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Solving Integro-Differential Equations by Using Numerical Techniques

Ahmed A. Hamoud *[1], **Nedal M. Mohammed**[2], **Kirtiwant P. Ghadle**[3], **and Satyawan L. Dhondge**[4]

[1,3]*Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, (M.S), India.*
[2]*Department of Computer Science, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, (M.S), India.*
[4]*Department of Mathematics, Deogiri Institute of Engineering & Management Studies, Aurangabad, (M.S), India.*

Abstract
In this article, we discussed different methods for solving mixed Volterra-Fredholm integro-differential equations, namely: Adomian decomposition method and modified decomposition method. Moreover, we prove the uniqueness results and convergence of the techniques. Finally, an example is included to demonstrate the validity and applicability of the proposed techniques.
Keywords: Adomian decomposition method, modified decomposition method, Volterra-Fredholm integro-differential equation, approximate solution.

1. INTRODUCTION

In recent years there has been a growing interest in the integro-differential equation. The integro-differential equations be an important branch of modern mathematics. It arises frequently in many applied areas which include engineering, electrostatics, mechanics, the theory of elasticity, potential, and mathematical physics [3, 4, 6, 10, 28].
In this work, we consider the mixed Volterra-Fredholm integro-differential equation of the second kind as follows:

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \int_a^x \int_\Omega K(x,t)G(t,u^{(l)}(t))dtdx, \tag{1}$$

with the initial conditions

$$u^{(r)}(a) = b_r, \quad r = 0,1,2,\ldots,(k-1), \tag{2}$$

where $u^{(j)}(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $a \leq x \leq b, \quad \Omega = [a,b]$, $f(x)$ and $p_j(x)$ are analytic functions, $G(t,u^{(l)}(t)), l \geq 0$ is nonlinear analytic function of u, and $b_r, \quad 0 \leq r \leq (k-1)$ are real finite constants.
Recently, Wazwaz (2001) presented an efficient and numerical procedure for solving boundary value problems for higher-order integro-differential equations. A variety of methods, exact, approximate and purely numerical techniques are available to solve nonlinear integro-differential equations. These methods have been of great interest to several authors and used to solve many nonlinear problems. Some of these techniques are Adomian decomposition method [4, 24], modified Adomian decomposition method [29], Variational iteration method [7, 31, 32] and homotopy perturbation method [29] and many methods for solving integro-differential equations [3, 2, 6, 16, 17, 18, 19, 20, 21, 27].

More details about the sources where these equations arise can be found in physics, biology, and engineering applications as well as in advanced integral equations. Some works based on an iterative scheme have been focusing on the development of more advanced and efficient methods for integro-differential equations such as the variational iteration method which is a simple and Adomian decomposition method [8, 9, 24, 30], and the modified decomposition method for solving Volterra-Fredholm integral and integro-differential equations which is a simple and powerful method for solving a wide class of nonlinear problems [24]. The Taylor polynomial solution of integro-differential equations has been studied in [28]. The use of Lagrange interpolation in solving integro-differential equations was investigated by Marzban [26]. The VIM has been successfully applied for solving integral and integro-differential equations [5, 24, 29].

A variety of powerful methods has been presented, such as the homotopy analysis method [29], homotopy perturbation method [6, 29], operational matrix with Block-Pulse functions method [3], variational iteration method [5, 29] and the Adomian decomposition method [4, 24, 29]. Some fundamental works on various aspects of modifications of the Adomian's decomposition method are given by Araghi [1]. The modified form of Laplace decomposition method has been introduced by Manafianheris [25]. Babolian et. al, [3], applied the new direct method to solve nonlinear Volterra-Fredholm integral and integro-differential equation using operational matrix with block-pulse functions. The Laplace transform method with the Adomian decomposition method to establish exact solutions or approximations of the nonlinear Volterra integro-differential

*Corresponding Author's Email: maths.aah@bamu.ac.in

equations, Wazwaz [30]. Recently, the authors have used several methods for the numerical or the analytical solutions of linear and nonlinear Volterra and Fredholm integro-differential equations [11, 12, 13, 14, 15, 22, 23, 24, 29].

In this work, our aim is to solve a general form of nonlinear Volterra-Fredholm integro-differential equations using four approximate methods, namely, Adomian decomposition method and modified Adomian decomposition method. Also, we prove the existence and uniqueness results and convergence of the techniques.

