Composite Non Smooth Mathematical Programming Problems with Equilibrium Constraints under Generalized Univexity

L. Venkateswara Reddy  
Professor  
Dept. of IT  
SVEC, Tirupati, India.

Anju Singh  
Research Scholar  
Dept. of Maths, Mewar University  
Gangrar, Chittorgarh (Rajasthan), India.

Dr. S.P. Pandey  
Professor  
Dept. of Maths, Maharshi University of Information Technology, IIM Road, Lucknow, India.

Abstract

Here, we will state and prove sufficient optimality conditions for a composite non smooth mathematical programming problem with respect of equilibrium constrains under generalized univexity condition. Further, we deduced Wolfe and Mond-Weir type dual models for the chosen problem using convexiticators. Also, we state and prove we are and strong dudity theorems.

Keywords: Composite program, duality, generalized Univexity, Convexifications

1. INTRODUCTION:

Convexificators concept was introduced by Demyanov [6]. This concept is used to generalize the concepts in optimization and the theory of non smooth analysis (Refer [16], [1], [32], [19], [20]). Further, the concept of Clarke sub differentials, Michel – Pinot sub – differentials and Treiman sub differentials of a Locally Lipschitz real valued functions for convexicators were introduced by Luc, DT. [20] and the latest development one can refer [16], [17], [18], [19].

Also, the concept of mathematical programming program with respect to equilibrium constraints (MPEC1) is usually studied for optimization problem in which the required constraints functions were defined by using complementary system or by using auxiliary parametric variational inequality. In the literature, various equilibrium phenomena were introduced to study applications on economics and engineering which was characterized either by a variational inequality or an optimization problem. This justifies the name mathematical programming problem with equilibrium constraints. This was studied for both smooth case [11], [36] and for the non smooth case ([29], [30] [37]). In another development, Luc et al. [20] introduced and studied a comprehensive study on mathematical programming with Equilibrium constraints. Consequently, Flegal and Kanzow [8,9] obtained the optimality conditions for MPEC1 by using FJ – conditions. Also in [9], Flegal and Kanzow introduced a new, constraint called “Slater type constraints qualifications and a new Abadie type constraint qualification for the MPEC1.

Further on, the concept of convexity and generalized play an important role in the field of optimization, control theory, Economics, Game Theory and so on. The most important generalization of convexity is invexity of function, which was introduced by Hanson [11] and the name coined by Craven [5]. Since last three and half decades, optimality and duality condition in convexity and generalized convexity (invexity) were introduced many researchers (see [4, 23]; [24], [26]. Duality results have many applications in Numerical Algorithm in the field of nonlinear programming problem for solving certain class of optimization programs. Consequently, the concept of duality helped the society to develop stopping rules and to solve primal and dual problems both in Linear and Nonlinear optimization problem.

In this context, Wolfe [35] and Mond-Weir [27] dual models were very popular in the field of nonlinear programming problem very recently, B.C. Joshi, et.al. [16] derived sufficient optimality conditions for global optimality for the chosen mathematical programming problem with equilibrium constraints under generalized univexity.

By make use of the above arguments in this paper, we introduce composite mathematical programming problem for equilibrium constraints by using generalized univexity assumptions. Also, we state and prove duality results of Wolfe and Mond – Weir types to the MPEC1.

This papers is organized is as follows. Section 2 gives elementary basic definitions and notations. Section 3 contains sufficient optimality condition for the chosen MPEC1 using generalized univexity. Finally, section 4 gives Weak and Strong duality results in the frame work of generalized convexifications with respect to generalized univexity condition.

2. NOTATIONS AND DEFINITIONS:

Definition 2.1: Let \( F: \mathbb{R}^n \rightarrow \mathbb{R}^+ \cup \{ +\infty \} \) be a generalized real-valued function, which admit convexifications at \( \bar{x} \in \mathbb{R}^n \) and \( \eta: \mathbb{R}^N \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a Kernal function, then, \( f \) is said to be:

a) \( \sigma^* - \nu - \rho \) univexity, with respect to \( \eta, \theta \) if for the very \( x \in \mathbb{R}^n \), we have

\[
\theta((x, \bar{x})(F(x) - F(\bar{x}))) \geq (\zeta_\eta^T(x, \bar{x})), \forall \in \sigma^*F(F'(\bar{x}))
\]
(b) $\partial^* - V - \rho -$ pseudo univexity with respect to $\eta, \theta$ if for every $x \in \mathbb{R}^n$, we have

$$\exists \in \partial^* F(F'(x)), (\xi, \eta^T(x, \tilde{x})) \geq 0$$

$$\Rightarrow F(F'(x)) \geq F(F'(\tilde{x})).$$

(iii) $\partial^* - V - \rho -$ quasi univexity at $\tilde{x}$ with respect to $\eta$ and $\theta$ if for every $x \in \mathbb{R}^n$, we have

