Exact solutions for nonlinear integro-partial differential equations using the \((G'/G, 1/G)\)-expansion method

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Abstract  
In this paper, we improve the \((G/G, 1/G)\)-expansion method to solve the nonlinear integro-partial differential equations. We use the proposed method to construct the traveling wave solutions for some nonlinear equations of evolution in mathematical physics via \((1+1)\)-dimensional Ito nonlinear integro-partial differential equation, first and second integral differential KP hierarchy equations. The \((G/G, 1/G)\) expansion method is compared between two different methods namely G/G expansion method and \((1/G)\) expansion method. This proposed method will be submitted to literature to extract exact solutions with arbitrary parameters which include solitary and periodic wave solutions of nonlinear integro-differential equations.

Keywords: the nonlinear integro-partial differential equations, KP hierarchy equations, the \((G/G, 1/G)\)-expansion method.

Introduction  
Nonlinear partial differential equations are useful in describing the various phenomena in disciplines and have many applications not only in physics but also in medical sciences, geometry, biology and chemistry. There are many ways to find accurate solutions for nonlinear partial differential equations (NPDEs) such as homogenous balance method [1-2], Darboux transform method [3-4], first integral method [5-6], tanh function method [7], modified simple equation method [8-10], auxiliary equations method [11-12], \((G'/G)\)-expansion method [13-14], \(F\)-expansion method [15-16], Jacobi elliptic function method [17-18] and so on. The \(G'/G\) expansion method was proposed by Wang et al. Guo and Zhou developed extended \(G'/G\) expansion method [19]. Recently Lü improved generalized \(G'/G\) expansion method [20]. In \(G'/G\) the trial auxiliary equation satisfies the second order linear equation as \(G'' + \lambda G' + \mu G = 0\). More recently, Li et al. [21] have presented the two variables \((G'/G, 1/G)\)-expansion method. In this paper, we have applied \((G'/G, 1/G)\)-expansion method to find the traveling wave solutions for some of the following nonlinear integro-partial differential equations:

(i) the \((1+1)\)-dimensional Ito nonlinear partial differential equation[22]:

\[
u_{tt} + \sigma_{xxt} + 3\left(2u_{xxx} + uu_{xx}\right) + 3u_{x} \int_{-\infty}^{x} u_{x} \, dx = 0 \quad (1.1)
\]

(ii) The first integral differential KP hierarchy equation[23]:

\[
v_{t} = \frac{1}{2}v_{xxx} + \frac{1}{2}v_{x}^{-1}\left[v_{xxx}\right] + 2v_{x} v_{x}^{-1}\left[v_{y}\right] + 4v_{x} v_{y} \quad (1.2)
\]

(iii) The second integral differential KP hierarchy equation[24-25]:

\[
v_{t} = \frac{1}{16}\nu_{xxxx} + \frac{3}{4}\nu_{x}^{-1}\left[v_{yy}\right] + \frac{5}{4}\nu_{x}^{-1}\left[v_{xy}\right] + \frac{5}{8}\nu_{x}^{-1}\left[v_{y}\right] + \frac{3}{16}\nu_{x}^{-1}\left[v_{yyy}\right] + \frac{3}{2}\nu_{x}^{-1}\left[v_{yy}\right] + \frac{3}{2}\nu_{x}^{-1}\left[v_{y}\right] + \frac{3}{8}\nu_{x}^{-1}\left[v_{x}\right] + \frac{15}{8}\nu_{x}^{-1}\left[v_{x}\right] \quad (1.3)
\]

where \(\partial_{x}^{-1} = \int_{0}^{x} dx\)

The contents of this chapter have submitted to publication in advanced difference equations.

The algorithm of \((G'/G, 1/G)\)-expansion method  
In this section we will describe the main steps of the \((G'/G, 1/G)\)-expansion method to search for the traveling wave solutions for nonlinear development equations. Suppose that the equation is given by:

\[
Q(v, v_{t}, v_{x}, v_{xy}, \ldots) = 0 \quad (2.1)
\]
Q is a polynomial in \( v(x, t) \) and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. The \((G'/G, 1/G)\)-expansion to solve Eq. (2.1) is summarized in the following steps:

**Step 1:** Firstly we use the transformation to convert the nonlinear integro-differential equations to nonlinear partial differential equations and using the traveling wave transformation:

\[
v = V(\xi), \quad \xi = k_x x - L t \tag{2.2}
\]
where \( k_i \ (i = 0, \ldots, m) \) and \( L \) are arbitrary constants.

The traveling wave transformation (2.2) permits us to reduce Eq. (2.1) to the following ODE's:

\[
Q(V, V', V'', \ldots) = 0 \tag{2.3}
\]

**Step 2:** Assume that the solution of ODE (2.3) can be expressed by a polynomial in \( \phi, \psi \) as follows:

\[
V(\xi) = \sum_{i=0}^{n} a_i \phi^i + \sum_{i=1}^{m} b_i \psi^{i-1} \phi \tag{2.4}
\]
where \( a_i (i = 0, 1, \ldots, n) \) and \( b_i (i = 1, 2, \ldots, m) \) are arbitrary constants. In the solution formula (2.4) \( \phi(\xi), \psi(\xi) \) are satisfying the following relations

\[
\phi(\xi) = \frac{G'(\xi)}{G(\xi)}, \quad \psi(\xi) = \frac{1}{G(\xi)} \tag{2.5}
\]
and the auxiliary equation is given by

\[
G''(\xi) + \lambda G(\xi) = \mu \tag{2.6}
\]

The Eq.(2.5) and Eq.(2.6) lets to get:

\[
\phi' = -\phi^2 + \mu \psi - \lambda, \quad \psi' = -\phi \psi \tag{2.7}
\]
The solution of the Eq.(2.6) in the following three cases:

**Case 1.** If \( \lambda < 0 \), the general solution of Eq. (2.6) is given by

\[
G = A_1 \sinh(\sqrt{-\lambda} \xi) + A_2 \cosh(\sqrt{-\lambda} \xi) + \mu / \lambda \tag{2.8}
\]
and

\[
\psi^2 = -\frac{\lambda(\phi^2 - 2\mu \psi + \lambda)}{\lambda^2 \sigma + \mu^2}, \tag{2.9}
\]
where \( A_1, A_2 \) are two arbitrary constants and \( \sigma = A_1^2 - A_2^2 \).

**Case 2.** If \( \lambda > 0 \), the general solution of Eq. (2.6) is given by

\[
G = A_1 \sin(\sqrt{\lambda} \xi) + A_2 \cos(\sqrt{\lambda} \xi) + \mu / \lambda \tag{2.10}
\]
and

\[
\psi^2 = \frac{\lambda(\phi^2 - 2\mu \psi + \lambda)}{\lambda^2 \sigma + \mu^2}, \tag{2.11}
\]
where \( \sigma = A_1^2 + A_2^2 \).