2. NONLINEAR MIXED VOLTERRA-FREDHOLM INTEGRO DIFFERENTIAL EQUATION OF SECOND KIND

We consider the nonlinear mixed Volterra-Fredholm integro-differential equation of the second kind as follows:

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \int_a^x \int_\Omega K(x,t)G(t,u^{(l)}(t))dtdx. \tag{3}$$

We can rewrite Eq.(3) as follows:

$$\begin{aligned} & p_k(x)u^{(k)}(x) + \sum_{j=0}^{k-1} p_j(x)u^{(j)}(x) \\ = \quad & f(x) + \int_a^x \int_\Omega K(x,t)G(t,u^{(l)}(t))dtdx, \end{aligned}$$

$$\begin{aligned} u^{(k)}(x) &= \frac{f(x)}{p_k(x)} + \int_a^x \int_\Omega \frac{K(x,t)G(t,u^{(l)}(t))}{p_k(t)}dtdx \\ & -\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)}u^{(j)}(x). \end{aligned} \tag{4}$$

Let us set L^{-1} is the multiple integration operator as follows:

$$L^{-1}(.) := \int_a^x \int_a^x \cdots \int_a^x \int_a^x (.) \underbrace{dxdx \ldots dxdx}_{k-times}. \tag{5}$$

From Eq.(4) and Eq.(5)

$$\begin{aligned} u(x) &= L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!}b_r \\ & +L^{-1}\{\int_a^x \int_\Omega \frac{K(x,t)G(t,u^{(l)}(t))}{p_k(t)}dtdx\} \\ & - L^{-1}\{\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)}u^{(j)}(x)\}. \end{aligned} \tag{6}$$

We can obtain the term $\sum_{r=0}^{k-1} \frac{(x-a)^r}{r!}b_r$ from the initial conditions. From [8], we have

$$\begin{aligned} & L^{-1}\{\int_a^x \int_\Omega \frac{K(x,t)G(t,u^{(l)}(t))}{p_k(t)}dtdx\} \\ = \quad & \int_a^x \int_\Omega \frac{(x-t)^k K(x,t)G(t,u^{(l)}(t))}{(k!) \quad p_k(t)}dtdx, \end{aligned} \tag{7}$$

also

$$\begin{aligned} & L^{-1}\{\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)}u^{(j)}(x)\} \\ = \quad & \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1}p_j(t)}{(k-1)! \quad p_k(t)}u^{(j)}(t)dt \end{aligned} \tag{8}$$

By substituting Eq.(7)and Eq.(8) in Eq.(6) we obtain

$$\begin{aligned} u(x) &= L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!}b_r \\ & + \int_a^x \int_\Omega \frac{(x-t)^k K(x,t)G(t,u^{(l)}(t))}{(k!) \quad p_k(t)}dtdx \\ & - \sum_{j=0}^{k-1} \int_a^x \frac{(x-t)^{k-1}p_j(t)}{(k-1)! \quad p_k(t)}u^{(j)}(t)dt. \end{aligned} \tag{9}$$

We set,

$$L^{-1}\{\frac{f(x)}{p_k(x)}\} + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!}b_r = F(x),$$

$$\int_\Omega \frac{(x-t)^k K(x,t)}{(k!) \quad p_k(t)}dx = K_1(x,t),$$

$$\frac{(x-t)^{k-1}p_j(x)}{(k-1)! \quad p_k(x)} = K_2(x,t).$$

So, we have one-dimensional nonlinear integro-differential equation as follows:

$$\begin{aligned} u(x) &= F(x) + \int_a^x K_1(x,t)G(t,u^{(l)}(t))dt \\ & -\sum_{j=0}^{k-1} \int_a^x K_2(x,t)u^{(j)}(t)dt. \end{aligned} \tag{10}$$

3. DESCRIPTION OF THE METHODS

3.1 Adomian Decomposition Method (ADM)

The ADM is applied to the following general nonlinear equation [1, 29]:

$$Lu + Ru + Nu = g(x), \tag{11}$$

where u is the unknown function, L is the highest-order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L and Nu represents the nonlinear terms and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(11) and using the given conditions we obtain

$$u = F(x) - L^{-1}(Ru) - L^{-1}(Nu), \tag{12}$$

where the function $F(x)$ represents the terms arising from integrating the source term $g(x)$. The nonlinear operator $Nu = G(u)$ is decomposed as

$$G(u) = \sum_{n=0}^{\infty} A_n, \tag{13}$$

where A_n; $n > 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i u_i)]]_{\lambda=0}.$$