$$0(x, \tilde{x})(F(F'(x)) - F(F'(\tilde{x})))$$

$$\leq (\xi, \eta^T(x, \tilde{x})) + \rho \| \theta(x, \tilde{x}) \|^2 \leq 0$$

$$\forall \in \partial^* F(F'(\tilde{x})).$$

3. OPTIMALITY:

In the sequel, we need the following notations from [16, 32]. They are

$I_r^0 = \{i \in I : \gamma^\alpha_r = 0, \psi > 0\}$

$I_r^\psi = \{i \in I : \gamma^\psi_r = 0, \gamma^\alpha_r = 0\}$

$\delta^\alpha_r = \{i \in \delta : \gamma^\alpha_r > 0\}$

$K^\psi_r = \{i \in K : \gamma^\psi_r > 0\}$

We will state and prove that the following optimality conditions with respect to $\partial^* - V - \rho -$ univexity.

**Theorem 3.1:** Suppose $\tilde{x}$ is a feasible - GA - Stationary point of MPEC1. Also, assume that $F(F')$ is $\partial^* - V - \rho -$ univexity at $\tilde{x}$ w.r.t. to the kernels $\eta$, $\theta$ and $g_j(G_j'(x))$, $(i \in I)$, $-h_m(H_m)$, $(m = 1, 2, \ldots, p)$

$$\theta_j(i \in \delta \cup \delta^\alpha_r \cup K^\psi_r) \text{ are } \partial^* - V - \rho -$ quasi univexity at $\tilde{x}$ with respect to the same common Kernels $\eta$ and $\theta$. If $I_r^0 \cup I_r^\psi \cup \delta^\alpha_r \cup K^\psi_r = \phi$, then $\tilde{x}$ is said to be global optimal solution of problem MPEC1.

**Proof:**

Suppose $x$ is any arbitrary feasible point

$$\Rightarrow g_j(G_j'(x)) \leq 0 = g_j(g_j'(\tilde{x})),$$

by definition of $\partial^* - V - \rho -$ univexity, we have

$$\big(\xi, \eta^T(x, \tilde{x})\big) + \rho \| \theta(x, \tilde{x}) \|^2 \leq 0, \forall \xi \in \partial g_j(G_j'(x)) \forall j \in I.$$

Similarly, we have

$$\big(T_m, \eta^T(x, \tilde{x})\big) + \rho \| \theta(x, \tilde{x}) \|^2 \leq 0$$

$$\forall T_m \in \partial^* h_m(H_m'(x)), \forall m \in \{1, 2, \ldots, q\}$$

$$\big(\lambda_m, \eta^T(x, \tilde{x})\big) + \rho \| \theta(x, \tilde{x}) \|^2 \leq 0$$

$$\forall \lambda_m \in \partial^* (-h_m(H_m'(x))), \forall m \in \{1, 2, \ldots, q\}$$

$$\xi^0, \eta^T(x, \tilde{x}) + \rho \| \theta(x, \tilde{x}) \|^2 \leq 0$$

$$\forall \xi^0 \in \partial^* \big((-\lambda_1')\big)(x), \forall \in S \cap I$$

$$\xi^0, \eta^T(x, \tilde{x}) + \rho \| \theta(x, \tilde{x}) \|^2 \leq 0$$

$$\forall \xi^0 \in \partial^* \big((-\eta_j')\big)(x), \forall \in I \cap K$$

Suppose $I^0_r \cup I^\psi_r \cup \delta^\alpha_r \cup K^\psi_r = \phi$, also multiply equation (3.7) to (3.11), by $T^g_i \geq 0$

$$\big(i \in I, T^g_i \geq T^h_i > 0, m = 1, 2, \ldots, q, T^h_i > 0, T^0_i > 0, i \in \delta \cup I, T^\psi_i > 0 i \in I \cup K, \text{ respectively and finally by sum rule, we have}$$

$$\bigg(\sum_{i=1}^{g} T^g_i \xi_i \bigg) + \sum_{m=1}^{q} T^h_m T^l_m + \gamma^h_m \lambda_m$$

$$+ \sum_{i=1}^{l} T^l_i \xi^0_i + \sum_{i=1}^{q} T^\psi_i \xi^\psi_i$$

$$\eta^T(x, \tilde{x}) + \rho \| \theta(x, \tilde{x}) \|^2 \leq 0,$$

for all $\xi^\psi_i \in \cap \partial^* g_j(x), T_m \in \cap \partial^* h_m(x)$,

$$\lambda_m \in \cap \partial^* (-h_m(x)), \xi^0 \in \cap \partial^* (-\theta_i(x)) \text{ and}$$

$$\xi^\psi \in \cap \partial^* (-\psi_j(x)),$$

which implies by using generalized GA-stationary of $\tilde{x}$, and also select $\xi \in \cap \partial^* F(F'(\tilde{x}))$, so that

$$<\xi, \eta^T(x, \tilde{x}) + \rho \| \theta(x, \tilde{x}) \|^2 \geq 0.$$
Also, by definition of \( \partial^* - V - \rho - \text{univexity of } F(F^t) \) at \( \tilde{x} \) with respect to the some common kernels of \( \eta \) and \( \theta \), we have
\[
b(x, \tilde{x})(F(F^t(x)) - F(F^t(\tilde{x})) \geq 0, \text{ for all feasible points of } x.
\]
Thus, by def, \( \tilde{x} \) is a global optimal point of MPEC1.