**Case 3.** If \( \lambda = 0 \), the general solution of Eq. (2.6) is

\[
G = \frac{\mu}{2} \xi^2 + A_1 \xi + A_2 \tag{2.12}
\]

Subsequently, the relation between \( \phi \) and \( \psi \) are given by:

\[
\psi^2 = \frac{\phi^2 - 2\mu \psi}{A_1^2 - 2\mu A_2} \tag{2.13}
\]

**Step 3** The balance number \( n \) is determined using the homogeneous balance between the non-linear terms and the higher order derivatives in the equation (2.3),

**Step 4:** Substituting (2.4) to (2.3) using (2.7) and (2.9), we will get many polynomials in \( \phi \) and \( \psi \). From equations (2.9), (2.11) and (2.13) the degree of \( \psi \) is not greater than 1. Setting the coefficient of \( \psi^2 \) \((i = 0, 2, j = 0, 1, \ldots, n)\) to be zero we produces a set of algebraic equations that can be solved by Maple software package, to obtain the values of \( a_i, b_i, L \) and \( k_i \).

**Step 5:** Substituting the values of \( a_i, b_i, L \) and \( k_i \) in (2.4) and (2.8), (2.10) and (2.12), we construct many different kind of the traveling wave solutions to Eq. (2.3).

**Application of (G'/G , 1/G)- expansion method**

In this section, we apply the proposed method \((G'/G, 1/G)\)-expansion method to construct the traveling wave solution to solve the following nonlinear partial differential equation of Ito \((1+1)\) and also to discuss the analytical solution of the Kadomtsev-Petviashvili hierarchy equations by using \((G'/G, 1/G)\)-expansion method:

**the (1+1)- dimensional Ito nonlinear partial differential equation**

In this subsection we will study the exact solutions to the following \((1+1)\)- dimensional Ito integro differential equation

\[
u_{tt} + u_{xxxx} + 3(u_x u_{xx} + u u_{xxx}) + 3 u_{xx} u_{xx} u_{x x} d^3 v = 0 \tag{3.1.1}
\]

We use the transformations \( u = V_x \) to reduce the integro-differential equation (3.1.1) to the following nonlinear partial differential equations

\[
u_{xxxx} + v_{xxxx} + 3(2 u_{xx} v_x + u v_{xxx}) + 3 v_{xxx} v_x = 0 \tag{3.1.2}
\]
The traveling wave transformation

\[
v = V(\xi), \quad \xi = x - kt \tag{3.1.3}
\]
leads to reduce Eq. (3.1.2) to the ODE in the form

\[
k V'''' - V(\xi) - 3[(V(\xi)]'' = 0 \tag{3.1.4}
\]

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Integrating Eq. (3.1.3) twice time, we have:

\[ kV'' - V''' - 3(V')^2 + C_1 = 0, \tag{3.1.5} \]

where \( C_1 \) is the integration constant further setting \( V' = u \), we have:

\[ ku'' - U'' - 3 U^2 + C_1 = 0. \tag{3.1.6} \]

Balancing the highest order derivative \( U'' \) with the nonlinear term \( U^2 \), we get \( n = 2 \).

We suppose that the Eq. (3.1.6) has the formal solution:

\[ U = a_0 + a_1 \phi + a_2 \phi^2 + (b_1 + b_2 \phi) \psi \tag{3.1.7} \]

where \( a_0, a_1, a_2, b_1 \) and \( b_2 \) are constants.

**Case 1.** when \( \lambda < 0 \) (hyperbolic function solutions):

Substituting (3.1.7) into (3.1.6) and using (2.7) and (2.9) we get the polynomials in \( \phi \) and \( \psi \), setting the coefficients of this polynomial to zero to obtain the following set of algebraic equations:

\[ \phi^4: -3a_0^2 - 6a_2 + \frac{2\phi_1^2}{\phi^2 + \mu^2} = 0, \]

\[ \phi^5: -6a_0 b_1 - 6b_2 = 0, \]

\[ \phi^4: -6a_1 a_2 - 2a_4 - \frac{4b_2 b_4}{\phi^2 + \mu^2} + \frac{6\phi_1 b_2}{\phi^2 + \mu^2} = 0, \]

\[ \phi^5: -6a_2 b_1 - 6a_1 b_2 + 10a_2 \mu - \frac{6b_2 b_4}{\phi^2 + \mu^2} = 0, \]

\[ \phi^4: -6a_0 a_1 - 2a_3 + k a_2 - 4a_2 \mu + \frac{8b_2^2}{\phi^2 + \mu^2} + \frac{6\phi_1 b_2}{\phi^2 + \mu^2} - \frac{b_4 b_4}{\phi^2 + \mu^2} = 0, \]

\[ \phi^5: -6a_0 b_1 - 6a_1 b_2 + 10a_2 \mu - \frac{6b_2 b_4}{\phi^2 + \mu^2} + \frac{6\phi_1 b_2}{\phi^2 + \mu^2} = 0, \]

\[ \psi^4: -6a_0 a_1 - 2a_3 + k a_2 - 4a_2 \mu + \frac{8b_2^2}{\phi^2 + \mu^2} + \frac{6\phi_1 b_2}{\phi^2 + \mu^2} - \frac{b_4 b_4}{\phi^2 + \mu^2} = 0, \]

\[ \psi^5: -6a_0 b_1 - 6a_1 b_2 + 10a_2 \mu - \frac{6b_2 b_4}{\phi^2 + \mu^2} + \frac{6\phi_1 b_2}{\phi^2 + \mu^2} = 0, \]

\[ C_1 = 0. \]

By solving the previous algebraic equations using the Maple software packages we get the following results:

**Result 1**

\[ a_0 = -\frac{4}{3} \lambda \pm \frac{1}{6} \sqrt{4 \lambda^2 - 3C_1}, \quad a_1 = 0, \]

\[ a_2 = -2, \quad b_1 = 0, \quad b_2 = 0, \]

\[ \mu = 0, \quad k = \pm 2 \sqrt{4 \lambda^2 - 3C_1}. \tag{3.1.8} \]

By substituting (3.1.8) into (3.1.7) with (3.1.3) by using (2.5) and (2.8), we get the exact solutions of Eq. (3.1.1) as follows:

\[ u_{12} = -\frac{4}{3} \lambda \pm \frac{1}{6} \sqrt{4 \lambda^2 - 3C_1} - \frac{2a_1 \cosh(\sqrt{\lambda} t) \cosh(\sqrt{\lambda} t) + a_2 \sinh(\sqrt{\lambda} t) \sinh(\sqrt{\lambda} t)}{(a_1 \cosh(\sqrt{\lambda} t) + a_2 \sinh(\sqrt{\lambda} t))^2} \tag{3.1.9} \]

In particular, if we put \( A_1 = 0 \) and \( A_2 > 0 \) in Eq. (3.1.9), we produce a solitary solution:

\[ u_{12} = -\frac{4}{3} \lambda \pm \frac{1}{6} \sqrt{4 \lambda^2 - 3C_1} + 2 \lambda \tanh(\sqrt{\lambda} t) \tag{3.1.10} \]

Setting \( \lambda = -0.2, \ C_1 = 0.5 \) we illustrate the behavior of the solitary wave solution (3.1.10) and its projection at \( t = 0 \), as we shown in the Figure 1.

![Fig. 1. The exact solutions \( u_{11} \) of Eq.(3.1.10) ](image)

While, if we put \( A_2 = 0 \) and \( A_1 > 0 \) in Eq. (3.1.9), we produce a solitary solution:

\[ u_{12} = -\frac{4}{3} \lambda \pm \frac{1}{6} \sqrt{4 \lambda^2 - 3C_1} + 2 \lambda \coth(\sqrt{\lambda} t) \tag{3.1.11} \]

Setting \( \lambda = -0.2, \ C_1 = 0.5 \) we illustrate the behavior of the solitary wave solution (3.1.11) and its projection at \( t = 0 \), as we shown in the Figure 2.