The Adomian polynomials were introduced in [29] as:

$$A_0 = G(u_0);\;\; A_1 = u_1 G^{'}(u_0);\;\; A_2 = u_2 G^{'}(u_0) + \frac{1}{2}u_1^2 G^{''}(u_0)$$

$$A_3 = u_3 G^{'}(u_0) + u_1 u_2 G^{''}(u_0) + \frac{1}{3}u_1^3 G^{'''}(u_0), ...$$

The standard decomposition technique represents the solution of u in Eq.(11) as the following series:

$$u = \sum_{n=0}^{\infty} u_n, \tag{14}$$

where, the components $u_0, u_1, ...$ are usually determined recursively by

$$\begin{aligned} u_0 &= F(x), \qquad (15)\\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n). \end{aligned}$$

Substituting (12) into (15) leads to the determination of the components of u. Having determined the components $u_1, u_2, ...$, the solution u in a series form defined by Eq.(14).
Now, we apply ADM to find the approximate solutions of Eq.(10), we can write the iterative formula as follows:

$$\begin{aligned} u_0(x) &= F(x), \qquad (16)\\ u_{n+1}(x) &= \int_a^x K_1(x,t)A_n dt - \sum_{j=0}^{k-1}\int_a^x K_2(x,t)B_{n_j} dt. \end{aligned}$$

The nonlinear terms $G(t, u^{(l)}(t))$ and $D^j(u(x))$, ($D^j = \frac{\partial^j}{\partial x^j}$ is derivative operator), are usually represented by an infinite series of the so called Adomian polynomials as follows:

$$G(t, u^{(l)}(t)) = \sum_{i=0}^{\infty} A_i, \quad D^j(u(x)) = \sum_{i=0}^{\infty} B_{i_j},$$

where A_i and B_{i_j} $(i \geq 0, j = 0, 1, \ldots, k-1)$ are the Adomian polynomials.

3.2 Modified Adomian Decomposition Method (MADM)

The modified decomposition method was introduced by Wazwaz [29]. This method is based on the assumption that the function $F(x)$ can be divided into two parts, namely $F_1(x)$ and $F_2(x)$. Under this assumption we set

$$F(x) = F_1(x) + F_2(x).$$

Consequently, the following modified recursive relation was developed:

$$\begin{aligned} u_0 &= F_1(x),\\ u_1 &= F_2(x) - L^{-1}(Ru_0) - L^{-1}(A_0),\\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 1. \end{aligned}$$

Now, we apply MADM to find the approximate solutions of Eq.(10), we can write the iterative formula as follows:

$$\begin{aligned} u_0(x) &= F_1(x),\\ u_1(x) &= F_2(x) + \int_a^x K_1(x,t)A_0 dt\\ &\quad - \sum_{j=0}^{k-1}\int_a^x K_2(x,t)B_{0_j} dt, \qquad (17)\\ u_{n+1}(x) &= \int_a^x K_1(x,t)A_n dt - \sum_{j=0}^{k-1}\int_a^x K_2(x,t)B_{n_j} dt. \end{aligned}$$

4. UNIQUENESS AND CONVERGENCE RESULTS

In this section the existence and uniqueness of the obtained solution and convergence of the methods are proved. Consider the Eq.(10), we assume $F(x)$ is bounded for all x in J and In Eq.(10), we assume $F(x)$ is bounded for all x in Ω and

$$\Big|K_1(x,z)\Big| \leq M_1, \;\; \Big|K_2(x,z)\Big| \leq M_{1j}, \quad j = 0, 1, \ldots, k-1.$$

Also, we suppose the nonlinear terms $G(u(x))$ and $D^j(u(x))$ are Lipschitz continuous with

$$\Big|G(x, u(x)) - G(x, u^*(x))\Big| \leq d\Big|u(x) - u^*(x)\Big|$$
$$\Big|D^j(u(x)) - D^j(u^*(x))\Big| \leq C_j\Big|u(x) - u^*(x)\Big|.$$

If we set,

$$\gamma = (b-a)(dM_1 + kCM), \quad C = max\Big|C_j\Big|, \quad M = max\Big|M_{1_j}\Big|.$$

Then the following theorems can be proved by using the above assumptions.
Theorem 1. Assume that the above assumptions are hold, and $0 < \gamma < 1$. Then Eq.(10) has a unique solution.

