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Convergence Analysis of Modified Triangular and Triangular Splitting Method for the Solution of Regularized Linear System-Circulant Matrices

Convergence Analysis of MTTS

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Abstract: In this paper, the homogeneous system $\pi\mathcal{Q} = 0$ is transformed to the non homogeneous regularized linear system $\mathcal{A}x = b$ by introducing small perturbation ϵ, and proved that the matrix $\mathcal{A} = \mathcal{Q}^T + \epsilon\mathcal{I}$ is positive definite for $\epsilon > 0$. The steady state probability vector π of an irreducible circulant rate matrix $\mathcal{Q}$ is computed, and also obtained the condition for the convergence of unique iterative solution by Modified Triangular and Triangular Splitting (MTTS) method proposed as in the cases of Traingualr and Triangular Splitting (TTS), Triangular and Skew-symmetric Splitting (TSS). Moreover, we prove some properties of circulant matrices. From the numerical results, we conclude that the steady state probability vector of proposed method converges rapidly to unique solution compare to TTS, and Jacobi methods.

Keywords: Self-Similarity, Circulant stochastic matrices, Steady state probability vector, TTS Method, MTTS Method, Convergence analysis.

INTRODUCTION

The investigation of the fractal characteristics of traffic processes comes from the presence of burstiness over several time scales. This leads to the introduction of the concept of self-similarity in teletraffic engineering emphasizing natural length of bursts. A traffic model is a mathematical description of specific traffic type represented in terms of stochastic matrix. Therefore it is important to capture packet arrival flows and describe them with suitable stochastic matrices. The use of stochastic rate matrix is of interest in wide range of applications of performance measures of queueing systems, neural networks, dynamical systems.

For computing these performance measures, it is the key importance to find out the steady state vector of the pertient linear system. Hence, the solution of the linear system is feasible and unique. Recently, there is a large amount of work devoted to solve the pertinent linear systems. The steady state vector of the linear system is obtained by direct or iterative methods [27], [17], [23], [4]. The well-known

Krylov subspace methods, and some preconditioning techniques in [10], [25], [8], [5], [2], [3], [6], [9], [15], [19]. The triangular and skew-symmetric (TSS) and triangular and symmetric splitting iteration methods has been developed and discussed for solving positive-definite linear systems on stochastic matrices [22], [29], and it is difficult to estimate the optimal parameter. Moreover, the triangular and triangular splitting iteration method is used for finding the solution of the pertinent system [31]. Hence, in this paper, we proposed new approach MTTS for determining the steady state vector of given system. Moreover, we find out the optimal parameter α , and convergence criteria of circulant stochastic rate matrix. While finding the convergence condition of the iterative solution, we obtain the value of that contraction factor α. In the earlier papers there is no fixed value for the contraction factor α and which is not suitable for long run systems. In this proposed method, we obtain the fixed value for the contraction factor α.

The rest of the paper is organized as follows: In section 2, the regularized preconditioned linear system of circulant rate matrix is considered and some properties of circulant matrices are discussed. In section 3, the MTTS iteration method discussed to solve the regularized linear system. In section 4, the convergence criteria for MTTS method discussed. In section 5, we demonstrate accuracy of the proposed method by means of numerical results. Finally, conclusions are made in section 6.

REGULARIZED PRECONDITIONED LINEAR SYSTEM AND SOME PROPERTIES OF CIRCULANT MATRICES

In this section, we define circulant matrix and some chrematistics of the regularized and positive definite circulant matrices.
Consider the circulant rate matrix

$$\mathcal{Q} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \dots & \sigma_n \\ \sigma_n & \sigma_1 & \sigma_2 & \dots & \sigma_{n-1} \\ \sigma_{n-1} & \sigma_n & \sigma_1 & \dots & \sigma_{n-2} \\ \dots & \dots & \dots & \ddots & \vdots \\ \sigma_2 & \sigma_3 & \sigma_4 & \dots & \sigma_1 \end{bmatrix}$$
$$= (\sigma_1, \sigma_2, \dots, \sigma_n)(say)$$