![Fig. 2. The exact solutions \( u_{12} \) of Eq. (3.1.11) ](image)

where \( \xi = x \pm 2 \sqrt{4 \lambda^2 - 3C_1} t \) and \( \sigma = A_1^2 - A_2^2 \).

**Result 2**

\[ a_0 = -\frac{11}{2} \lambda \pm \frac{1}{6} \sqrt{\lambda^2 - 12C_1}, \quad a_1 = 0, \]

\[ a_2 = -2, \quad b_1 = 2 \mu, \quad b_2 = 0, \]

\[ k = \pm \sqrt{\lambda^2 - 12C_1}. \tag{3.1.12} \]
By substituting (3.1.12) into (3.1.7) with (3.1.3), (2.5) and (2.8), we get the exact solutions of Eq. (3.1.1) as follows:

$$u_2 = -rac{11}{6} \lambda + \frac{1}{6} \sqrt{\lambda^2 - 12C_1} - \frac{2(A_1 \cosh(\sqrt{\lambda} z) + A_2 \sinh(\sqrt{\lambda} z))}{(A_2 \cosh(\sqrt{\lambda} z) + A_1 \sinh(\sqrt{\lambda} z))}$$  \hspace{1cm} (3.1.13)

In particular, if we put $\mu = 0$, $A_1 = 0$ and $A_2 > 0$ in Eq. (3.1.13), we produce a solitary solution:

$$u_{2s} = -\frac{11}{6} \lambda + \frac{1}{6} \sqrt{\lambda^2 - 12C_1} + 2\lambda \tanh^2(\sqrt{\lambda} \xi)$$  \hspace{1cm} (3.1.14)

While, if we put $\mu = 0$, $A_2 = 0$ and $A_1 > 0$, then we produce a solitary solution:

$$u_{2s} = -\frac{11}{6} \lambda \pm \frac{1}{6} \sqrt{\lambda^2 - 12C_1} + 2\lambda \coth^2(\sqrt{\lambda} \xi)$$  \hspace{1cm} (3.1.15)

The behavior of the exact solution (3.1.13) and its projection at $t = 0$, when $\lambda = -0.2$, $C_1 = 0.5$, $\mu = 0.1$ as we shown in the Figure 3.

Case 2. when $\lambda > 0$ (trigonometric function solutions):

Substituting (3.1.7) into (3.1.6) and using (2.7) and (2.11), we get the polynomials in $\phi$ and $\psi$. Setting the coefficients of this polynomial to be zero to obtain a set of algebraic

$$\phi_1^*: -3a_2^2 - 6a_2 - \frac{2b_2}{\lambda^2 \sigma - \mu^2} = 0,$$
$$\phi_2^*: -6a_2a_2 - 2a_1 - \frac{4b_1^2}{\lambda^2 \sigma - \mu^2} - \frac{2b_2}{\lambda^2 \sigma - \mu^2} = 0,$$
$$\phi_3^*: -6a_2b_1 - 2b_2 - 6a_2b_2 + 10a_2 \mu - \frac{6b_1^2}{\lambda^2 \sigma - \mu^2} = 0,$$
$$\phi_4^*: -6a_2a_2 - 2a_1 + k_2a_2 - 2a_2 \mu - \frac{6b_1^2}{\lambda^2 \sigma - \mu^2} + \frac{2b_2}{\lambda^2 \sigma - \mu^2} = 0,$$
$$\phi_5^*: -6a_2a_2 - 2a_1 - k_2a_2 - 2a_2 \mu - \frac{6b_1^2}{\lambda^2 \sigma - \mu^2} - \frac{2b_2}{\lambda^2 \sigma - \mu^2} = 0.$$  \hspace{1cm} (3.1.16)

Solving the system (3.1.16) using the Maple or Mathematica package we have:

Result 1

$$a_0 = -\frac{3}{2} \lambda \pm \frac{1}{2} \sqrt{4\lambda^2 - 3C_1}, \hspace{1cm} a_1 = 0,$$
$$a_2 = -2, \hspace{1cm} b_1 = 0, \hspace{1cm} b_2 = 0,$$
$$\mu = 0, \hspace{1cm} k = \pm 2\sqrt{4\lambda^2 - 3C_1}.$$  \hspace{1cm} (3.1.17)

By substituting (3.1.17) into (3.1.7) with (3.1.3) by using (2.5) and (2.10), we get the exact solutions of Eq. (3.1.1) has the form:

$$u_2 = -\frac{4}{5} \lambda \pm \frac{1}{5} \sqrt{4\lambda^2 - 3C_1} - \frac{2(A_1 \cos(\sqrt{\lambda} z) + A_2 \sin(\sqrt{\lambda} z))}{(A_1 \sin(\sqrt{\lambda} z) + A_2 \cos(\sqrt{\lambda} z))}$$  \hspace{1cm} (3.1.18)

In particular, if we put $A_1 = 0$ and $A_2 > 0$ in Eq. (3.1.18), we produce a solitary solution:

$$u_{2s} = -\frac{4}{5} \lambda \pm \frac{1}{5} \sqrt{4\lambda^2 - 3C_1} - 2\lambda \tan^2(\sqrt{\lambda} \xi).$$  \hspace{1cm} (3.1.19)

Setting $\lambda = 0.2$, $C_1 = 0.5$, we illustrate the behavior of the solitary wave solution (3.1.19) and its projection at $t = 0$ as shown in the Figure 4.
While, if we put $A_2 = 0$ and $A_1 > 0$, then we produce a solitary solution:

$$u_{32} = -\frac{11}{6} \lambda \pm \frac{1}{\sqrt{3}} \sqrt{\lambda^2 - 3C_1} - 2\lambda \cot^2(\sqrt{\lambda} \xi)$$  \hspace{1cm} (3.1.20)

Setting $\lambda = 0.2$, $C_1 = 0.5$, we illustrate the behavior of the solitary wave solution (3.1.20) and its projection at $t = 0$ as shown in the Figure 5.

**Fig. 5.** The exact solutions $u_{32}$ of Eq. (3.1.20)

where $\xi = x \pm 2\sqrt{4\lambda^2 - 3C_1} t$ and $\sigma = A_1^2 + A_2^2$

**Result 2**

$$a_0 = -\frac{11}{6} \lambda \pm \frac{1}{\sqrt{3}} \sqrt{\lambda^2 - 12C_1}, \quad a_1 = 0, \quad a_2 = -2$$

$$b_1 = 2\mu, \quad b_2 = 0, \quad k = \pm \sqrt{\lambda^2 - 12C_1}$$  \hspace{1cm} (3.1.21)

By substituting (3.1.21) into (3.1.7) with (3.1.3), (2.5) and (2.10), we get the exact solutions of Eq. (3.1.1) as follows:

$$u_4 = -\frac{11}{6} \lambda \pm \frac{1}{\sqrt{3}} \sqrt{\lambda^2 - 12C_1} + \frac{2(A_1 \cos(\sqrt{\lambda} \xi) + A_2 \sin(\sqrt{\lambda} \xi) \lambda)^k}{(A_1 \sin(\sqrt{\lambda} \xi) + A_2 \cos(\sqrt{\lambda} \xi) + \frac{1}{\sqrt{3}} \xi)^k}$$  \hspace{1cm} (3.1.22)

The behavior of the exact solution (3.1.22) and its projection at $t = 0$, when $\lambda = 0.2$, $C_1 = 0.5$, $\mu = 0.1$ as we shown in the Figure 6.