Proof. Let u and u^* be two different solutions of Eq.(10) then

$$\begin{aligned}
&\Big| u(x) - u^*(x) \Big| \\
= & \Big| \int_a^x K_1(x,t) G(t, u^{(l)}(t)) dt - \sum_{j=0}^{k-1} \int_a^x K_2(x,t) D^j(u(t)) dt \\
- & \int_a^x K_1(x,t) G(t, u^{(l)^*}(t)) dt + \sum_{j=0}^{k-1} \int_a^x K_2(x,t) D^j(u^*(t)) dt \Big| \\
\leq & \Big| \int_a^x K_1(x,t) \Big[G(t, u^{(l)}(t)) - G(t, u^{(l)^*}(t)) \Big] dt \\
- & \sum_{j=0}^{k-1} \int_a^x K_2(x,t) \Big[D^j(u(t)) - D^j(u^*(t)) \Big] dt \Big| \\
\leq & \int_a^x \Big| K_1(x,t) \Big| \Big| G(t, u^{(l)}(t)) - G(t, u^{(l)^*}(t)) \Big| dt \\
+ & \sum_{j=0}^{k-1} \int_a^x \Big| K_2(x,t) \Big| \Big| D^j(u(t)) - D^j(u^*(t)) \Big| dt \\
\leq & M_1 d \Big| u(x) - u^*(x) \Big| (b-a) + kMC \Big| u(x) - u^*(x) \Big| (b-a) \\
\leq & (b-a)(M_1 d + kMC) \Big| u(x) - u^*(x) \Big| \\
= & \gamma \Big| u(x) - u^*(x) \Big| .
\end{aligned}$$

So,

$$\Big| u(x) - u^*(x) \Big| \leq \gamma \Big| u(x) - u^*(x) \Big|,$$

from which we get $(1-\gamma) \Big| u - u^* \Big| \leq 0$. Since $0 < \gamma < 1$, so $\Big| u - u^* \Big| = 0$. Therefore, $u = u^*$, and this completes the proof.

Theorem 2. If the series solution $u(x) = \sum_{i=0}^{\infty} u_i(x)$ obtained by the using ADM or MADM is convergent, then it converges to the exact solution of the Eq.(10) when $0 < \gamma < 1$ and $\|u_1(x)\| < \infty$.

Proof. Denote as $(C[J], \|.\|)$ the Banach space of all continuous functions on J with the norm $\|f(x)\| = max \Big| f(x) \Big|$ for all x in J. Define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that $s_n = \sum_{i=0}^{n} u_i(x)$ is a Cauchy sequence in this Banach space [1]:

$$\begin{aligned}
\|s_n - s_m\| &= \max_{\forall x \in J} \Big| s_n - s_m \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=0}^{n} u_i(x) - \sum_{i=0}^{m} u_i(x) \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} u_i(x) \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} \Big[\int_a^x K_1(x,t) A_i dt \\
&\quad - \sum_{j=0}^{k-1} \int_a^x K_2(x,t) L_{i_j} dt \Big] \Big| \\
&= \max_{\forall x \in J} \Big| \int_a^x K_1(x,t) \Big(\sum_{i=m}^{n-1} A_i \Big) dt \\
&\quad - \sum_{j=0}^{k-1} \int_a^x K_2(x,t) \Big(\sum_{i=m}^{n-1} L_{i_j} \Big) dt \Big| .
\end{aligned}$$