$$\text{for } \sigma_1 > 0, \ \sigma_i \leq 0, \ 2 \leq i \leq n \quad (1)$$

and $\sum_{i=1}^{n} \sigma_i = 0$. Each row and column sum is zero. Hence the matrix $\mathcal{Q}$ is a doubly stochastic matrix.
M-Matrix. Any matrix $\mathcal{A} \in \mathcal{R}^{n\times n}$ of the form $\mathcal{A} = s\mathcal{I} - \mathcal{B}, s > 0, \mathcal{B} \geq 0$ is called an M-matrix if $s \geq \rho(\mathcal{B})$. If $s > \rho(\mathcal{B})$ then $\mathcal{A}$ is non-singular M-matrix, otherwise $\mathcal{A}$ is singular M-matrix.

Definition 1. The matrix $\mathcal{A} = [a_{ij}] \in \mathcal{R}^{n\times n}$ is said to be diagonal dominant (DD), if $|a_{ii}| \geq \sum_{i\neq j} |a_{ij}|, i \in \mathcal{N}$, and $\mathcal{A}$ is called strictly diagonal dominant if $|a_{ii}| > \sum_{i\neq j} |a_{ij}|, i \in \mathcal{N}$, where, $\mathcal{N} = \{1, 2, 3, \cdots n\}$.

Definition 2. A non-symmetric matrix $\mathcal{A} \in \mathcal{R}^{n\times n}$ is positive definite if its symmetric part is positive definite.

The MTTS iteration method cannot be directly applied to find the steady state vector of the linear system wherein the coefficient matrix is circulant stochastic rate matrix. Hence, we consider a regularized linear system as follows [29]. Consider the homogeneous linear system $\pi\mathcal{Q} = 0$, where, $\mathcal{Q} = \mathcal{I} - \mathcal{P}$ and $\mathcal{P}$ is circulant stochastic probability matrix. Taking transpose on both sides of said homogeneous system, we obtain, $\mathcal{Q}^T\pi^T = 0$, and rewritten as $\bar{\mathcal{A}}x = 0$, where $x = \pi^T$, and $\bar{\mathcal{A}} = \mathcal{Q}^T$ has zero row and column sum, positive diagonal entries, non-positive off diagonal entries and has the one dimensional null space is singular. Hence, we modified the given homogeneous system of equations into regularized linear system by introducing small perturbation ϵ. Then there exists a non negative constant $\epsilon > 0$ such that the preconditioned linear system is

$$\mathcal{A}x = (\bar{\mathcal{A}} + \epsilon\mathcal{I})x = (\mathcal{Q}^T + \epsilon\mathcal{I})x = e_n = b. \quad (2)$$

where e_n is a vector given by $e_n = [0, 0, \dots, 1]^T$. The steady-state probability distribution vector is then obtained by normalizing the solution x. For proving the regularized preconditioned matrix $\mathcal{A}$ is positive define, the matrix must be non-symmetric. If $\mathcal{Q}$ is circulant matrix with distinct elements of order n then $\mathcal{Q}$ is non-symmetric for $n \geq 3$. If the matrix $\mathcal{Q}$ is non-symmetric apparently the matrix $\mathcal{A}$ is non-symmetric.

By the definition, for proving the matrix $\mathcal{Q}$ is positive definite, it is enough to prove that symmetric part of the matrix $\mathcal{Q}$ is positive definite. i.e. $\frac{\mathcal{Q}+\mathcal{Q}^T}{2}$ is positive definite.

Lemma 3. If $\Re$ is non-negative matrix and $\Re_i$ is the row sum of $\Re$ then $\rho(\Re) = \Re_i(\mathcal{A})$.