**Fig. 6.** The exact solutions $u_4$ of Eq. (3.1.22)

In particular, if we put $\mu = 0$, $A_1 = 0$ and $A_2 > 0$ in Eq. (3.1.22), we produce the solitary wave solution as following:

$$u_{42} = -\frac{11}{6} \lambda \pm \frac{1}{\sqrt{3}} \sqrt{\lambda^2 - 12C_1} - 2\lambda \tan^2(\sqrt{\lambda} \xi)$$  \hspace{1cm} (3.1.23)

While, if we put $\mu = 0$, $A_2 = 0$ and $A_1 > 0$, we produce the solitary solution

$$u_{42} = -\frac{11}{6} \lambda \pm \frac{1}{\sqrt{3}} \sqrt{\lambda^2 - 12C_1} - 2\lambda \cot^2(\sqrt{\lambda} \xi)$$  \hspace{1cm} (3.1.24)

where $\xi = x < \pm \sqrt{\lambda^2 - 12C_1} t$ and $\sigma = A_1^2 + A_2^2$

**Case 3.** when $\lambda = 0$

By substituting (3.1.7) into (3.1.6) and using (2.7) and (2.13) we get the polynomials in $\phi$ and $\psi$ By equating coefficients of equation with zero, we obtain a set of idioms which, by solving them, we obtain the following results:

$$a_1 = 0, \quad a_2 = -1, \quad b_1 = \mu, \quad b_2 = \pm \sqrt{-2\mu A_2 + A_1^2}, \quad k = 6a_0$$

$$C_1 = -3a_0^2$$  \hspace{1cm} (3.1.25)

where $a_0, A_1, A_2$ and $\mu$ are arbitrary constants. By substituting (3.1.25) into (3.1.7) with (3.1.3), (2.5) and (2.12), we get the rational traveling wave solutions of Eq. (3.1.1) has the form:

$$u_5 = a_0 - \frac{4(\mu A_1 + A_2)^2 + 2(\mu A_2 + A_1 + A_2) + 4(\mu A_1 + A_2) - 2\mu A_1 + A_2^2)}{(\mu A_2 + A_1 + A_2)^2}$$  \hspace{1cm} (3.1.26)

Setting $a_0 = 0.2$, $\mu = 0.1$, $A_1 = 1$ and $A_2 = 2$ we illustrate the behavior of the rational wave solution (3.1.26) and its projection at $t = 0$ as shown in the Figure 7.

**Fig. 7.** The exact solutions $u_5$ of Eq. (3.1.26)

where $\xi = x - 6a_0 t$

**The generalized KP hierarchy equations:**

In this subsection we will study the exact solutions to the following the first and second generalized KP hierarchy equations integro differential equation

$$v_x = \frac{1}{16} v_{xxxx} + \frac{5}{2} \theta_{x}^{2} \theta_{y}^{2} + 2v_{yy} \theta_{x}^{2} + 4v_{xy}$$  \hspace{1cm} (4.1)

and

$$v_x = \frac{1}{16} v_{xxxx} + \frac{5}{2} \theta_{x}^{2} \theta_{y}^{2} + 2v_{yy} \theta_{x}^{2} + 4v_{xy}$$  \hspace{1cm} (4.2)

where $\theta_{x}^{2} = \int_{0}^{x} dx$

\[2453\]
The KP equation can be used to model water waves of long wavelength with weakly non-linear restoring forces and frequency dispersion. If surface tension is weak compared to gravitational forces, $\lambda = +1$ is used; if surface tension is strong, then $\lambda = -1$. Because of the asymmetry in the way $x$- and $y$-terms enter the equation, the waves described by the KP equation behave differently in the direction of propagation ($x$-direction) and transverse ($y$) direction; oscillations in the $y$-direction tend to be smoother (be of small-deviation).

The KP equation can also be used to model waves in ferromagnetic media, as well as two-dimensional matter-wave pulses in Bose–Einstein condensates.

The first equation of integral differential KP hierarchy equations

We use the transformations $v = u_{xx}$ to reduce the integro-differential equation (4.1) to the following partial differential equations

$$u_{xxt} = \frac{1}{2} u_{xxxx} + \frac{1}{2} u_{yyyy} + 2u_{xx} u_{xy} + 4u_{xx} u_{xxy} \quad (4.1.1)$$

The traveling wave transformation

$$u = U(\xi), \quad \xi = x + y - kt, \quad (4.1.2)$$

leads to reduce Eq. (4.1.1) to the following ODE in the form

$$(k + \frac{1}{2}) u^{\prime\prime} + \frac{3}{2} u^{(4)} + 3(U^\prime)^2 + C_1 = 0, \quad (4.1.3)$$

where $C_1$ is the integration constant. When we take the conversion $U = W$, Eq. (4.1.3) is written on the formula:

$$(k + \frac{1}{2}) W^{\prime\prime} + \frac{3}{2} W^{(4)} + 3W^2 + C_1 = 0 \quad (4.1.4)$$

Balancing the highest order derivative $W^{\prime\prime}$ with the nonlinear term $W^2$, we get $n = 2$.

Consequently the solution of Eq. (4.1.3) has the formal solution:

$$W = a_0 + a_1 \phi + a_2 \phi^2 + (b_1 + b_2 \phi) \psi \quad (4.1.5)$$

where $a_0, a_1, a_2, b_1$ and $b_2$ are constants.

**Case 1.** When $\lambda < 0$, we substitute (4.1.5) into (4.1.4) and using (2.7) and (2.9) and setting the coefficient of $\phi$ and $\psi$ to be zero, we obtain the following set of algebraic equations:

$$\phi^2: 6a_0 a_2 + 3a_2^2 + (k + \frac{1}{2})a_2 + \frac{1}{2}a_2 - \frac{1}{2} \lambda \frac{b_1}{\lambda^2 \sigma + \mu^2} + \frac{b_1 \lambda \mu}{(\lambda^2 \sigma + \mu^2)^2} - \frac{a_2 \lambda^2}{\lambda^2 \sigma + \mu^2} = 0, \quad (4.1.6)$$

$$\psi^2: 6a_0 a_2 + 3a_2^2 + (k + \frac{1}{2})a_2 + \frac{1}{2}a_2 - \frac{1}{2} \lambda \frac{b_1}{\lambda^2 \sigma + \mu^2} + \frac{b_1 \lambda \mu}{(\lambda^2 \sigma + \mu^2)^2} - \frac{a_2 \lambda^2}{\lambda^2 \sigma + \mu^2} = 0, \quad (4.1.7)$$

Substituting (4.1.7) into (4.1.6), (2.5), (2.8), (2.9), we get the traveling wave exact solutions of Eq. (4.1) can be represented in the following form:

$$v_1 = -\frac{1}{2} \left(2 \lambda \pm \sqrt{\lambda^2 + 3C_1}\right) - \frac{A_1 \cosh(\sqrt{-\lambda} t) - A_2 \sinh(\sqrt{-\lambda} t)}{(A_1 \sinh(\sqrt{-\lambda} t) + A_2 \cosh(\sqrt{-\lambda} t))^2} \quad (4.1.8)$$

In particular, if we put $A_1 = 0$ and $A_2 > 0$ in Eq. (3.2.9), We produce a solitary solution:

$$v_{11} = -\frac{1}{2} \left(2 \lambda \pm \sqrt{\lambda^2 + 3C_1}\right) + \lambda \tanh(\sqrt{-\lambda} t) \quad (4.1.9)$$

Setting $\lambda = -0.2$, $C_1 = 0.5$, $y = 3$ we illustrate the behavior of the solitary wave solution and its projection at $t = 0$ as shown in Figure 8.

