Some Applications of Multivalent Functions Defined by Extended Fractional Differintegral Operator

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Abstract

In the present paper an extended fractional differintegral operator \( \Omega_{\alpha}^{(\lambda,p)} \), suitable for the study of multivalent functions is introduced. The various results obtained here for each of these function classes include coefficient bound, inclusion relation for \((k,\theta)\)-neighborhood of subclass of analytic and multivalent functions with negative coefficient, Hadamard products, Integral means. Further, results based on partial sums of functions belonging to the class are derived.

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1. INTRODUCTION

Let \( S_{p} \) denotes a class of functions of the form:

\[
f(z) = z^{p} + \sum_{n=1}^{\infty} a_{n} z^{n} (p < k; p, k \in \mathbb{N} = \{1,2,\ldots\}),
\]

which are analytic and p-valent in the open unit disk \( U = \{ z : |z| < 1 \} \). A function \( f \) belong to the class \( S_{p} \) is said to be p-valent starlike of order \( \alpha \) in \( U \) if and only if

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, (0 \leq \alpha < p; z \in U). \tag{2}
\]

Also a function \( f \) belonging to the class \( S_{p} \) is said to be p-valent convex of order \( \alpha \) in \( U \) if and only if

\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, (0 \leq \alpha < p; z \in U). \tag{3}
\]

We denote by \( S_{p}^{*}(\alpha) \) the class of all functions in \( S_{p} \) which are p-valent starlike of order \( \alpha \) in \( U \) and by \( K_{p}(\alpha) \) the class of all functions in \( S_{p} \) which are p-valent convex of order \( \alpha \) in \( U \). We denote that

\[
S_{p}(0) = S_{p}, S_{p}^{*}(\alpha) = S^{*}(\alpha), K_{p}(0) = K_{p}, K_{p}(\alpha) = K(\alpha), \text{ and}
\]

\[
f(z) \in K_{p}(\alpha) \iff \frac{zf'(z)}{p} \in S_{p}^{*}(\alpha). \tag{4}
\]

The classes \( S_{p}^{*}(\alpha) \) and \( K_{p}(\alpha) \) were studied by Patil and Thakare [24], Aouf [1] and Owa [20] for \( f \in S_{p} \) given by (1) and \( g \in S_{p} \) given by

\[
g(z) = z^{p} + \sum_{n=k}^{\infty} b_{n} z^{n}, (b_{n} \geq 0). \tag{5}
\]

The Hadamard product (or convolution) of \( f \) and \( g \) is given by

\[
(f * g)(z) = z^{p} + \sum_{n=k}^{\infty} a_{n} b_{n} z^{n} = (g * f)(z). \tag{6}
\]

If \( f(z) \) and \( g(z) \) are analytic in \( U \), we say that \( f(z) \) is subordinate to \( g(z) \), written symbolically as

\[
f \prec g \quad \text{in} \quad U \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U),
\]

If there exists a Schwarz function \( w(z) \), which (by definition) is analytic in \( U \) with \( w(0) = 0 \) and \( \overline{\text{w}(w(z))} \) in \( U \) such that \( f(z) = g(w(z)), z \in U \), it is known that

\[
f(z) \prec g(z) \quad (z \in U) \quad \Rightarrow \quad f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

In particular, if the function \( g(z) \) is univalent in \( U \), then we have the following equivalence (see [17], [18])

\[
f(z) \prec g(z) \quad (z \in U) \quad \Leftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

Furthermore, \( f(z) \) is said to be subordinate to \( g(z) \) in the disk if the Schwaz lemma that if \( f(z) \prec g(z) \) in \( U \), then \( f \prec g \) in \( U \), for every \( r(0 < r < 1) \).