Proof. Let

$$\Re = \begin{bmatrix} 0 & \frac{\sigma_2+\sigma_n}{2} & \frac{\sigma_3+\sigma_{n-1}}{2} & \dots & \frac{\sigma_2+\sigma_n}{2} \\ \frac{\sigma_2+\sigma_n}{2} & 0 & \frac{\sigma_2+\sigma_n}{2} & \dots & \frac{\sigma_3+\sigma_{n-1}}{2} \\ \frac{\sigma_3+\sigma_{n-1}}{2} & \frac{\sigma_2+\sigma_n}{2} & 0 & \dots & \frac{\sigma_4+\sigma_{n-2}}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_2+\sigma_n}{2} & \frac{\sigma_3+\sigma_{n-1}}{2} & \frac{\sigma_4+\sigma_{n-2}}{2} & \dots & 0 \end{bmatrix} \geq 0.$$

be a non-negative circulant matrix.

$$\begin{aligned} &\Rightarrow \min_{i\neq j} r_{\mathcal{A}}(i,j) = \max_{i\neq j} r_{\mathcal{A}}(i,j) \\ &\quad = \sigma_2+\sigma_3+\sigma_4+\ldots+\sigma_n, \\ &\Rightarrow \sigma_2+\sigma_3+\sigma_4+\ldots+\sigma_n \leq \rho(\Re) \\ &\quad \leq \sigma_2+\sigma_3+\sigma_4+\ldots+\sigma_n, \\ &\Rightarrow \rho(\Re) = \sigma_2+\sigma_3+\sigma_4+\ldots+\sigma_n. \end{aligned}$$

■

Theorem 4. If $\mathcal{Q} \in \mathcal{R}^{n\times n}$ is a circulant stochastic rate matrix given in (1) then there exists $\epsilon > 0$ such that the matrix $\mathcal{A} = \bar{\mathcal{A}} + \epsilon\mathcal{I} = \mathcal{Q}^T + \epsilon\mathcal{I}$ is positive definite *iff* $(\sigma_1+\epsilon) > \rho(\Re)$.

Proof. For proving the matrix $\mathcal{A}$ is positive definite it is enough to prove that the symmetric part of the matrix $\frac{\mathcal{A}+\mathcal{A}^T}{2}$ is positive definite.

Consider the $\mathcal{Q}$ given by (1). Now,

$$\frac{\mathcal{A}+\mathcal{A}^T}{2} = \begin{bmatrix} \sigma_1+\frac{\epsilon}{2} & \frac{\sigma_2+\sigma_n}{2} & \frac{\sigma_3+\sigma_{n-1}}{2} & \dots & \frac{\sigma_2+\sigma_n}{2} \\ \frac{\sigma_2+\sigma_n}{2} & \sigma_1+\frac{\epsilon}{2} & \frac{\sigma_2+\sigma_n}{2} & \dots & \frac{\sigma_3+\sigma_{n-1}}{2} \\ \frac{\sigma_3+\sigma_{n-1}}{2} & \frac{\sigma_2+\sigma_n}{2} & \sigma_1+\frac{\epsilon}{2} & \dots & \frac{\sigma_4+\sigma_{n-2}}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_2+\sigma_n}{2} & \frac{\sigma_3+\sigma_{n-1}}{2} & \frac{\sigma_4+\sigma_{n-2}}{2} & \dots & \sigma_1+\frac{\epsilon}{2} \end{bmatrix}$$
$$= (\sigma_1+\epsilon)\mathcal{I} - \Re,$$

where

$$\Re = \begin{bmatrix} 0 & \frac{\sigma_2+\sigma_n}{2} & \frac{\sigma_3+\sigma_{n-1}}{2} & \dots & \frac{\sigma_2+\sigma_n}{2} \\ \frac{\sigma_2+\sigma_n}{2} & 0 & \frac{\sigma_2+\sigma_n}{2} & \dots & \frac{\sigma_3+\sigma_{n-1}}{2} \\ \frac{\sigma_3+\sigma_{n-1}}{2} & \frac{\sigma_2+\sigma_n}{2} & 0 & \dots & \frac{\sigma_4+\sigma_{n-2}}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_2+\sigma_n}{2} & \frac{\sigma_3+\sigma_{n-1}}{2} & \frac{\sigma_4+\sigma_{n-2}}{2} & \dots & 0 \end{bmatrix} \geq 0.$$

Let $\rho(\Re)$ denote the spectral radius of the non-negative matrix $\Re$.