**Fig. 8.** The exact solutions $v_{11}$ of Eq. (4.1.9)
While, if we put $A_2 = 0$ and $A_1 > 0$ in Eq. (4.1.8), We produce a solitary solution:

$$v_{12} = -\frac{1}{2}(2\lambda \pm \sqrt{\lambda^2 + 3C_1}) + \lambda \coth^2\left(\sqrt{\lambda} \xi\right)$$

(4.1.10)

Setting $\lambda = -0.1$, $C_1 = 0.5$, $y = 3$ we illustrate the behavior of the solitary wave solution and its projection at $t = 0$ as shown in Figure 9.

![Figure 9](image)

Fig. 9. The exact solutions $v_{12}$ of Eq. (4.1.10)

where $\xi = x + y - \left(\frac{1}{2} \pm 2\sqrt{\lambda^2 + 3C_1}\right)t$ and $\sigma = A_1^2 - A_2^2$.

Result 2

$$a_0 = -\frac{1}{2}\left(5\lambda \pm \sqrt{\lambda^2 + 48C_1}\right), \quad a_1 = 0, \quad a_2 = -\frac{1}{2}, \quad b_1 = \frac{1}{2} \mu,$$

$$b_2 = \pm \frac{1}{2\lambda}\left(\frac{\lambda^2\sigma + \mu^2}{2A_1 \sinh(\sqrt{\lambda} t) + A_2 \cosh(\sqrt{\lambda} t)} + \frac{1}{2}\right)$$

$$k = -\frac{1}{2}\left(1 \pm \sqrt{\lambda^2 + 48C_1}\right)$$

(4.1.11)

By substituting (4.1.11) into (4.1.5) and using (2.5) and (2.8), we get the traveling wave solutions of (4.1.1) take the forms:

$$u_2 = \frac{1}{\Pi_{2}}\left[\frac{2A_1 \sinh(\sqrt{\lambda} t) + A_2 \cosh(\sqrt{\lambda} t)}{A_1 \sinh(\sqrt{\lambda} t) + A_2 \cosh(\sqrt{\lambda} t)}\right]$$

(4.1.12)

The behavior of the exact solution (4.1.12) and its projection at $t = 0$, when $\lambda = -0.1$, $C_1 = 0.5$, $\mu = 0.2$ as we shown in the Figure 10.

![Figure 10](image)

Fig. 10. The exact solutions $v_{12}$ of Eq. (4.1.12)

In particular, if we put $\mu = 0$, $A_1 = 0$ and $A_2 > 0$ in Eq. (4.1.12), we produce a solitary solution

$$v_{21} = -\frac{1}{2\lambda}\left(5\lambda \pm \sqrt{\lambda^2 + 48C_1}\right) + \frac{1}{2}\lambda \left(\cosh^2\left(\sqrt{-\lambda} \xi\right) + \frac{1}{2}\lambda \cosh\left(\sqrt{-\lambda} \xi\right) \csch\left(\sqrt{-\lambda} \xi\right)\right)$$

(4.1.13)

The behavior of the real solitary wave solution (4.1.13) and its projection at $t = 0$ when $\lambda = -0.1$, $C_1 = 0.5$, $y = 3$ are shown in the Figure 11.

![Figure 11](image)

Fig. 11. The exact solutions $v_{21}$ of Eq. (4.1.13)

While, if we put $\mu = 0$, $A_2 = 0$ and $A_1 > 0$, then ,we produce the solitary solution:

$$v_{22} = -\frac{1}{2\lambda}\left(5\lambda \pm \sqrt{\lambda^2 + 48C_1}\right) + \frac{1}{2}\lambda \left(\cosh^2\left(\sqrt{-\lambda} \xi\right) + \frac{1}{2}\lambda \cosh\left(\sqrt{-\lambda} \xi\right) \csch\left(\sqrt{-\lambda} \xi\right)\right)$$

(4.1.14)

where $\xi = x + y + \frac{1}{2}\left(1 \pm \sqrt{\lambda^2 + 48C_1}\right)t$ and $\sigma = A_1^2 - A_2^2$

Case 2. when $\lambda > 0$:

Substituting (4.1.5) into (4.1.4) and using (2.7) and (2.11), we get the polynomials in $\phi$ and $\psi$, setting the coefficients of this polynomial of $\phi$ and $\psi$ to be zero to obtain a set of algebraic equations:

$$\phi^4: 3\sigma_2 + 3\sigma_3 + \frac{3i\lambda_1}{\lambda^2\sigma + \mu^2} = 0,$$

$$\phi^3 \psi: b_2 + 6a_2 b_2 = 0,$$

$$\phi^2 \psi: 6a_1 \sigma_2 + a_1 + \frac{2b_2 \mu}{\lambda^2\sigma + \mu^2} + \frac{6\lambda_1 b_2}{\lambda^2\sigma + \mu^2} - \frac{b_2 \mu}{\lambda} = 0,$$

$$\phi \psi: 6a_0 \sigma_2 + 3\sigma_3 + \left(k + \frac{1}{2}\right) a_2 + 4a_2 \lambda + \frac{2b_3 \mu^2}{\lambda^2\sigma - \mu^2} + \frac{14b_3 \mu^2}{\lambda^2\sigma - \mu^2} - \frac{b_3 \mu^2}{\lambda^2\sigma - \mu^2} + \frac{2a_0 \mu^2}{\lambda^2\sigma - \mu^2} = 0,$$

$$\phi \theta: b_2 \lambda - \frac{b_2 \mu}{\lambda^2\sigma - \mu^2} + \frac{6a_1 \mu}{\lambda^2\sigma - \mu^2} + \frac{6a_0 b_2}{\lambda^2\sigma - \mu^2} + \left(k + \frac{1}{2}\right) b_2 + \frac{4b_3 \mu^2}{\lambda^2\sigma - \mu^2} + \frac{2b_3 \mu^2}{\lambda^2\sigma - \mu^2} + \frac{2a_0 \mu^2}{\lambda^2\sigma - \mu^2} = 0.$$
Using the Maple software package to solve the system of algebraic equations (4.1.15) to get the following results:

**Result 1**

\[
\phi: \alpha_2 \lambda + 6 \alpha_3 \alpha_1 + (k + \frac{1}{2}) \alpha_1 - b_2 \mu \sigma + b_2 \mu^2 + \frac{6i \beta_1 \beta_2}{\lambda^2 \sigma - \mu^2} - \frac{2b_1 \mu^2}{\lambda^2 \sigma - \mu^2} = 0,
\]

\[
\psi: -2a_2 \mu \lambda + 6a_2 b_1 + (k + \frac{1}{2}) b_1 + \frac{1}{2} b_1 \lambda - \frac{3 \beta_1 \mu^2}{\lambda^2 \sigma - \mu^2} - \frac{6i \beta_1 \mu^2}{\lambda^2 \sigma - \mu^2} = 0,
\]

(4.1.15)

\[
\phi^3: \alpha_2 \lambda^2 + 3 \alpha_3 \lambda \left( k + \frac{1}{2} \right) \alpha_0 + \frac{a_2 \mu^2 \lambda^2}{\lambda^2 \sigma - \mu^2} - \frac{b_1 \mu^2}{\lambda^2 \sigma - \mu^2} + \frac{3 \beta_1 \mu^2}{\lambda^2 \sigma - \mu^2} + C_1 = 0,
\]

(4.1.16)

Using the Maple software to solve the system of algebraic equations (4.1.15) to get the following results:

**Result 1**

\[
\alpha_0 = -\frac{1}{12} \left( 5\lambda \pm \sqrt{\lambda^2 + 48C_1} \right), \quad \alpha_1 = 0,
\]

\[
\alpha_2 = -\frac{1}{2}, \quad b_1 = 0, \quad b_2 = \pm \frac{1}{2} \sqrt{\lambda \sigma}, \quad \mu = 0, \quad k = -\frac{1}{2} \left( 1 \pm \sqrt{\lambda^2 + 48C_1} \right)
\]

(4.1.17)

Equations (4.1.16), (4.1.5), (2.5) and (2.10) lead to write the traveling wave solutions of (4.1.1) as follows:

\[
v_2 = -\frac{1}{12} \left( 5\lambda \pm \sqrt{\lambda^2 + 48C_1} \right) - \frac{A_1 \cos(\sqrt{\lambda} \xi) - A_2 \sin(\sqrt{\lambda} \xi)}{2(A_4 \sin(\sqrt{\lambda} \xi) + A_5 \cos(\sqrt{\lambda} \xi))^2} \left( A_4 \cos(\sqrt{\lambda} \xi) - A_2 \sin(\sqrt{\lambda} \xi) - \sqrt{A_1^2 + A_2^2} \right)
\]

(4.1.18)

In particular, if we put \(A_1 = 0\) and \(A_2 > 0\) in Eq. (4.1.17), we produce a solitary solution

\[
v_{21} = -\frac{1}{12} \left( 5\lambda \pm \sqrt{\lambda^2 + 48C_1} \right) - \frac{1}{2} \left( \tan^2(\sqrt{\lambda} \xi) + \tan(\sqrt{\lambda} \xi) \sec(\sqrt{\lambda} \xi) \right)
\]

(4.1.19)

The behavior of the solitary wave solution (4.1.18) and its projection at \(t = 0\) when \(\lambda = 0.2, C_1 = 0.25, y = 3\) are shown in the Figure 12.

**Result 2**

\[
\alpha_0 = -\frac{1}{12} (11\lambda \pm \sqrt{\lambda^2 + 48C_1}), \quad \alpha_1 = 0,
\]

\[
\alpha_2 = -1, \quad b_1 = \mu, \quad b_2 = 0, \quad k = -\frac{1}{2} \left( 1 \pm \sqrt{\lambda^2 + 48C_1} \right)
\]

(4.1.20)

Substituting Eq. (4.1.20) into (4.1.5) and using (2.5), (2.8), we get the traveling wave solution of Eq. (4.1.1) as follows:

\[
v_4 = -\frac{1}{12} \left( 11\lambda \pm \sqrt{\lambda^2 + 48C_1} \right) - \frac{A_1 \sin(\sqrt{\lambda} \xi) + A_2 \cos(\sqrt{\lambda} \xi) + \sqrt{A_1^2 + A_2^2}}{\left( A_4 \sin(\sqrt{\lambda} \xi) + A_5 \cos(\sqrt{\lambda} \xi) \right) \left( 1 \pm \sqrt{\lambda^2 + 48C_1} \right)}
\]

(4.1.21)

In particular, if we put \(\mu = 0, \lambda = 0, A_2 > 0\) in Eq. (4.1.21), we produce a solitary solution

\[
v_{41} = -\frac{1}{12} \left( 11\lambda \pm \sqrt{\lambda^2 + 48C_1} \right) - \lambda \tan^2 \left( \sqrt{\lambda} \xi \right)
\]

(4.1.22)

We illustrate the behavior solution (4.1.22) and its projection at \(t = 0\) when \(\lambda = 0.2, C_1 = 0.25, y = 3\) as shown in the Figure 13.

**Fig. 13.** The exact solutions \(v_{41}\) of Eq. (4.1.22)

While, if we put \(\mu = 0, A_2 = 0, A_1 > 0\), then we produce a solitary solution

\[
v_{42} = -\frac{1}{12} \left( 11\lambda \pm \sqrt{\lambda^2 + 48C_1} \right) - \lambda \cot^2 \left( \sqrt{\lambda} \xi \right)
\]

(4.1.23)

where \(\xi = x + y + \frac{1}{2} \left( 1 \pm \sqrt{\lambda^2 + 48C_1} \right) t\) and \(\sigma = A_1^2 + A_2^2\).

**Case 3.** when \(\lambda = 0\)

By substituting (4.1.5) into (4.1.4) and using (2.7) and (2.13), we get the polynomials in \(\phi\) and \(\psi\). Equating coefficients of this polynomial to be zero, we obtain a set of algebraic
equations. Solve this system of algebraic equations to obtain the following results:

\[
\begin{align*}
\alpha_1 &= 0, & \alpha_2 &= -\frac{1}{2}, & b_1 &= \frac{1}{2}\mu, \\
b_2 &= \pm \frac{1}{2} \sqrt{-2\mu A_2 + A_1^2}, & c_1 &= 3a_0^2, \\
k &= -6a_0 - \frac{1}{2}.
\end{align*}
\]

Equations (4.1.24), (4.1.5) (2.5) and (2.12), lead to write the rational traveling wave solutions as the following form:

\[
\psi = a_0 + \frac{\mu}{\mu^2 + 2A_2 + 2A_3} \left( \frac{\mu(\xi + A_2)^2 + (\mu^2 + A_2 + A_3)}{(\mu^2 + 2A_2 + 2A_3)^2} \right)
\]

(4.1.25)

The behavior solution (4.1.25) and its projection at \( t = 0 \) when \( a_0 = 0.3, \mu = 0.2, A_1 = 1, A_2 = 2 \) are shown in the Figure 14.

Fig. 14. The exact solutions \( \psi \) of Eq. (4.1.25)

where \( \xi = x + y + \left(6a_0 + \frac{1}{2}\right)t \).