Recently, Patel & .Mishra [23] (see also Aouf et al. [4], Liu [14], Liu and Patel [15], Sharma et al. [33], Srivastava et al. [30], Supramaniam et al. [34], Zhi-Gang Wang and Lei Shi [35]) introduced and investigated an extended fractional differintegral operator \( \Omega_{\alpha}^{(\lambda,p)} f(z) : S_{p} \rightarrow S_{p}^{*} \) for a function
(2) for a real number \(\lambda (-\infty < \lambda < p+1)\) by
\[
\Omega_z^{(\lambda,p)} f(z) = z^p + \sum_{n=0}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)} a_nz^n
\]
where \(a_n = C_{n,p} a_n\) and \(C_{n,p} = \frac{\Gamma(n+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(n+p+1-\lambda)}\).

We also note that
\[
\Omega_z^{(0,p)} f(z) = f(z), \quad \Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p}
\]
and in general
\[
\Omega_z^{(k,p)} f(z) = \frac{\Gamma(p+1-k)}{\Gamma(p+1)} z^k D_z^k f(z)
\]
for \(k \in N, k < p+1\).

Where \(D_z^k f(z)\) is, respectively, the fractional integral of order \(-\lambda\) when \(-\infty < \lambda < 0\) and the fractional derivative of \(f(z)\) of order \(\lambda\) when \(0 \leq \lambda < p+1\).

Further, we define the class \(TS_p^{(\beta,\gamma,\xi)} (\lambda, l, \gamma, \beta, \xi)\) by
\[
TS_p^{(\beta,\gamma,\xi)} (\lambda, l, \gamma, \beta, \xi) = S_p^{(\beta,\gamma,\xi)} (\lambda, l, \gamma, \beta, \xi) \cap T_p
\]

We note that:
For \(\lambda = 0\), in (11), the class \(TS_p^{(\beta,\gamma,\xi)} (\lambda, l, \gamma, \beta, \xi)\) reduces to the class \(T_p^{(\beta,\gamma,\xi)} (\lambda, l, \gamma, \beta, \xi)\), which for \(p = 1\) reduces to \(T(\gamma, \beta, \xi)\) studies by Kulkarni [9].

In this paper, we aim at proving coefficient inequality, neighborhood, partial sums, integral means, and modified Hadamard product involving the extended fractional differintegral operator \(\Omega_z^{(\lambda,p)}\).

2. COEFFICIENT INEQUALITY

Unless otherwise mentioned, we shall assume in the remainder of this paper that
\[0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, 0 \leq \gamma < \frac{p}{2}, n \geq k, p < k,\] and \(C_{n,p}^{\lambda}\) is given by (8) with \(-\infty < \lambda < p+1\) and \(z \in U\).
Theorem 2.1 Let the function $f$ be defined by (12). Then $f$ is in the class $TS_p^\lambda(\beta, \gamma, \xi)$ if and only if
\[
\sum_{n=k}^\infty (n-p)(1-\beta) + 2\xi\beta(n-\gamma)C_{n,p}^\lambda a_n \leq 2\beta\xi(p-\gamma).
\] (13)

Proof. Assume that inequality (13) holds true. We find from (12) that
\[
2^\xi \left[ z\left(\Omega_z^{(\lambda,p)} f(z)\right) \right] - p \left(\Omega_z^{(\lambda,p)} f(z)\right) - \beta \left[ 2^\xi \left[ z\left(\Omega_z^{(\lambda,p)} f(z)\right) \right] - \gamma \left(\Omega_z^{(\lambda,p)} f(z)\right) \right] - \left\{ z\left[ \left(\Omega_z^{(\lambda,p)} f(z)\right) \right] - p \left(\Omega_z^{(\lambda,p)} f(z)\right) \right\}
\]
\[
= \sum_{n=k}^\infty (n-p)C_{n,p}^\lambda a_n z^n - \beta \left[ \sum_{n=k}^\infty (n-\gamma)C_{n,p}^\lambda a_n z^n \right] + \sum_{n=k}^\infty (n-p)C_{n,p}^\lambda a_n z^n
\]
\[
\leq \sum_{n=k}^\infty \left[ (n-p) + 2^\xi \beta(n-\gamma) - \beta(n-p) \right]C_{n,p}^\lambda a_n - 2\beta\xi(p-\gamma)
\]
\[
= \sum_{n=k}^\infty (n-p)(1-\beta) + 2^\xi\beta(n-\gamma)\left(C_{n,p}^\lambda a_n - 2\beta\xi(p-\gamma)\right) \leq 0.
\]