Since $\frac{\mathcal{A}+\mathcal{A}^T}{2} = \epsilon\mathcal{I} + \frac{\mathcal{Q}+\mathcal{Q}^T}{2}$,
$= (\epsilon+\sigma_1) - \Re$,
and $\rho(\Re) = \sigma_2+\sigma_3+\sigma_4+\ldots+\sigma_n$,
$= \sigma_1$.
$\Rightarrow (\sigma_1+\epsilon) > \rho(\Re)$.
$\mathcal{A}$ is positive definite.

On the other hand, we assume that $\mathcal{A}$ is positive definite. We complete the theorem along the lines of the theorem 1 [18]. If $\mathcal{A}$ is positive definite then $\frac{\mathcal{A}+\mathcal{A}^T}{2} = (\sigma_1+\epsilon)\mathcal{I} - \Re$, is positive definite. For any set of indices $N \subseteq \{1,2,\ldots .n\}$, $\sum_{j=N} \sigma_{kj} < \sigma_1+\epsilon$ for $k = 1,2,\ldots .n$
$\Rightarrow (\sigma_1+\epsilon) > \rho(\Re)$. ■

Corollary 5. [18] Let $D = \rho I - A$ be a positive definite M-matrix of order n, $A = (a_{ij}, i = 1,\ldots,n,\ \ j = 1,\ldots,n)$. Then $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} < \rho n;\ a_{ij}+a_{ji} < 2\rho$.

Corollary 6. Let $\mathcal{A}$ be a positive definite M-matrix of order n. Then $(\sigma_1+\epsilon) > \frac{2}{n}(\sigma_2+\sigma_3+\sigma_4+\ldots+\sigma_n)$.

Proof. Along the lines of the corollary 5, $2(\sigma_2+\sigma_3+\sigma_4+\ldots+\sigma_n) < n(\sigma_1+\epsilon)$,
$\Rightarrow\ \sigma_1+\epsilon > \frac{2}{n}(\sigma_2+\sigma_3+\sigma_4+\ldots+\sigma_n)$.
For proving the given system possesses an unique solution here, we prove some properties of circulant matrices. These properties are used in the convergence analysis of regularized linear system to get the unique solution.
Let $\Re_i(\mathcal{A}) = \sum_{j=1, j\neq i}^{n} |a_{ij}|$ be non diagonal row sum of a matrix $\mathcal{A} = [a_{ij}] \in \mathcal{R}^{n\times n}$.
The spectral radius $\rho(\mathcal{A})$ of the non-negative matrix is obtained by Gersogorin theorem [28],

$$r_{\mathcal{A}}(i,j) = \frac{1}{2}\{a_{ii}+a_{jj}+[(a_{ii}-a_{jj})^2+4\Re_i(\mathcal{A})\Re_j(\mathcal{A})]^{\frac{1}{2}}\}$$

for $i \neq j, 1 \leq i, j \leq n$, and improved Perronï¿½Frobenius result [28]

$$\min_{i\neq j} r_{\mathcal{A}}(i,j) \leq \max_{i\neq j} r_{\mathcal{A}}(i,j).$$

■

Lemma 7. If $\mathcal{Q} \in \mathcal{R}^{n\times n}$ is a circulant stochastic rate matrix then $\mathcal{Q}$ is diagonally dominant matrix.

Proof. From the equation (2), we have

$$\mathcal{Q} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \dots & \sigma_n \\ \sigma_n & \sigma_1 & \sigma_2 & \dots & \sigma_{n-1} \\ \sigma_{n-1} & \sigma_n & \sigma_1 & \dots & \sigma_{n-2} \\ \dots & \dots & \dots & \ddots & \vdots \\ \sigma_2 & \sigma_3 & \sigma_4 & \dots & \sigma_1 \end{bmatrix}$$

is a circulant stochastic rate matrix.
$\Rightarrow \sum_{i=1}^{n} \sigma_i = 0,$
$\Rightarrow \sigma_1 = \sum_{i=2}^{n} \sigma_i = 0, \sigma_i > 0,$
$\Rightarrow \sigma_1 \geq \sum_{i=1}^{n} \sigma_i.$
$\Rightarrow \mathcal{Q}$ is diagonally dominant matrix. ■

Corollary 8. If $\mathcal{A} \in \mathcal{R}^{n\times n}$ is regularized circulant matrix then $\mathcal{A}$ is strictly diagonally dominant.