### 4. DUALITY RESULTS

Here, we will discuss Wolfe and Mond-Weir type dual problems under generalized \( \partial^* - V - \rho - \text{univexity} \) with convexificators.

The Wolfe type dual programming problem for MPEC1 is:
\[
(CWD) \max \left( u, \mu \right) \left\{ F \left( F^t(\mu) + \sum_{j=1}^{q} \mu_j^g g_j \left( G_j(\mu) \right) \right) \right\} + \sum_{m=1}^{q} \lambda_m^b h_m \left( H_m(u) \right) - \sum_{j=1}^{q} \left[ \mu_j^v \theta_j \left( \theta^j(u) \right) + \mu_j^v \left( \psi_j^v \left( \psi_j^v(u) \right) \right) \right]
\]
subject to conditions:
\[
0 \in \text{con } \partial^* F(F^t(u)),
\]
\[
+ \sum_{j=1}^{q} \mu_j^v \text{con } \partial^* g_j \left( G_j(u) \right)
\]
\[
+ \sum_{m=1}^{q} \left[ \mu_m^h \text{con } \partial^* h_m \left( H_m(u) \right) \right]
\]
\[
+ \gamma_m^h \text{con } \partial^* \left( -h_m \left( H_m(u) \right) \right)
\]
\[
+ \sum_{j=1}^{q} \left[ \mu_j^v \text{con } \partial^* \left( -\theta_j \right) \left( u \right) \right]
\]
\[
+ \mu_j^v \text{cond } \partial^* \left( -\psi_j \left( u \right) \right)
\]
\[
\mu_j^g \geq 0, \mu_m^h, \gamma_m^h \geq 0, m = h
\]
\[
\mu_j^v, \nu_i^v \geq 0, i = 1, \ldots, p
\]
\[
\mu_k^0 = \mu_k^v = \mu_i^0 = 0, \forall i \in I, \nu_i^0 = 0, \gamma_i^v = 0
\]

**Theorem 4.1 (Wear Duality)**

Let \( \tilde{x} \) be a feasible solution for the problem (MPEC1), \( (u, \tau) \) be feasible for the dual problem (CWD) and the corresponding index sets \( I_{\ell}, \delta, I^t, K \) are defined accordingly.

Suppose that \( F(F^t), g_j \left( G_j \right), \left( J \in I_{\ell} \right), \pm h_m \left( H_m \right) \) \((m = 1, 2, \ldots, q)\), \(- \theta_j \left( \theta^j \right)\), \( \left( j \in \delta \cup I^t \right), \psi_j \left( \psi_j^v \right) \left( g \in 1 \cup K \right) \) admit bounded upper semi-regular convexificators and are \( \partial^* - V - \rho - \text{univexity} \) functions at \( u \), with respect to the some common kernels \( \eta \) and \( \theta \).

If \( ^t \ \Gamma^t \ \Gamma^t \ \Gamma^t \ \Gamma^t \ \Gamma^t \ = \ \phi \) then any \( w \) feasible for the problem MPEC1, we have
\[
b(u, w) \left[ F(F^t(u)) - F(F^t(w)) \right] \geq \sum_{j=1}^{q} \lambda_j^g g_j \left( G_j(u) \right)
\]
\[
+ \sum_{m=1}^{q} \lambda_m^b h_m \left( H_m(u) \right)
\]
\[
- \sum_{k=1}^{l} \left[ \lambda_k^g \theta_k \left( \theta^k(u) \right) + \lambda_k^v \psi_k \left( \psi_k^v(u) \right) \right]
\]

**Proof:**

Suppose \( w \) is any feasible point for the composite problem MPEC1.