Hence by the maximum modulus theorem, we have $f \in TS_p^\lambda(\beta, \gamma, \xi)$ conversely, let $f \in TS_p^\lambda(\beta, \gamma, \xi)$. Then
\[
\left| \frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)}{\Omega_z^{(\lambda,p)} f(z)} - p \right| < \beta
\]
\[
2^\xi \left| z\left(\Omega_z^{(\lambda,p)} f(z)\right) - \gamma \left(\Omega_z^{(\lambda,p)} f(z)\right) \right| - \left| z\left(\Omega_z^{(\lambda,p)} f(z)\right) - p \left(\Omega_z^{(\lambda,p)} f(z)\right) \right|< \beta
\]
that is,
\[
\sum_{n=k}^\infty (n-p)C_{n,p}^\lambda a_n z^n < \beta
\]
\[
2^\xi \left[ (p-\gamma)z^n - \sum_{n=k}^\infty (n-\gamma)C_{n,p}^\lambda a_n z^n \right] + \sum_{n=k}^\infty (n-p)C_{n,p}^\lambda a_n z^n
\]
Now $\Re\{f(z)\} \leq |f(z)|$ for all $z$, we have
\[
\Re\left\{ \frac{\sum_{n=k}^\infty (n-p)C_{n,p}^\lambda a_n z^n}{2^\xi \left[ (p-\gamma)z^n - \sum_{n=k}^\infty (n-\gamma)C_{n,p}^\lambda a_n z^n \right] + \sum_{n=k}^\infty (n-p)C_{n,p}^\lambda a_n z^n} \right\} < \beta
\] (15)

Choose value of $z$ on the real axis so that $\frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)}{\Omega_z^{(\lambda,p)} f(z)}$ is real. Then upon clearing the denominator in (15) and letting $z \to 1$ through real values, we have
\[
\sum_{n=k}^\infty (n-p)C_{n,p}^\lambda a_n z^n \leq \beta
\]
\[
2^\xi \left[ (p-\gamma) - \sum_{n=k}^\infty (n-\gamma)C_{n,p}^\lambda a_n \right] + \sum_{n=k}^\infty (n-p)C_{n,p}^\lambda a_n
\]
That is
\[
\sum_{n=k}^{\infty} \left( (n-p)(1-\beta)+2\xi\beta(n-\gamma) \right) C_{n,p}^k a_n \leq 2\beta \xi (p-\gamma)
\]
This is the required condition, which completes the proof of theorem 2.1.

**Corollary 2.2** Let the function \( f \) be defined by (12). Then \( f \) is in the class \( TS^\lambda_p(\beta,\gamma,\xi) \) if and only if

\[
\sum_{n=k}^{\infty} \Psi_{(p,n)}^\lambda (\beta,\gamma,\xi) a_n \leq 1.
\]

(16)

Where, \( \Psi_{(p,n)}^\lambda (\beta,\gamma,\xi) = \frac{[(n-p)(1-\beta)+2\xi\beta(n-\gamma)]C_{n,p}^k}{2\beta \xi (p-\gamma)} \).

(17)

0 < \beta \leq 1, \frac{1}{2} \leq \xi \leq 1, 0 \leq \gamma < \frac{p}{2}, n \geq k, p < k, -\infty < \lambda < p + 1.

**Corollary 2.3** Let the function \( f \) defined by (12) is in the class \( TS^\lambda_p(\beta,\gamma,\xi) \) then we have

\[
a_n \leq \frac{2\xi\beta(p-\gamma)}{[(n-p)(1-\beta)+2\xi\beta(n-\gamma)]C_{n,p}^k}, (n \geq k)
\]

(18)

The result is sharp for the function \( f \) given by

\[
f(z) = z^p - \frac{2\xi\beta(p-\gamma)}{[(n-p)(1-\beta)+2\xi\beta(n-\gamma)]C_{n,p}^k} z^n, n \geq k.
\]

(19)

3. NEIGHBORHOOD FOR THE CLASS \( TS^\lambda_p(\beta,\gamma,\xi) \)

Next, following the earlier investigations by Goodman [8], Rucheweyh [26], and others including Srivastava et al. [29], Orhan (121 and [20]), Altinas et al. [2] (see also [11], [16], [31], [3]), we define the \((k, \delta)\)-neighborhood of functions in the family \( TS^\lambda_p(\beta,\gamma,\xi) \).