Proof. Let $\mathcal{A} = \mathcal{Q}^T + \epsilon\mathcal{I}$

$$\mathcal{A} = \begin{bmatrix} \sigma_1+\epsilon & \sigma_2 & \sigma_3 & \dots & \sigma_n \\ \sigma_n & \sigma_1+\epsilon & \sigma_2 & \dots & \sigma_{n-1} \\ \sigma_{n-1} & \sigma_n & \sigma_1+\epsilon & \dots & \sigma_{n-2} \\ \dots & \dots & \dots & \ddots & \vdots \\ \sigma_2 & \sigma_3 & \sigma_4 & \dots & \sigma_1+\epsilon \end{bmatrix},$$

Since $\mathcal{Q}$ is circulant stochastic matrix, then
$\Rightarrow \sum_{i=1}^{n} \sigma_i = 0$ with $\sigma_1 > 0$ and $\sigma_i < 0$ for $2 < i < n$,
$\sigma_1 = \sigma_2 + \sigma_3 + \sigma_4 + \ldots + \sigma_n.$
$\Rightarrow \sigma_1 + \epsilon > \sigma_2 + \sigma_3 + \sigma_4 + \ldots + \sigma_n,$
$\Rightarrow \sigma_1 + \epsilon > \sum_{i=2}^{n} \sigma_i,$
$\Rightarrow \mathcal{A}$ is strictly diagonal dominant matrix. ■

Lemma 9. [21] If $\mathcal{A}$ is diagonally dominant and non-singular then $sgn(det\mathcal{A}) = sgn(\prod_i a_{ii})$.

Theorem 10. If $\mathcal{A}$ is circulant matrix with diagonally dominant and positive definite then $sgn(det\mathcal{A})$ is positive.

Proof. From corollary 8,

$$\mathcal{A} = \mathcal{Q}^T + \epsilon\mathcal{I} \text{ is a diagonally dominant.}$$

$$\Rightarrow sgn(det\mathcal{A}) = sgn(\prod_i a_{ii}),$$
$$= sgn(\sigma_1+\epsilon.\sigma_1+\epsilon....\sigma_1+\epsilon),$$
$$= sgn((\sigma_1+\epsilon)^n),$$

Since $\mathcal{A}$ is positive definite with $\sigma_1 + \epsilon$ is positive, then $sgn(det\mathcal{A})$ is positive. ■

Lemma 11. [14] Let $\alpha_i > 0 (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \alpha_i \geq 1$. Let $a_{ij} > 0, (i = 1, 2, \ldots, n)$, then

$$\sum_{j=1}^{n} a_{1j}^{\alpha_1} a_{2j}^{\alpha_2} \cdots a_{nj}^{\alpha_n} \leq \left(\sum_{j=1}^{n} a_{1j}\right)^{\alpha_1} \left(\sum_{j=1}^{n} a_{2j}\right)^{\alpha_2} \cdots \left(\sum_{j=1}^{n} a_{nj}\right)^{\alpha_n}.$$

Theorem 12. Let $\mathcal{P} = [a_{ij}]$, for $i = 1, 2, 3, \ldots, n$ be a circulant stochastic probability matrix such that $a_{ij} > 0$ for $i = 1, 2, 3, \ldots, n$, then,

$$\sum_{j=1}^{n} a_{1j}^{\alpha_1} a_{2j}^{\alpha_2} \cdots a_{nj}^{\alpha_n} \leq \left(\sum_{j=1}^{n} a_{1j}\right)^{\alpha_1} \left(\sum_{j=1}^{n} a_{2j}\right)^{\alpha_2} \cdots \left(\sum_{j=1}^{n} a_{nj}\right)^{\alpha_n} \leq 1.$$

Proof. From Lemma 11, we have,

$$\sum_{j=1}^{n} a_{1j}^{\alpha_1} a_{2j}^{\alpha_2} \cdots a_{nj}^{\alpha_n} \leq \left(\sum_{j=1}^{n} a_{1j}\right)^{\alpha_1} \left(\sum_{j=1}^{n} a_{2j}\right)^{\alpha_2} \cdots \left(\sum_{j=1}^{n} a_{nj}\right)^{\alpha_n}$$