By def, we have
\[
g_j \left( G_j \left( W \right) \right) \leq 0
\]
and \( h_m \left( H_m \left( w \right) \right) = 0, \forall m = 1, 2, \ldots, q. \)

Since \( F(F^t) \) is \( \partial^* - V - \rho - \text{univexity} \) at \( u \) with respect to the some common kernels \( \eta \) and \( \theta \), then, we get
\[
b(u, w) \left[ F(F^t(u)) - F(F^t(w)) \right] \]
\[
\geq \left( \xi, \eta^T(u, w) + \rho \left| \left| \theta(u, w) \right| \right| \right)^T \]
\[
\forall \in \partial^* F(F^t(u))
\]

We can write
\[
b(u, w) \left[ g_j \left( G_j(u) \right) - g_j \left( G_j(w) \right) \right] \]
\[
\geq \left( \xi_j^g, \eta^T(u, w) + \rho \left| \left| \theta(u, w) \right| \right| \right)^T, \]
\[
\forall \xi_j^g \in \partial^* g_j \left( G_j(u) \right), j \in I_{\ell}
\]
From equation (2.2), there exists \( \xi_j \in \text{con} \partial^+ \left( F \left( F^I \left( u \right) \right) \right) \),
\( \xi_j^g \in \text{con} \partial^+ \left( G_j \left( u \right) \right) \), \( \lambda_m \in \text{con} \partial^+ \left( -h_m \left( H_m \left( u \right) \right) \right) \), and \( \mu_m \in \text{con} \partial^+ \left( -\theta \left( \theta_j \left( w \right) \right) \right) \), respectively. Then, finally, adding the equations (4.3) – (4.8), we have

\[
\begin{align*}
\mathbf{b} (u, w) & \left[ h_m \left( H_m \left( u \right) - h_m \left( H_m \left( w \right) \right) \right) \right] (\lambda_m) = D^* h_m \left( H_m \left( u \right) \right), \\
\lambda_m & \geq (\lambda_m, \eta^g (u, w) + \rho \| \theta (u, w) \|^2 ) \quad \forall m = 1, 2, \ldots, q \\
\mathbf{b} (u, w) & \left[ -h_m \left( H_m \left( u \right) - h_m \left( H_m \left( w \right) \right) \right) \right] (\mu_m) \geq (\xi_j^g, \eta^g (u, w) + \rho \| \theta (u, w) \|^2 ) \quad \forall \mu_m \in \partial^+ \left( -h_m \left( H_m \left( u \right) \right) \right) \forall m = 1, 2, \ldots, q \\
\mathbf{b} (u, w) & \left[ -\theta_j \left( \theta_j \left( u \right) \right) + \theta_j \left( \theta_j \left( w \right) \right) \right] \geq (\xi_j^g, \eta^g (u, w) + \rho \| \theta (u, w) \|^2 ) \quad \forall \epsilon_j^g \partial^+ \left( -\theta_j \left( \theta_j \left( w \right) \right) \right), \\
\forall j \in \delta \cup I, \\
\mathbf{b} (u, w) & \left[ -\psi_k \left( \psi_k \left( u \right) \right) + \psi_k \left( \psi_k \left( w \right) \right) \right] \geq (\xi_j^g, \eta^g (u, w) + \rho \| \theta (u, w) \|^2 ) \quad \forall \epsilon_j^g \partial^+ \left( -\psi_k \left( \psi_k \left( w \right) \right) \right), \quad \forall j \in I \cup K.
\end{align*}
\]

If \( I^*_j \cap I^*_j \cap S^*_j \cup K^*_j = \rho \) and then multiply the equations (4.4), (4.8) with \( \lambda_j^h \geq 0 \ (i \in I^*_j) \), \( \lambda_m ^h > 0 \)

\( \lambda_j^h \geq 0 \ (j \in I \cup K) \), respectively and then finally adding the equations (4.3) – (4.8), we have