**Definition 3.1** For \( f \in T_p \) of the form (12) and \( \delta \geq 0 \) we define a \((k, \delta)\)-neighborhood of a function \( f(z) \) by

\[
N_{k,\delta}(f) = \left\{ g : g \in T_p, g(z) = z^p - \sum_{n=k}^{\infty} c_n z^n \& \sum_{n=k}^{\infty} n |a_n - c_n| \leq \delta \right\}.
\]

In particular, for the function, \( h(z) = z^p \)

We immediately have

\[
N_{k,\delta}(h) = \left\{ g : g \in T_p, g(z) = z^p - \sum_{n=k}^{\infty} c_n z^n \& \sum_{n=k}^{\infty} n |c_n| \leq \delta \right\}.
\]

**Theorem 3.2** The class \( TS^\lambda_p(\beta,\gamma,\xi) \subset N_{k,\delta}(h) \) where \( \delta = \frac{(k+1-2p)}{\Psi_{(k,p)}^\lambda (\beta,\gamma,\xi)} \).

**Proof** For the function \( f(z) \in TS^\lambda_p(\beta,\gamma,\xi) \) of the form (12), corollary 1 immediately yields

\[
[(k-p)(1-\beta)+2\xi\beta(k-\gamma)]C_{k,p}^\lambda \sum_{n=k}^{\infty} a_n \leq 2\beta \xi (p-\gamma).
\]
\[
\sum_{n=k}^{\infty} a_n \leq \frac{2 \beta \xi (p - \gamma)}{(k - p)(1 - \beta) + 2 \xi \beta (k - \gamma)} C_{k,p}^\lambda \Psi_{(k,p)}^\lambda (\beta, \gamma, \xi)
\]

(20)

On the other hand, we also find from (16) and (20) that

\[
C_{k,p}^\lambda \sum_{n=k}^{\infty} a_n \leq 2 \beta \xi (p - \gamma) + [(1 - p)(1 - \beta) - 2 \xi \beta (k - \gamma)] C_{k,p}^\lambda \sum_{n=k}^{\infty} a_n
\]

\[
\leq 2 \beta \xi (p - \gamma) + [(1 - p)(1 - \beta) - 2 \xi \beta (k - \gamma)] \frac{2 \beta \xi (p - \gamma)}{(k - p)(1 - \beta) + 2 \xi \beta (k - \gamma)} C_{k,p}^\lambda
\]

\[
\leq \frac{2 \beta \xi (p - \gamma)(k + 1 - 2p)}{(k - p)(1 - \beta) + 2 \xi \beta (k - \gamma)} C_{k,p}^\lambda = \frac{(k + 1 - 2p)}{\Psi_{(k,p)}^\lambda (\beta, \gamma, \xi)} = \delta,
\]

Which in view of definition 3.1, proves Theorem 3.

4. PARTIAL SUMS

Following the earlier works by Silverman [27], N.C. Cho et al. [5] and others (see also [25], [13]), in this section we investigate the ratio of real parts of functions involving (12) and their sequence of partial sums defined by

\[
f_1(z) = z^p; \quad f_n(z) = z^p - \sum_{n=k}^{\infty} a_n z^n, r \in N
\]

(21)

And determine sharp lower bounds for

\[
\Re \left\{ \frac{f(z)}{f_n(z)} \right\}, \Re \left\{ \frac{f_n(z)}{f(z)} \right\}, \Re \left\{ \frac{f'(z)}{f_n'(z)} \right\}, \Re \left\{ \frac{f'(z)}{f(z)} \right\}.
\]

Theorem 4.1 If \( f \) of the form (12) satisfies condition (13), then

\[
\Re \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{\Psi_{(p,k+r)}^\lambda (\beta, \gamma, \xi) - 1}{\Psi_{(p,k+r)}^\lambda (\beta, \gamma, \xi)}
\]

(22)

and

\[
\Re \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{\Psi_{(p,k+r)}^\lambda (\beta, \gamma, \xi)}{\Psi_{(p,k+r)}^\lambda (\beta, \gamma, \xi) + 1}
\]

(23)

Where \( \Psi_{(p,n)}^\lambda (\beta, \gamma, \xi) \) is given by (17).