From [29], we have

$$\sum_{i=m}^{n-1} A_i = G(s_{n-1}) - G(s_{m-1}),$$

$$\sum_{i=m}^{n-1} L_{i_j} = D^j(s_{n-1}) - D^j(s_{m-1}),$$

So,

$$\begin{aligned}
&\|s_n - s_m\| \\
= & \max_{\forall x \in J} \Big| \int_a^x K_1(x,t)(G(s_{n-1}) - G(s_{m-1})) dt \\
& - \sum_{j=0}^{k-1} \int_a^x K_2(x,t)(D^j(s_{n-1}) - D^j(s_{m-1})) dt \Big| \\
\leq & \max_{\forall x \in J} \Big(\int_a^x \Big| K_1(x,t) \Big| \Big| G(s_{n-1}) - G(s_{m-1}) \Big| dt \\
& + \sum_{j=0}^{k-1} \Big(\int_a^x \Big| K_2(x,t) \Big| \Big| D^j(s_{n-1}) - D^j(s_{m-1}) \Big| dt \Big) \Big) \\
\leq & \max_{\forall x \in J} \Big(M_1 d \Big| s_{n-1} - s_{m-1} \Big| (b-a) \\
& + \sum_{j=0}^{k-1} (M_{1_j} C_j) \Big| s_{n-1} - s_{m-1} \Big| (b-a) \Big) \\
\leq & M_1 d \|s_{n-1} - s_{m-1}\| (b-a) + kMC \|s_{n-1} - s_{m-1}\| (b-a) \\
= & (M_1 d + kMC)(b-a) \|s_{n-1} - s_{m-1}\| \\
= & \gamma \|s_{n-1} - s_{m-1}\|
\end{aligned}$$

Let $n = m+1$, then

$$\|s_n - s_m\| \leq \gamma \|s_m - s_{m-1}\| \leq \gamma^2 \|s_{m-1} - s_{m-2}\| \leq \ldots$$
$$\leq \gamma^m \|s_1 - s_0\|.$$

So,

$$\begin{aligned}
& \|s_n - s_m\| \\
\leq\ & \|s_{m+1} - s_m\| + \|s_{m+2} - s_{m+1}\| + \cdots + \|s_n - s_{n-1}\| \\
\leq\ & [\gamma^m + \gamma^{m+1} + \cdots + \gamma^{n-1}]\|s_1 - s_0\| \\
\leq\ & \gamma^m[1 + \gamma + \gamma^2 + \cdots + \gamma^{n-m-1}]\|s_1 - s_0\| \\
\leq\ & \gamma^m(\frac{1-\gamma n-m}{1-\gamma})\|u_1(x)\|.
\end{aligned}$$

Since $0 < \gamma < 1$, we have $(1 - \gamma^{n-m}) < 1$, then

$$\|s_n - s_m\| \leq \frac{\gamma m}{1-\gamma}\|u_1(x)\|.$$

But $\|u_1(x)\| < \infty$ (since $F(x)$ is bounded), so, as $m \longrightarrow \infty$, then $\|s_n - s_m\| \longrightarrow 0$. We conclude that s_n is a Cauchy sequence in $C[J]$, therefore the series is convergence and the proof is complete.

5. NUMERICAL EXAMPLE

In this section, we present the semi-analytical techniques based on ADM and MADM to solve Volterra-Fredholm integro-differential equations:

Example 1.

Consider the Volterra-Fredholm integro-differential equation as follow:

$$u'''(x)+u(x)\sin x^2 = x^2\sin x^2 - \frac{1}{3}x^3 + \int_0^x\int_0^1 xtu'(t)dxdt, \tag{18}$$

with the initial conditions

$$u''(0) = u'(0) = u(0) = 0.$$

The exact solution is $u(x) = x^2, \quad \epsilon = 10^{-2}$

Table 1: Numerical Results of the Example 1.

x	Exact	$ADM_{n=10}$	$MADM_{n=6}$
0.1	0.01	0.054634	0.054475
0.2	0.04	0.034863	0.024638
0.4	0.16	0.083404	0.094205
0.6	0.36	0.282315	0.294133
0.8	0.64	0.561304	0.574032

6. CONCLUSION

In this work, the ADM and MADM have been successfully employed to obtain the approximate solutions of a mixed Volterra-Fredholm integro-differential equation. Moreover, we proved the existence and uniqueness results and convergence of the techniques. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results of these methods are similar approximately, MADM is the easiest, the most efficient and convenient.

Table 2: Numerical Results of the Absolute Error

x	Er($ADM_{n=10}$)	Er($MADM_{n=6}$)
0.1	0.074634	0.064475
0.2	0.074863	0.064638
0.4	0.076596	0.065795
0.6	0.077685	0.065867
0.8	0.078696	0.065968

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