\[
\begin{align*}
\mathbf{b} (u, w) & \left[ F \left( F^I \left( u \right) \right) - F \left( F^I \left( w \right) \right) \right] \geq b (u, w) \left[ \sum_{j \in I^*_j} \lambda_j^h \left( G_j \left( u \right) \right) - \sum_{j \in I^*_j} \lambda_j^h \left( G_j \left( w \right) \right) \right] + \sum_{m=1}^q \lambda_m^h \left( H_m \left( u \right) \right) - \sum_{m=1}^q \lambda_m^h \left( H_m \left( w \right) \right) - \sum_{m=1}^q \gamma_m^h \left( H_m \left( u \right) \right) + \sum_{m=1}^q \gamma_m^h \left( H_m \left( w \right) \right) \\
& - \sum_{k=1}^q T_k^0 \left( \theta \left( \theta_j \left( w \right) \right) \right) + \sum_{k=1}^q T_k^0 \left( \theta \left( \theta_j \left( w \right) \right) \right) - \sum_{k=1}^q T_k^\psi \left( \psi_k \left( u \right) \right) + \sum_{k=1}^q T_k^\psi \left( \psi_k \left( w \right) \right) \\
& \geq \left( \xi_j^g + \sum_{j \in I^*_j} \lambda_j^h \left( G_j \left( u \right) \right) + \sum_{m=1}^q \lambda_m^h \left( H_m \left( u \right) \right) + \sum_{m=1}^q \mu_m^h \left( H_m \left( u \right) \right) \right) \sum_{k=1}^q \left( T_k^0 \left( \theta \left( \theta_j \left( w \right) \right) \right) + T_k^\psi \left( \psi_k \left( u \right) \right) \right) \eta^g \left( u, w \right) + \rho \| \theta (u, w) \|^2 \\
& \geq \left( \xi_j^g + \sum_{j \in I^*_j} \lambda_j^h \left( G_j \left( u \right) \right) + \sum_{m=1}^q \lambda_m^h \left( H_m \left( u \right) \right) + \sum_{m=1}^q \mu_m^h \left( H_m \left( u \right) \right) \right) \sum_{k=1}^q \left( T_k^0 \left( \theta \left( \theta_j \left( w \right) \right) \right) + T_k^\psi \left( \psi_k \left( u \right) \right) \right) \eta^g \left( u, w \right) + \rho \| \theta (u, w) \|^2 \\
& \geq \left( \xi_j^g + \sum_{j \in I^*_j} \lambda_j^h \left( G_j \left( u \right) \right) + \sum_{m=1}^q \lambda_m^h \left( H_m \left( u \right) \right) + \sum_{m=1}^q \mu_m^h \left( H_m \left( u \right) \right) \right) \sum_{k=1}^q \left( T_k^0 \left( \theta \left( \theta_j \left( w \right) \right) \right) + T_k^\psi \left( \psi_k \left( u \right) \right) \right) \eta^g \left( u, w \right) + \rho \| \theta (u, w) \|^2 \\
\end{align*}
\]
Thus, we have
\[
b(u, v)\left(F(F^*(n)) - F(F^*(w))\right) + \sum_{j=1}^{n} \lambda_j^h g_j(G_j(u)) - \sum_{j=1}^{n} \lambda_j^v g_j(G_j(w)) + \sum_{m=1}^{q} \gamma_m^h h_m(H_m(u)) - \sum_{m=1}^{q} \gamma_m^v h_m(H_m(w))
\]
\[-\sum_{m=1}^{q} \mu_m^h h_m(H_m(u)) + \sum_{m=1}^{q} \mu_m^v h_m(H_m(w)) + \sum_{k=1}^{l} T_k^0 \theta_k(\theta_k(u)) + \sum_{k=1}^{l} T_k^v \psi_k(\psi_k(u)) - \sum_{k=1}^{l} T_k^v \psi_k(\psi_k(w)) \geq 0\]

Now, by applying the feasibility condition of MPEC1, i.e. \(g_i(G_i(u)) \leq 0, h_m(H_m(u)) = 0\), \(\theta_i(\theta_i(u)) \geq 0, \psi_i(\psi_i(u)) \geq 0\), hence it follows that
\[
b(u, w)\left[F(F(u)) - F(F^*(w))\right] - \sum_{j=1}^{n} \lambda_j^h g_j(G_j(u))
\]
\[-\sum_{m=1}^{q} \lambda_m^h h_m(H_m(u)) + \sum_{m=1}^{q} \gamma_m^h h_m(H_m(w)) + \sum_{k=1}^{l} T_k^0 \theta_k(\theta_k(u)) + \sum_{k=1}^{l} T_k^v \psi_k(\psi_k(u)) \geq 0\]

Hence, it follows that
\[
b(u, w)\left[F(F(u)) - F(F^*(w))\right] \geq \sum_{j=1}^{n} g_j(G_j(u)) + \sum_{m=1}^{q} \lambda_m^h h_m(H_m(u)) - \sum_{k=1}^{l} [T_k^0 \theta_k(\theta_k(u)) + T_k^v \psi_k(\psi_k(u))]
\]

Hence proved.

**Theorem 4.2 (Strong Duality)**

If \(\tilde{u}\) be a local optimal solution for the problem MPEC1 and also assume that \(F(F^*)\) is locally Lipschitz near \(\tilde{u}\). Let us suppose that \(F(F^*)\) and \(\tilde{u}\) admit bounded upper semi-regular convexificators and are \(\partial^* V - \rho -\) univex function at \(\tilde{u}\) with respect to the same common Kernel \(\eta\) and \(\theta\). According to [16, 30], if Gs – AC on holds at \(\tilde{u}\) then there exists \(\tilde{\mu} = (\tilde{\mu}^g, \tilde{\mu}^h, \tilde{\mu}^0, \tilde{\mu}^v) \in \mathbb{R}^{k+p+2l}\) such that \(\tilde{\mu}\) is an optimal solution of the dual (CWD) and the corresponding objective values of MPEC1 and CWD are equal.