Since $\mathcal{P}$ is doubly stochastic matrix, $\sum_{j=1}^{n} a_{1j} = \Re_i$ for $i = 1, 2, 3, \ldots, n$.

$$\Rightarrow \sum_{j=1}^{n} a_{1j}^{\alpha_1} a_{2j}^{\alpha_2} \cdots a_{nj}^{\alpha_n} \leq \Re_1^{\alpha_1} \Re_2^{\alpha_2} \cdots \Re_n^{\alpha_n} = 1.1...1\,(n\ times) = 1,$$

$$\Rightarrow \sum_{j=1}^{n} a_{1j}^{\alpha_1} a_{2j}^{\alpha_2} \cdots a_{nj}^{\alpha_n} \leq \left(\sum_{j=1}^{n} a_{1j}\right)^{\alpha_1} \left(\sum_{j=1}^{n} a_{2j}\right)^{\alpha_2} \cdots \left(\sum_{j=1}^{n} a_{nj}\right)^{\alpha_n} \leq 1.$$

■

Corollary 13. Let $\mathcal{P} = [a_{ij}]$, for $i = 1, 2, 3, \cdots n$ be a circulant stochastic probability matrix such that $a_{ij} > 0$ for $i = 1, 2, 3, \cdots n$, then $a_{1j}^{\alpha_1} a_{2j}^{\alpha_2} \cdots a_{nj}^{\alpha_n} \leq \frac{1}{n}$.

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THE MTTS ITERATION METHOD

In this section, we find the steady state probability vector and convergence analysis of the regularized linear system (2) by applying the MTTS iteration method. Splitting iteration methods for (2) require efficient splitting of the regularized matrix $\mathcal{A}$. The matrix $\mathcal{A}$ can also be split into its Triangular and Triangular parts as

$$\mathcal{A} = [£ + \mathcal{D}] + [\mathcal{U}], \tag{3}$$

where

$$\mathcal{D} = \begin{bmatrix} \sigma_1 & 0 & 0 & \dots & 0 \\ 0 & \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1 \end{bmatrix},$$

$$£ = \begin{bmatrix} \frac{\epsilon}{2} & 0 & 0 & \dots & 0 \\ \sigma_2 & \frac{\epsilon}{2} & 0 & \dots & 0 \\ \sigma_3 & \sigma_2 & \frac{\epsilon}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \dots & \frac{\epsilon}{2} \end{bmatrix},$$

and

$$\mathcal{U} = \begin{bmatrix} \frac{\epsilon}{2} & \sigma_n & \sigma_{n-1} & \dots & \sigma_2 \\ 0 & \frac{\epsilon}{2} & \sigma_n & \dots & \sigma_3 \\ 0 & 0 & \frac{\epsilon}{2} & \dots & \sigma_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\epsilon}{2} \end{bmatrix}.$$

Here, $\mathcal{D}$ is a diagonal matrix with the diagonal elements of the matrix $\mathcal{A}$ and $£, \mathcal{U}$ are lower, upper triangular matrices of $\mathcal{A}$ respectively of regularized linear system.

Let $\mathcal{T} = £ + \mathcal{D}$ and $\mathcal{T}' = \mathcal{U}$, then the splitting

$$\mathcal{A} = \mathcal{T} + \mathcal{T}', \tag{4}$$

is called the Modified Triangular and Triangular splitting (MTTS) method for circulant stochastic matrices.
Where

$$\mathcal{T} = \begin{bmatrix} \sigma_1 + \frac{\epsilon}{2} & 0 & 0 & \dots & 0 \\ \sigma_2 & \sigma_1 + \frac{\epsilon}{2} & 0 & \dots & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 + \frac{\epsilon}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \dots & \sigma_1 + \frac{\epsilon}{2} \end{bmatrix},$$

and

$$\mathcal{T}' = \begin{bmatrix} \frac{\epsilon}{2} & \sigma_n & \sigma_{n-1} & \dots & \sigma_2 \\ 0 & \frac{\epsilon}{2} & \sigma_n & \dots & \sigma_3 \\ 0 & 0 & \frac{\epsilon}{2} & \dots & \sigma_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\epsilon}{2} \end{bmatrix}.$$

i.e. $\mathcal{T} \in \mathcal{R}^{n\times n}$ is a lower triangular matrix with its diagonal elements positive, and $\mathcal{T}' \in \mathcal{R}^{n\times n}$ is a upper triangular matrix with its diagonal elements positive and its off diagonal elements negative. We can establish the following MTTS iteration method for the regularized linear system (2).