Proof. In order to prove (22), it is sufficient to show that

\[
\Psi_{(p,k+r)}^\lambda (\beta, \gamma, \xi) \left| \frac{f(z)}{f_n(z)} \right| \leq \frac{1 + z}{1 - z} (z \in U).
\]
We can write

\[ \Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \left[ \frac{f(z)}{f_n(z)} - \left( \frac{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) - 1}{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi)} \right) \right] \]

\[ = \Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \left[ \frac{1 - \sum_{n=k}^{\infty} a_n z^{-p}}{1 - \sum_{n=k+r}^{\infty} a_n z^{-p}} - \left( \frac{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) - 1}{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi)} \right) \right] \]

\[ = \Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \left[ \frac{1 - \sum_{n=k}^{\infty} a_n z^{-p} - \sum_{n=k+r}^{\infty} a_n z^{-p}}{1 - \sum_{n=k+r}^{\infty} a_n z^{-p}} - \left( \frac{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) - 1}{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi)} \right) \right] \]

\[ = \frac{1 + w(z)}{1 - w(z)}. \]

Then

\[ w(z) = \frac{-\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \sum_{n=k+r}^{\infty} a_n z^{-p}}{2 - 2 \sum_{n=k}^{\infty} a_n - \Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \sum_{n=k+r}^{\infty} a_n} \]

Obviously \( w(0) = 0 \) and

\[ |w(z)| \leq \frac{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \sum_{n=k+r}^{\infty} a_n}{2 - 2 \sum_{n=k}^{\infty} a_n - \Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \sum_{n=k+r}^{\infty} a_n} \]

Now, \(|w(z)| \leq 1\) if and only if

\[ 2\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \sum_{n=k+r}^{\infty} a_n \leq 2 - 2 \sum_{n=k}^{\infty} a_n, \]

which is equivalent to

\[ \sum_{n=k}^{\infty} a_n - \Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \sum_{n=k+r}^{\infty} a_n \leq 1. \]

In view of (13), this is equivalent to showing that

\[ \sum_{n=k}^{\infty} \left[ \Psi^\lambda_{(p,n)}(\beta, \gamma, \xi) - 1 \right] a_n + \sum_{n=k+r}^{\infty} \left[ \Psi^\lambda_{(p,n)}(\beta, \gamma, \xi) - \Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) \right] a_n \geq 0. \]

Thus we have completed the proof of (22), the proof of (23) is similar to (22) and will be omitted.

**Theorem 4.2** If \( f(z) \) of the form (12) satisfies (13), then

\[ \Re \left[ \frac{f(z)}{f_n(z)} \right] \geq \frac{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi) - k - 1}{\Psi^\lambda_{(p,k+r)}(\beta, \gamma, \xi)} \]

and
\[ \mathcal{R} \left\{ f_n(z) \right\} \geq \frac{\Psi^{k}_{(p,k+r)}(\beta, \gamma, \xi)}{\Psi^{k}_{(p,k+r)}(\beta, \gamma, \xi) + k + 1} \]

(25)

Where \( \Psi^{k}_{(p,k+r)}(\beta, \gamma, \xi) \) is given by (17).

5. INTEGRAL MEANS

The following subordination result due to Littewood [12] will be required in our investigation. The integral means of analytic functions was studied in [25], [19].

**Lemma 5.1** if \( f(z) \) and \( g(z) \) are analytic in \( U \) with, \( f(z) \prec g(z) \), then

\[ \int_{0}^{2\pi} |f(re^{i\theta})|^\mu \ d\theta \leq \sum_{k=0}^{2\pi} |g(re^{i\theta})|^\mu \ d\theta, \]

where \( \mu > 0, z = re^{i\theta} \) & \( 0 < r < 1 \).