**Proof:**

If \(\tilde{u}\) be local optimal solution for the problem MPEC1 and the Generalized Slater and ACQ is satisfied at \(\tilde{u}\) and also by using corollary 4.6 of [1, 16], there exists \(\tilde{\mu} = (\tilde{\mu}^g, \tilde{\mu}^h, \tilde{\mu}^0, \tilde{\mu}^v) \in \mathbb{R}^{k+p+2l}\) such that \(\tilde{\mu}\) is an optimal solution of the dual (CWD) and the corresponding GS – stationary conditions for By applying theorem 4.1, we get the problem MPEC1 are satisfied, it follows that, there exists
\[
\tilde{\xi}^0 \in \text{con} \partial^* F(F^*(\tilde{u})),
\]
\[
\tilde{\xi}^g \in \text{con} \partial^* g_j(G_j(\tilde{u})), \tilde{\lambda}_m \in \text{con} \partial^* h_m(H_m(\tilde{u}))
\]
\[
\tilde{\mu}_m \in \text{con} \partial^* (-h_m(H_m(\tilde{u}))),
\]
\[
\tilde{\xi}^0 \in \text{con} \partial^* (-\theta_k(\theta_k(\tilde{u}))),
\]
\[
\tilde{\xi}^g + \sum_{\eta_j} \tilde{\xi}^g_j + \sum_{\mu_m} \tilde{\lambda}_m + \tilde{\mu}^0 T_m^0 + \tilde{\mu}^v T_m^v \geq 0
\]
\[
\tilde{\mu}^g_0 \geq 0, \tilde{\mu}^h_0 \geq 0, m = 1, 2, \ldots, p
\]
\[
T_k^0 \geq 0, k = 1, \ldots, l
\]
\[
\forall j \in I, \tilde{\xi}^0_k = T_k^v = 0
\]
\[
\therefore (\tilde{u}, \tilde{\mu}) \text{ is feasible for the (CWD).}
\]
\( b(u, \mu) \left( F(F'(\bar{u})) - F(F'(\mu)) \right) \geq \sum_{j=1}^{\bar{u}} g_j(G_j(\bar{u})) + \sum_{m=1}^{\bar{u}} h_m(H_m(\bar{u})) - \sum_{k=1}^{\bar{u}} T_k^0 \theta_k \left( \theta_k^l(\bar{\mu}) \right) + T_k^y \psi_k \left( \psi_k^l(\bar{\mu}) \right) \) (4.9)

Here \( \lambda_{m}^b \) represents \( \mu_{m}^b - T_{m}^b \) for any feasible solution \( (\mu, T) \) for the considered dual (CWD).

Now by applying the feasibility condition of the problem MPEC1 and its corresponding dual CWD, \( \exists j \in I \), \( g_j(G_j(\bar{u})) = 0 \)

\( h_m(H_m(\bar{u})) = 0, (m = 1, 2, \ldots, q), \theta_k \left( \theta_k^l(\bar{\mu}) \right) = 0 \)

\( \forall k \in \delta \cup I, \) and \( \psi_k \left( \psi_k^l(\bar{\mu}) \right) = 0, \forall k \in I \cup K, \) it follows that

\[ b(u, \bar{u}) \left( F(F'(u)) - F(F'(\bar{u})) \right) \geq \sum_{j=1}^{\bar{u}} g_j(G_j(\bar{u})) + \sum_{m=1}^{\bar{u}} h_m(H_m(\bar{u})) - \sum_{k=1}^{\bar{u}} T_k^0 \theta_k \left( \theta_k^l(\bar{\mu}) \right) + T_k^y \psi_k \left( \psi_k^l(\bar{\mu}) \right) \] (4.10)

Applying (4.9) and (4.10) and the indices \( \bar{\lambda}_{m}^b, T_{m}^b - \bar{\mu}_{m}^b, \) we set

\[ b(\bar{u}, \bar{\mu}) \left( \left[ F(F'(\bar{u})) - F(F'(\bar{\mu})) \right] + \sum_{j=1}^{\bar{u}} g_j(G_j(\bar{u})) + \sum_{m=1}^{\bar{u}} h_m(H_m(\bar{u})) - \sum_{k=1}^{\bar{u}} T_k^0 \theta_k \left( \theta_k^l(\bar{\mu}) \right) + \sum_{j=1}^{\bar{u}} g_j(G_j(\bar{u})) + \sum_{m=1}^{\bar{u}} h_m(H(\bar{v})) + \sum_{k=1}^{\bar{u}} T_k^0 \theta_k \left( \theta_k^l(\bar{\mu}) \right) + \sum_{k=1}^{\bar{u}} T_k^y \psi_k \left( \psi_k^l(\bar{\mu}) \right) \right) \]

subject to conditions:

\[ O \in \text{con} \ \delta^* F(F'(u)) + \sum_{j=1}^{\bar{u}} g_j(G_j(u)) \]

\[ + \sum_{m=1}^{\bar{u}} h_m(H_m(u)) + T_{m}^b \text{con} \ \delta^* \left( -h_{m}H_{m}(u) \right) \]

\[ + \sum_{k=1}^{\bar{u}} \theta_k \left( \theta_k^l(u) \right) + \lambda_{k}^y \text{con} \ \delta^* \left( -\psi_k \left( \psi_k^l(u) \right) \right) \]

\[ g_j(G_j(u)) \geq 0 \ (j \in I), h_m(H_m(u)) = 0, m = 1, 2, \ldots, q, \theta_k \left( \theta_k^l(u) \right) \leq 0 \ (k \in \delta \cup I), \]

\[ \psi_k \left( \psi_k^l(u) \right) \leq 0 \ (k \in I \cup K), \]

\[ \lambda_{j}^g \geq 0, \mu_{m}^b, T_{m}^b \geq 0, m = 1, 2, \ldots, q \]
\( \lambda^0_k, \lambda^v_k, \mu^0_k, \mu^v_k \geq 0, k = 1, 2, ..., l. \)