Given an initial guess $x^{(0)} \in \mathcal{R}^n$, consider the iterative scheme

$$\begin{aligned} (\alpha\mathcal{I} + \mathcal{T})x^{(k+1/2)} &= (\alpha\mathcal{I} - \mathcal{T}')x^{(k)} + b, \\ (\alpha\mathcal{I} + \mathcal{T}')x^{(k+1)} &= (\alpha\mathcal{I} - \mathcal{T})x^{(k+1/2)} + b, \end{aligned} \tag{5}$$

for $k = 0, 1, 2, \dots$, where α is a given positive constant. The above iterative scheme can be written as $x^{(k+1)} = \mathcal{M}(\alpha)x^{(k)} + \mathcal{N}(\alpha)b$, for $k = 0, 1, 2, \dots$, where

$$\mathcal{M}(\alpha) = (\alpha\mathcal{I} + \mathcal{T}')^{-1}(\alpha\mathcal{I} - \mathcal{T})(\alpha\mathcal{I} + \mathcal{T})^{-1}(\alpha\mathcal{I} - \mathcal{T}'), \tag{6}$$

is the iteration matrix of the MTTS iteration method, and $\mathcal{N}(\alpha) = 2\alpha(\alpha\mathcal{I}+\mathcal{T}')^{-1}(\alpha\mathcal{I}+\mathcal{T})^{-1}$. If $\rho(\mathcal{M}(\alpha)) < 1$ then the iterative method MTTS is convergent.

Lemma 14. [2] If $\mathcal{T}$ be a triangular matrix with positive diagonal elements then $\mathcal{T}$ is positive definite.

Lemma 15. [2] If $\mathcal{T}'$ be a triangular matrix with positive diagonal elements then $\mathcal{T}'$ is positive definite matrix.

Theorem 16. [2] Let $\mathcal{W}(\alpha) = (\alpha\mathcal{I} - \mathcal{T})(\alpha\mathcal{I} + \mathcal{T})^{-1}$. If $\mathcal{T} \in \mathcal{R}^{n\times n}$ is a positive-definite matrix, then we have $\|\mathcal{W}(\alpha)\|_2 < 1, \forall\alpha > 0$.

Corollory 17. [2] Let $\mathcal{W}'(\alpha) = (\alpha\mathcal{I} - \mathcal{T}')(\alpha\mathcal{I} + \mathcal{T}')^{-1}$. If $\mathcal{T}' \in \mathcal{R}^{n\times n}$ is a positive-definite matrix, then we have $\|\mathcal{W}'(\alpha)\|_2 < 1, \forall\alpha > 0$.

Theorem 18. Let $\mathcal{A} \in \mathcal{R}^{n\times n}$ be the regularized matrix defined in (2), and splitting into triangular and triangular matrices given in (4). Then the spectral radius of the iterative matrix $\mathcal{M}(\alpha) < 1$.

Proof. Let $\mathcal{A} \in \mathcal{R}^{n\times n}$ be the regularized matrix defined in (2), and splitting into the form (4). Let $\mathcal{M}(\alpha)$ be the iteration matrix given in (6). Then the iteration matrix $\mathcal{M}(\alpha)$ is similar to the matrix $\overline{\mathcal{M}(\alpha)} = (\alpha\mathcal{I} - \mathcal{T})(\alpha\mathcal{I} + \mathcal{T})^{-1}(\alpha\mathcal{I} - \mathcal{T}')(\alpha\mathcal{I} + \mathcal{T}')^{-1} = \mathcal{W}(\alpha)\mathcal{W}'(\alpha)$, where $\mathcal{W}(\alpha) = (\alpha\mathcal{I} - \mathcal{T})(\alpha\mathcal{I} + \mathcal{T})^{-1}$, and $\mathcal{W}'(\alpha) = (\alpha\mathcal{I} - \mathcal{T}')(\alpha\mathcal{I} + \mathcal{T}')^{-1}$.