Application of Lemma 5.1 to function \( f(z) \) in the class \( TS^A_p(\beta, \gamma, \xi) \) gives the following result using known procedures.

**Theorem 5.2** Let \( f(z) \in TS^A_p(\beta, \gamma, \xi) \) and \( f_2(z) = z^n - \frac{1}{\Psi^{A}_{(p,n)}(\beta, \gamma, \xi)} z^n \) where \( \Psi^{A}_{(p,n)}(\beta, \gamma, \xi) \) is given by (17), if \( f(z) \) satisfies

\[ \sum_{n=k}^{\infty} |a_n| \leq \left| \frac{1}{\Psi^{A}_{(n,p)}(\beta, \gamma, \xi)} \right| \]

(26)

Then for \( \mu > 0 \) and \( z = re^{i\theta}, (0 < r < 1) \),

\[ \int_{0}^{2\pi} |f(z)|^\mu \ d\theta \leq \int_{0}^{2\pi} |f_2(z)|^\mu \ d\theta. \]

(27)

**Proof.** By putting \( z = re^{i\theta}, (0 < r < 1) \), we see that

\[ \int_{0}^{2\pi} |f(z)|^\mu \ d\theta = r^{\mu p} \int_{0}^{2\pi} \left| 1 - \sum_{n=k}^{\infty} a_n z^{-n-p} \right|^\mu \ d\theta. \]

And

\[ \int_{0}^{2\pi} |f_2(z)|^\mu \ d\theta = r^{\mu p} \int_{0}^{2\pi} \left| 1 - \frac{1}{\Psi^{A}_{(n,p)}(\beta, \gamma, \xi)} z^{-n-p} \right|^\mu \ d\theta. \]

Applying lemma (5.1), we have to show that

\[ 1 - \sum_{n=k}^{\infty} a_n z^{-n-p} < 1 - \frac{1}{\Psi^{A}_{(n,p)}(\beta, \gamma, \xi)} z^{-n-p}, \]

Let us define the function \( w(z) \) by

\[ 1 - \sum_{n=k}^{\infty} a_n z^{-n-p} = 1 - \frac{1}{\Psi^{A}_{(n,p)}(\beta, \gamma, \xi)} (w(z))^{-n-p} \]

(28)
or by

$$\frac{1}{\psi_{(n,p)}(\beta,\gamma,\xi)}(w(z))^{\alpha-\beta} = \sum_{n=k}^{\infty} a_n z^{\alpha-\beta}$$

(29)

Since, for $z = 0$, $\frac{1}{\psi_{(n,p)}(\beta,\gamma,\xi)}(w(0))^{\alpha-\beta} = 0$,

there exists an analytic function $w(z)$ in $U$ such that $w(0) = 0$.

Next, we prove the analytic function $w(z)$ satisfies $|w(z)| < 1$ ($z \in U$) for

$$\sum_{n=k}^{\infty} |a_n| \leq \left| \frac{1}{\psi_{(n,p)}(\beta,\gamma,\xi)} \right|$$

By the equality (27), we know that

$$\left| \frac{1}{\psi_{(n,p)}(\beta,\gamma,\xi)}(w(z))^{\alpha-\beta} \right| \leq \sum_{n=k}^{\infty} a_n z^{\alpha-\beta} < \sum_{n=k}^{\infty} |a_n|,$$

For $z \in U$, hence,

$$\left| \frac{1}{\psi_{(n,p)}(\beta,\gamma,\xi)}(w(z))^{\alpha-\beta} - \sum_{n=k}^{\infty} a_n \right| < 0.$$  

(30)

Letting $t = |w(z)|$ ($t \geq 0$) in (30), we define the function $G(t)$ by

$$G(t) = \left| \frac{1}{\psi_{(n,p)}(\beta,\gamma,\xi)}(t)^{\alpha-\beta} - \sum_{n=k}^{\infty} a_n \right| \quad (t \geq 0).$$