\( \lambda^0_k = \lambda^v_k = T_k = T^v_k = 0, \forall i \in I, \)  \((5.1)\)

Here \( \lambda = (\lambda^0_k, \lambda^v_k, T^h_k, T^v_k) \in R^{k+p+2l}. \)

and \( T = (T^h_k, T^v_k) \in R^{p+1-l} = R^{p+1}. \)

**Theorem 5.1 (Weak Duality)**

Suppose \( \bar{u}_i \) is a feasible solution for the problem MPEC, and \( (v_i, \tau_i) \) be a feasible solution for the corresponding dual (CMWD) and also the index sets \( I_g, \delta, I, K \) be defined accordingly. Also, suppose that the composite functions

\[
F(F^i), g_j(G_j), \pm h_m (H_m), (m = 1, 2, ..., q),
\]

- admits bounded upper semi - regular convexificators and are \( \delta^* - V - \rho - \) univex functions at \( v_i \), with respect to the some common kernals \( \eta_i \) and \( \theta_i \).

If we denote \( T^0_k \cup T^v_k \cup \delta^*_k \cup K^* = \phi \), then for any \( u_i \) is feasible for the problem MPEC1, then

\[
F(F^i(u_i)) \geq F(F^i(\bar{u}_i))
\]

**Proof:** Proof is similar to theorem 4.1

**Theorem 5.2 (Strong Duality)**

If \( \bar{u}_i \) is a local optimal solution for the problem MPEC1 and let \( F(F^i) \) be locally Lipschitz nearer at \( \bar{u}_i \). Also let us suppose that

\[
F(F^i), g_j(G_j), j \in I_g, \pm h_m (H_m),
\]

- is a feasible, solution for the considered problem MPEC, \( (\bar{v}_i, \bar{\tau}_i) \) is another feasible solution for the corresponding dual CMWD and the index sets \( I_g, \delta, W_{k1}, K_k \) defined accordingly. Also, let us suppose that \( F(F^i) \) is \( \delta^* - V - \rho \) pseudounivex at \( v_i \), with respect to the some common kernals \( \eta_i \) and \( \theta_i \) and

\[
g_j(G_j)(j \in I_g), \pm h_m (H_m) \)

admit bounded upper semi-regular convexificators and are \( \delta^* - V - \rho \) quasi univex functions as with respect to the some common Kernels \( \eta_i \) and \( \theta_i \). Let us suppose, if \( W_{g} \cup W_{v} \cup K_{g} = \psi \), then for any other \( u_2 \) feasible for the problem MPEC1, it follows that \( F(F^i(\bar{u}_i)) \geq F(F^i(\bar{v}_i)) \).

**Proof:** Let us assume that \( \bar{u}_i \) as some feasible point,

\[
\exists \left(F(F^i(\bar{u}_i)) \leq F(F^i(\bar{v}_i))\right)
\]

Then by definition of \( \delta^* - V - \rho \) pseudounivexity of \( F(F^i) \) at \( \bar{v}_i \) with respect to some common kernals \( \eta_i \) and \( \theta_i \), we have

\[
(\xi_i^g, \eta_i^g, v_i, v_i) + \rho_i \| \theta_i (\xi_i^g, \eta_i^g, v_i) \|^2 < 0
\]

\[
\forall \xi_i^g \in \delta^* F(F^i(\bar{v}_i))
\]

But from (4.2), there exists \( \xi_i^g \) con \( \delta^* F(F^i(\bar{v}_i)) \),

\[
\xi_i^g \in \delta^* F(F^i(\bar{v}_i)) \]

\[
\xi_i^g \in \delta^* \left( g_j(G_j(\bar{v}_i)) \right), \lambda^g_m \in \delta^* h_m (H_m (\bar{v}_i)), \mu^g_m \in \delta^* (-h_m (H_m (\bar{v}_i))), \xi_i^g \in \delta^* \left( -\theta_k (\theta_k) \right) (\bar{v}_i), \]

and \( \xi_k^v \in \delta^* \left( \psi_k (\psi_k) (\bar{v}_i) \right), \) \( \mu \)

\[
- \sum_{j \in I_g} \xi_j^g \xi_j^g - \sum_{m=1}^q \left( \tau_m^h \tau_m^h + \gamma_m^h \mu_m^h \right)
\]

\[
- \sum_{j \in I_g, W_{k1}} \xi_j^v \xi_j^v - \sum_{j \in I_g, W_{k1}} \xi_j^v \xi_j^v \in \delta^* F(F^i(\bar{v}_i))
\]

