $E_1 = e_1, e_2, e_3, \ldots, e_q$ be the set of newly introduced vertices in the construction of $L_\mu(G)$. For each pair $\{e_i, e_j\}$ of adjacent edges of $G$, we have an edge $e_i e_j$ in $L(G)$. Corresponding to this edge $e_i e_j$, there are edges $e_i e_j, e_i e_j', e_j e_j$ in $L_\mu(G)$. Among these, $e_i e_j'$ and $e_j e_j'$ are independent in $L_\mu(G)$. Thus, each pair of adjacent edges of $G$ gives rise to two independent edges in $L_\mu(G)$. That is, each edge of $L_\mu(G)$ gives rise to two independent edges in $L_\mu(G)$. So, $\beta_1(L_\mu(G))$ independent edges of $L(G)$ give rise to $2 \beta_1(L_\mu(G))$ independent edges in $L_\mu(G)$. Hence $\beta_1(L_\mu(G)) \geq 2 \beta_1(L_\mu(G))$. By Theorem 4.1, it follows that $2 \left[ \frac{q}{2} \right] + 1 \leq \beta_1(L_\mu(G))$. Since $L_\mu(G)$ has $2q + 1$ vertices and $\alpha_1(L_\mu(G)) + \beta_0(L_\mu(G)) = 2q + 1$, we have $\alpha_0(L_\mu(G)) \leq 2 \left[ \frac{q}{2} \right] + 1$. Now, if $L(G)$ contains a spanning odd cycle $C_q : e_1 e_2 \ldots e_q e_1; q = 2k + 1, k \geq 1$, then the subset $N(G)$ of $E(L_\mu(G))$, where $N = \{e_1, e_2, e_3, \ldots, e_{q-1} e_q' e_1\}$ also forms an edge cover of $L_\mu(G)$ with $|N| = q$. In this case, $q < 2 \alpha_1(L(G))$. Also, $L_\mu(G)$ has at least $q$ independent edges of $E(L_\mu(G))$ to cover these independent vertices in $L_\mu(G)$. Therefore, $\alpha_1(L_\mu(G))$ cannot be less than $q + 1$. Hence $q + 1 \leq \alpha_1(L_\mu(G))$.

Thus we have,
$$q + 1 \leq \alpha_1(L_\mu(G)) \leq 2 \left[ \frac{q}{2} \right] + 1 \text{ and } 2 \left[ \frac{q}{2} \right] + 1 \leq \beta_1(L_\mu(G)) \leq q + 1.$$ 

5 Domination number and Chromatic number of line Mycielskian graph of a graph

A set $D$ of vertices in a graph $G$ is a dominating set if every vertex in $V(G) - D$ is adjacent to at least one vertex of $D$. A dominating set $D$ is said to be a minimal dominating set, if no proper subset of $D$ is a dominating set. The minimum cardinality of a dominating set is called the domination number of $G$ and is denoted by $\gamma(G)$. A set $S$ of edges of a graph $G$ is an edge dominating set if every edge in $E(G) - S$ is adjacent to at least one edge of $S$. A coloring of a graph is an assignment of colors to its vertices(or edges) so that no two adjacent vertices(or adjacent edges) have the same color. The chromatic number $\chi(G)$ of a graph $G$ is the minimum number of colors required to assign the colors to the vertices of $G$ in such a way that no two adjacent vertices of $G$ receive the same color. The edge chromatic number $\chi'(G)$ of a graph $G$ is the minimum number of colors required to assign the colors to the edges of $G$ in such a way that no two adjacent edges of $G$ receive the same color.
to the edges of $G$ in such a way that no two adjacent edges of $G$ receive the same color.

In this section, we derive an expression for domination number of line Mycielskian graph of a graph $\gamma(L_\mu(G))$ and chromatic number of line Mycielskian graph of a graph $\chi(L_\mu(G))$.

**Theorem 5.1.** If $G$ is a $(p, q)$ connected graph with $p \geq 3$, then

$$\gamma(L_\mu(G)) = \gamma(L(G)) + 1.$$  

**Proof.** Since $V[L_\mu(G)] = E(G) \cup E_1(G) \cup \{e\}$ and $D$ be a minimum edge dominating set of $G$. Since the root vertex $e$ is adjacent to every vertex of $E_1(G)$ in $L_\mu(G)$, the set $D \cup \{e\}$ is a minimum dominating set of $L_\mu(G)$. Hence,

$$\gamma(L_\mu(G)) = \gamma(L(G)) + 1.$$  

**Theorem 5.2.** For a connected $(p, q)$ graph $G$ with order $p \geq 3$, the chromatic number of line Mycielskian graph of a graph is

$$\chi(L_\mu(G)) = \chi(L(G)) + 1.$$  

**Proof.** Let $G$ be a connected $(p, q)$ graph. For each edge $e_i$ in $G$, let $e'_i$ be the new vertex chosen in the construction of $L_\mu(G)$. Since $\chi^{-1}(G) = \chi(L(G))$, $\chi^{-1}(G)$ coloring of $G$ can be extended to $\chi(L_\mu(G))$ coloring of $L(G)$. Also, we can assign the same color to an edge $e_i$ and $e'_i$ in $L_\mu(G)$. Hence,

$$\chi(L_\mu(G)) = \chi(L(G)) + 1.$$  

6 Acknowledgement

This research is supported by University Research Studentship (URS), No.KU/Sch/URS/2017-18/467, dated 3rd July 2018, Karnatak University Dharwad, Karnataka, India.

References