If $G(1) \geq 0$, then we have $t < 1$ for $G(t) < 0$. Therefore, for $|w(z)| < 1$ ($z \in U$), we need

$$G(1) = \left| \frac{1}{\psi_{(n,p)}(\beta,\gamma,\xi)} - \sum_{n=k}^{\infty} a_n \right| \geq 0,$$

that is,

$$\sum_{n=k}^{\infty} |a_n| \leq \left| \frac{1}{\psi_{(n,p)}(\beta,\gamma,\xi)} \right|.$$  

Consequently, if the inequality (26) holds true, there exists an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ ($z \in U$), such that $f(z) = f_2(w(z))$. This completes the proof of Theorem (5).

6. MODIFIED HADAMARD PRODUCT

For the functions $f_j(z) = z^p - \sum_{n=k}^{\infty} a_{n,j} z^n$ ($a_{n,j} \geq 0; j = 1, 2; p, k \in N$),

$$f_1 * f_2 = f_2(w(z))$$

(31)

We denote by $(f_1, f_2)$ the modified Hadamard product of functions $f_1$ and $f_2$, that is,
\[(f_1 * f_2)(z) = z^p - \sum_{n=k}^{\infty} a_{n,1} a_{n,2} z^n.\]  
(32)

**Theorem 6.1** Let the functions \(f_j (j = 1, 2)\), defined by (31) be in the class \(TS^\lambda_p (\beta, \gamma, \xi)\) then \((f_1 * f_2) \in TS^\lambda_p (\beta, \mu, \xi)\) where

\[
\mu = p - \frac{2\beta \xi (p-\gamma)^2 (k-p) [(1-\beta) + 2\beta \xi]}{[(k-p)(1-\beta) + 2\beta \xi (k-\gamma)]^2 C_{n,p}^{\lambda} - 4\beta^2 \xi^2 (p-\gamma)^2}
\]  
(33)

The result is sharp.

**Theorem 6.2** Let the function \(f_j (j = 1, 2)\) defined by (31), \(f_1 \in TS^\lambda_{n,p} (\beta, \mu_1, \xi)\) and \(f_2 \in TS^\lambda_{n,p} (\beta, \mu_2, \xi)\).

Then \((f_1 * f_2) \in TS^\lambda_{n,p} (\beta, \mu, \xi)\), where

\[
\mu = p - \frac{2\xi \beta (p-\mu_1)(p-\mu_2)(k-p) [(1-\beta) + 2\beta \xi]}{A_1(\mu_1, p, \beta, \xi, k) A_2(\mu_2, p, \beta, \xi, k) C_{n,p}^{\lambda} - 4\xi^2 \beta^2 (p-\mu_1)(p-\mu_2)}
\]  
(34)

And

\[
A_1(\mu_1, p, \beta, \xi, k) = [(k-p)(1-\beta) + 2\beta \xi (k-\mu_1)]
\]  
(35)

\[
A_2(\mu_2, p, \beta, \xi, k) = [(k-p)(1-\beta) + 2\beta \xi (k-\mu_2)]
\]  
(36)

**Theorem 6.3** Let the functions \(f_j (j = 1, 2)\) defined by (31) are in the class \(TS^\lambda_p (\beta, \gamma, \xi)\). Then the function

\[h(z) = z^p - \sum_{n=k}^{n} (a_{n,1}^2 + a_{n,2}^2) z^n\]  
(37)

Belongs to the class \(TS^\lambda_p (\beta, \tau, \xi)\), where

\[
\tau = p - \frac{4\beta \xi (p-\gamma)^2 (n-p) [(1-\beta) + 2\beta \xi]}{[(n-p)(1-\beta) + 2\beta \xi (n-\gamma)]^2 C_{n,p}^{\lambda} - 8\beta^2 \xi^2 (p-\gamma)^2}
\]  
(38)

The result is sharp for the functions \(f_j (j = 1, 2)\) defined by (31).

**Conflict of Interest**

The authors confirm that there is no conflict of interest to declare for this publication.

**REFERENCES**