(5.3)

Applying (5.2) one can obtain

\[
\left( \sum_{j \in I_g} \xi_j^g \xi_j^g + \sum_{m=1}^q \left( \tau_m^h \tau_m^h + \gamma_m^h \mu_m^h \right) + \sum_{j \in I_g, W_{k1}} \xi_j^v \xi_j^v \right)
\]

\[
\eta_i^g \left( \bar{v}_i, \bar{v}_i \right) + \rho_i \| \theta_i (\bar{v}_i, \bar{v}_i) \|^2 > 0
\]

(5.4)

Now for each \( j \in I_g, g_j(G_j(\bar{u}_i)) \leq 0 \leq g_j(G_j(\bar{v}_i)) \).

By definition of \( \delta^* - V - \rho \) quasivenixity, we get

\[
(\xi_i^g, \eta_i^g, v_i, v_i) + \rho_i \| \theta_i (\bar{v}_i, \bar{v}_i) \|^2 \]

\[
\leq 0, \forall j \in I_g
\]

(5.5)
Similarly, we can write the remaining as,
\[
\begin{align*}
\langle \tau \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0, \forall \mu \in \partial^* \tau \mu, F(F^l(\bar{v})), \\
\langle \mu \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0, \forall \mu \in \partial^* (-h_m(H_m)\bar{v}), \\
\forall m = 1, 2, \ldots, q \quad (5.6)
\end{align*}
\]
\[
\begin{align*}
\langle \zeta_j \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0, \forall \zeta_j \in \partial^* (-\zeta_k(\psi_k(\bar{v}))), \forall k \in \delta_i \cup W_i. \\
\langle \zeta_k \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0, \forall \zeta_k \in \delta_i \cup W_i \quad (5.7)
\end{align*}
\]
Above equations (5.3) – (5.8) given
\[
\begin{align*}
\langle \zeta_j \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0, \forall \zeta_j \in I_g, \\
\langle \tau \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0, \forall \mu \in 1, 2, \ldots, q. \\
\langle \zeta_k \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0, \forall \zeta_k \in k_i \cup W_i \quad (5.8)
\end{align*}
\]
By hypothesis, since \( w_\gamma U \delta \cup K_i = \phi \), we set
\[
\begin{align*}
\langle \sum_{j=1}^q \tau^j \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0 \quad (5.8)
\end{align*}
\]
Hence, we obtain as
\[
\begin{align*}
\langle \sum_{j=1}^q \tau^j \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0 \quad (5.8)
\end{align*}
\]
Which is a contradiction to equation
\[
\begin{align*}
\langle \sum_{j=1}^q \tau^j \eta \bar{u}, \bar{v} \rangle + \rho \eta \theta \bar{u}, \bar{v} \rangle \leq 0 \quad (5.8)
\end{align*}
\]
Hence \( b(\bar{u}, \bar{v}) \langle F(F^l(\bar{u}))) - (F(F^l(\bar{v}))) \rangle \geq 0 \)
\[
\Rightarrow F(F^l(\bar{u})) \geq (F(F^l(\bar{v})))
\]
Thus, the result is proved.
Theorem (5.4) (Strong Duality)

Suppose $\tilde{u}$ is a local optimal solution for the problem MPEC1 and also let $F(F')$ be a locally Lipschitz near at $\tilde{u}_1$. Let in suppose that $F(F')$ is $\partial^* - V - \rho -$ pseudounivex at $\tilde{u}$ with respect to the some common kernels at $\eta_i$ and $\theta_j$, then $g_j \left( G_j \right) \left( j \in I_\sigma \right) \in h_m \left( H_m \right)\left( m = 1, \ldots, q, \right)$, $-\theta_j \left( \theta' \right) \left( k \in \delta_i \right)$ $\left( K \in W_i \cup K_j \right)$ admit bounded upper semi-regular convenificators and are $\partial^* - V - \rho$ quasi univex functions at $\tilde{u}_i$ with respect to the some common kernels $\eta_i$ and $\theta_j$. Also, let us suppose that, if $GS - ACQ$ of [1, 16] holds at $\tilde{u}_i$, then $\exists \tau_i \in (\tilde{u}_i, \tau_i)$ is an optimal solution of the dual (CMWD), therefore, the corresponding objective values are equal.

Proof: This theorem can be proved similar to the theorem 4.2 by making use of the assumption stated in theorem 5.4

CONCLUSION

In this paper we derived generalized duality theorems of Wolfe type and Mond-Weir type with respective to generalized univexity. These results are the generalizations of [16, 31].

REFERENCES