The Second equation of two members of integral differential KP hierarchy equations:

In this section, we study the traveling exact solutions of the integral differential equations (1.3). Notice that the equation (1.3) contains one, two, and three integral operators. The traveling wave transformation (4.1.2) transfer the integral differential equations (1.3) to the following ordinary differential equation:

\[
\left(k + \frac{5}{16}\right)U' + \frac{1}{16}U^{(5)} + \frac{29}{4}U^{(3)} + \frac{11}{2}U'U' + \frac{1}{2}U''U'' + \frac{5}{4}(UU')' + \frac{5}{8}U'''' = 0
\]

(4.2.1)

By using integration, we have

\[
\left(k + \frac{5}{16}\right)U + \frac{1}{16}U^{(5)} + \frac{125}{4}U' + \frac{5}{2}U^2 + \frac{5}{2}U^2' + \frac{1}{4}U'' + \frac{5}{8}U'' + C_1 = 0
\]

(4.2.2)

where \( C_1 \) is the integration constant. Balancing the highest order derivative \( U^{(4)} \) with the nonlinear term \( U^2 \), we get \( n = 2 \). Consequently, we suppose the solution of (4.2.2) has the formal

\[
U = a_0 + a_1\phi + a_2\psi^2 + (b_1 + b_2\phi)\psi
\]

(4.2.3)

where \( a_0, a_1, a_2, b_1 \) and \( b_2 \) are constants.

Case 1. When \( \lambda < 0 \), Substituting (4.2.3) into (4.2.2) and using (2.7), (2.9), we get the polynomials in \( \phi \) and \( \psi \). Setting the coefficients of \( \phi \) and \( \psi \) to be zero, we obtain a set of algebraic equations, which can be solved by using the Maple software packages to get the following results:

Result 1

\[
\begin{align*}
\alpha_0 &= -\frac{1}{2} - \frac{5}{4}, & \alpha_1 &= 0, & \alpha_2 &= -\frac{3}{2}, \\
b_1 &= \frac{3}{2}\mu, & b_2 &= \pm \frac{3}{2} \sqrt{(-3\mu^2 + \mu^2 + 2\lambda)}, & c_1 &= \frac{1}{16} + \frac{1}{64}\lambda^2(1 - 7), & k &= \frac{25}{16} + \frac{7}{8}\lambda.
\end{align*}
\]

(4.2.4)

By substituting (4.2.4) into (4.2.3) with (2.5) and (2.8), we get the traveling wave solutions of Eq. (4.2.1) as follows:

\[
\psi_3 = -\frac{1}{2} - \frac{5}{4} \lambda - \frac{3(\lambda - \frac{1}{2}) \left[ a_1 \cosh\left(\sqrt{\lambda} \xi\right) + a_2 \sinh\left(\sqrt{\lambda} \xi\right) \right]^2}{2 \left( a_1 \sinh\left(\sqrt{\lambda} \xi\right) + a_2 \cosh\left(\sqrt{\lambda} \xi\right) \right)^2} - \frac{1}{\lambda} \left[ a_1 \cosh\left(\sqrt{\lambda} \xi\right) + a_2 \sinh\left(\sqrt{\lambda} \xi\right) \right] \left[ a_1 \sinh\left(\sqrt{\lambda} \xi\right) + a_2 \cosh\left(\sqrt{\lambda} \xi\right) \right]
\]

(4.2.5)

In particular, if we put \( A_1 = 0, \mu = 0 \) and \( A_2 > 0 \) in Eq. (4.2.5), we produce a solitary solution:

\[
\psi_1 = \frac{1}{2} - \frac{5}{4} \lambda + \frac{3}{2} \lambda \left( \tanh^2\left(\sqrt{-\lambda} \xi\right) + i \tanh\left(\sqrt{-\lambda} \xi\right) \text{sech}\left(\sqrt{-\lambda} \xi\right) \right)
\]

(4.2.6)

The behavior of the solitary wave solution (4.2.6) when \( \lambda = -0.2, \mu = 3 \) and its projection at \( t = 0 \) are shown in the Figure 15.

Fig. 15. The exact solutions \( \psi_1 \) of Eq. (4.2.6)
While, if we put $A_2 = 0, \mu = 0$ and $A_1 > 0$ in Eq. (4.2.5), we produce a solitary solution:

$$v_{12} = \frac{1}{2} - \frac{5}{4} \lambda^2 + \frac{3}{2} \lambda \cosh^2 \left( \sqrt{-\lambda} \xi \right) + i \cosh \left( \sqrt{-\lambda} \xi \right) \cosh \left( \sqrt{-\lambda} \xi \right) . \quad (4.2.7)$$

The behavior of the solitary wave solution (4.2.7) when $\lambda = -0.2$, $y = 3$ and its projection at $t = 0$ are shown in the Figure 16.

**Fig. 16.** The exact solutions $v_{12}$ of Eq. (4.2.7) where $\xi = x + y + \left( \frac{25}{16} + \frac{7}{32} \lambda^2 \right) t$ and $\sigma = A_1^2 - A_2^2$.

**Result 2**

$$\alpha_1 = 0, \quad \alpha_2 = -\frac{1}{2}, \quad b_1 = \frac{1}{2} \mu, \quad b_2 = \pm \frac{1}{32} \sqrt{-\lambda(\lambda^2 \sigma + \mu^2)},$$

$$k = \frac{\sqrt{2} a_0}{16} - \frac{11}{16} \lambda^2 - \frac{5}{16} a_0 \lambda - \frac{15}{2} a_0 - \frac{25}{8} \lambda \quad (4.2.8)$$

In this case the solitary wave solution take the following

$$v_2 = v_0 + \frac{2}{3} \left( \frac{a_0 \sinh(\sqrt{\xi}) + a_2 \cos(\sqrt{\xi})}{a_0 \cosh(\sqrt{\xi}) + a_2 \sin(\sqrt{\xi})} \right)$$

$$\cdot \frac{\left( a_0 \cosh(\sqrt{\xi}) + a_2 \sin(\sqrt{\xi}) \right) \cos(\sqrt{\xi}) \sin(\sqrt{\xi}) - \left( a_0 \sinh(\sqrt{\xi}) + a_2 \cos(\sqrt{\xi}) \right) \sin(\sqrt{\xi}) \cos(\sqrt{\xi})}{\left( a_0 \cosh(\sqrt{\xi}) + a_2 \sin(\sqrt{\xi}) \right) \cos(\sqrt{\xi}) \sin(\sqrt{\xi}) - \left( a_0 \sinh(\sqrt{\xi}) + a_2 \cos(\sqrt{\xi}) \right) \sin(\sqrt{\xi}) \cos(\sqrt{\xi})}$$

$$\cdot \frac{1}{21} \left( a_0 \sinh(\sqrt{\xi} \xi) + a_2 \cos(\sqrt{\xi}) \right)$$

$$\cdot \frac{1}{21} \left( a_0 \cos(\sqrt{\xi} \xi) + a_2 \sin(\sqrt{\xi}) \right) \quad (4.2.9)$$

The behavior of the traveling wave solution (4.2.9) and its projection at $t = 0$ when $\lambda = -0.1, \alpha_0 = 0.3, \mu = 0.2$ and $y = 3$ are shown in the Figure 17.

**Fig. 17.** The exact solutions $v_2$ of Eq. (4.2.9)

In particular, if we put $\mu = 0, A_1 = 0$ and $A_2 > 0$ in Eq. (4.2.9), we produce a solitary solution:

$$v_{21} = a_0 + \frac{1}{2} \lambda \left( \tanh^2 \left( \sqrt{-\lambda} \xi \right) + \tan \left( \sqrt{\lambda} \xi \right) \sec \left( \sqrt{\lambda} \xi \right) \right) \quad (4.2.10)$$

Also if we put $\mu = 0, A_2 = 0$ and $A_1 > 0$, then we produce a solitary solution:

$$v_{21} = a_0 + \frac{1}{2} \lambda \left( \tanh^2 \left( \sqrt{-\lambda} \xi \right) + \tan \left( \sqrt{\lambda} \xi \right) \sec \left( \sqrt{\lambda} \xi \right) \right) \quad (4.2.11)$$

where

$$\xi = x + y + \left( \frac{15}{16} a_0^2 + \frac{21}{16} \lambda^2 + \frac{5}{16} a_0 \lambda + \frac{15}{8} a_0 + \frac{25}{8} \lambda \right) t$$

and $\sigma = A_1^2 - A_2^2$.

**Case 2.** When $\lambda > 0$

Substituting (4.2.3) into (4.2.2) and using (2.7), (2.11), we get the polynomials in $\phi$ and $\psi$. Setting the coefficients of $\phi$ and $\psi$ to be zero, we obtain a set of algebraic equations, which can be solved by using the Maple software packages to get the following results:

**Result 1**

$$a_0 = -\frac{1}{4} \lambda - \frac{3}{2}, \quad a_1 = 0, \quad a_2 = -\frac{3}{2},$$

$$b_1 = \frac{3}{2} \mu, \quad b_2 = \pm \frac{1}{32} \sqrt{-\lambda(\lambda^2 \sigma + \mu^2)},$$

$$k = \frac{25}{16} - \frac{7}{32} \lambda^2 \quad (4.2.12)$$

By substituting (4.2.12) into (4.2.5) and using (2.5) and (2.10), we get the exact solutions of Eq. (1.3) as follows:

$$v_3 = \frac{5}{4} \lambda - \frac{1}{4} \left( a_1 \sinh(\sqrt{\xi}) + a_2 \cos(\sqrt{\xi}) \right)$$

$$\left( a_1 \cosh(\sqrt{\xi}) - a_2 \sin(\sqrt{\xi}) \right) \cosh(\sqrt{\xi}) \sin(\sqrt{\xi}) - \left( a_1 \sinh(\sqrt{\xi}) - a_2 \cos(\sqrt{\xi}) \right) \sin(\sqrt{\xi}) \cos(\sqrt{\xi})$$

$$\cdot \frac{1}{21} \left( a_1 \sinh(\sqrt{\xi}) + a_2 \cos(\sqrt{\xi}) \right)$$

$$\cdot \frac{1}{21} \left( a_1 \cos(\sqrt{\xi}) + a_2 \sin(\sqrt{\xi}) \right) \quad (4.2.13)$$

In particular, if we put $A_1 = 0, \mu = 0$ and $A_2 > 0$ in Eq. (4.2.13), we produce a solitary solution:

$$v_{31} = -\frac{1}{4} - \frac{5}{4} \lambda - \frac{3}{4} \lambda \left( \tan^2 \left( \sqrt{\lambda} \xi \right) + \tan \left( \sqrt{\lambda} \xi \right) \sec \left( \sqrt{\lambda} \xi \right) \right) \quad (4.2.14)$$

The behavior of the solitary wave solution (4.2.14) and its projection at $t = 0$ when $\lambda = 0.2, y = 3$ are shown in the Figure 18.
While, if we put $A_2 = 0$, $\mu = 0$ and $A_1 > 0$, then we produce a solitary solution:

$$v_{22} = -\frac{1}{2} - \frac{5}{4} \lambda + \frac{3}{2} \lambda \left( \cot^2(\sqrt{\lambda} \xi) - \cot(\sqrt{\lambda} \xi) \csc(\sqrt{\lambda} \xi) \right)$$

\hspace{1cm} (4.2.15)

where $\xi = x + y - \left(\frac{15}{16} - \frac{7}{16} \lambda^2\right)t$ and $\sigma = A_1^2 + A_2^2$.

**Result 2**

$$a_1 = 0, \quad a_2 = -\frac{1}{2}, \quad b_1 = \frac{1}{2} \mu,$$

$$b_2 = \pm \frac{1}{2} \sqrt{-\lambda (-\lambda^2 \sigma + \mu^2)},$$

\hspace{1cm} (4.2.16)

Substituting (4.2.16) into (4.2.5) using (2.5) and (2.8), we get the traveling wave exact solutions of Eq. (1.3) as follows:

$$v_4 = a_0 + \frac{1}{2} \mu \sin(\sqrt{\lambda} \xi) + \frac{1}{2} \mu \cos(\sqrt{\lambda} \xi) + \frac{1}{2} \mu$$

\hspace{1cm} (4.2.17)

The behavior of the traveling wave solution (4.2.17) and its projection at $t = 0$ when $\lambda = 0.2$, $a_0 = 0.3$, $y = 3$ are shown in the Figure 19.

Fig. 18. The exact solutions $v_{22}$ of Eq. (4.2.14)

In particular, if we put $\mu = 0$, $A_1 = 0$ and $A_2 > 0$ in Eq. (4.2.17), we produce a solitary solution:

$$v_{41} = a_0 - \frac{1}{2} \lambda \left( \tan^2(\sqrt{\lambda} \xi) - \tan(\sqrt{\lambda} \xi) \sec(\sqrt{\lambda} \xi) \right)$$

\hspace{1cm} (4.2.18)

The behavior of the solitary wave solution (4.2.18) and its projection at $t = 0$ when $\lambda = 0.2$, $a_0 = 0.3$, $y = 3$ are shown in the Figure 20.

Fig. 20. The exact solutions $v_{41}$ of Eq. (4.2.18)

While, if we put $\mu = 0$, $A_2 = 0$ and $A_1 > 0$, then we produce a solitary solution:

$$v_{42} = a_0 - \frac{1}{2} \lambda \left( \cot^2(\sqrt{\lambda} \xi) + \cot(\sqrt{\lambda} \xi) \csc(\sqrt{\lambda} \xi) \right)$$

\hspace{1cm} (4.2.19)

Consequently by substituting (4.2.20) into (4.2.5) using (2.5) and (2.12), we get the exact solutions of Eq. (1.3) as follows:

$$v_5 = -\frac{1}{2} \mu^2 + 2 \mu \xi + \frac{1}{2} \mu + \frac{1}{16} \lambda^2 - 2 \mu \lambda$$

\hspace{1cm} (4.2.21)

The behavior of the traveling wave solution (4.2.21) and its projection at $t = 0$ when $\mu = 0.2$, $A_1 = 1$, $A_2 = 2$, $y = 3$ are shown in the Figure 21.
Conclusion
The $(G'/G, 1/G)$-expansion method is one of the direct methods to solve the nonlinear partial differential equations. This method is combined between two different methods $(G'/G)$-expansion method and $(1/G)$-expansion method. We improved the $(G'/G, 1/G)$ -expansion method to solve some of nonlinear problem in mathematical physics namely (1+1)-dimensional Ito nonlinear integro-partial differential equation, first and second integral differential KP hierarchy equations. This proposed method will be submitted to literature to extract exact solutions with arbitrary parameters which include solitary and periodic wave solutions of nonlinear integro-differential equations. This method is powerful and effective method to solve more complicated nonlinear evolution equations in mathematical physics.

References