Since the triangular matrices $\mathcal{T}$ and $\mathcal{T}'$ of the regularized preconditioned matrix $\mathcal{A}$ given in (3) are positive definite, then from the lemma 14, and corollory 15, we have $\|\mathcal{W}(\alpha)\|_2 < 1, \|\mathcal{W}^{'}(\alpha)\|_2 < 1, \forall\alpha > 0$.

$$\begin{aligned}\therefore \rho(\mathcal{M}(\alpha)) &= \rho(\overline{\mathcal{M}(\alpha)}) \\ &= \|(\alpha\mathcal{I} - \mathcal{T})(\alpha\mathcal{I} + \mathcal{T})^{-1}(\alpha\mathcal{I} - \mathcal{T}^{'})(\alpha\mathcal{I} + \mathcal{T}^{'})^{-1}\|_2 \\ &= \|\mathcal{W}(\alpha)\mathcal{W}^{'}(\alpha)\|_2, \\ \Rightarrow \rho(\mathcal{M}(\alpha)) &= \rho(\overline{\mathcal{M}(\alpha)}) = \|\mathcal{W}(\alpha)\|_2\|\mathcal{W}^{'}(\alpha)\|_2 < 1. \qquad (7)\end{aligned}$$

Therefore, the MTTS iteration method converges to unique solution of the regularized linear system (2). ■

CONVERGENCE CONDITION FOR MTTS METHOD

In this section, we find out the optimal parameter and discuss the inexact triangular and triangular (ITT) iterative method by using the krylov subspace method as the inner iterative process of the MTTS iteration method for the Markov process [2]. Here, we find out the contraction factor α on the lines of [2,29]. From the TSS, HSS, PSS iterative methods and the theoretical approach in the above cases, we observed that the iterative solution of (2) by modified triangular and triangular splitting method of the preconditioned matrix $\mathcal{A}$ converge for any contraction factor α . The iterative solution not only based the contraction factor α, but also based on the parameter ϵ. For the fixed value of the contraction factor α, and different arbitrary values of the parameter ϵ, we get the different solutions of the MTTS iteration method for the numerical solutions of the Markov process. Therefore, for the fast convergence in the MTTS iteration method, it is the key importance to choose the values of ϵ. Since the pre conditioned matrix $\mathcal{A}$ of the regularized linear system (2) splits into triangular matrices given (3).

Consider

$$\begin{aligned}\mathcal{A} &= [£ + \mathcal{D}] + [\mathcal{U}] = \mathcal{T}_1 + \mathcal{T}_1^{'} \\ &= [\mathcal{U} + \mathcal{D}] + [£] = \mathcal{T}_2 + \mathcal{T}_2^{'},\end{aligned}$$

where $\mathcal{T}_i$ and $\mathcal{T}_i^{'}, (i = 1, 2)$ are triangular matrices. Along the lines of [29], we find out the contraction factor as follows:

Let $\mathcal{G}_1 = £ + \mathcal{D}$ and $\mathcal{G}_2 = \mathcal{U} + \mathcal{D}$ then $\mathcal{G}_i(i = 1, 2)$ are lower and upper triangular matrices such that

$$[\mathcal{G}_i(\alpha\mathcal{I} + \mathcal{D})^{-1}]^n = [(\alpha\mathcal{I} + \mathcal{D})^{-1}\mathcal{G}_i]^n = 0 \quad \text{for } i = 1, 2.$$

Consider

$$\begin{aligned}(\alpha\mathcal{I} + \mathcal{T}_i)^{-1} &= [(\alpha\mathcal{I} + \mathcal{D}) + \mathcal{G}_i]^{-1} \\ &= (\alpha\mathcal{I} + \mathcal{D})^{-1}[\mathcal{I} + \mathcal{G}_i(\alpha\mathcal{I} + \mathcal{D})^{-1}]^{-1}, \\ &= (\alpha\mathcal{I} + \mathcal{D})^{-1}\sum_{j=0}^{n-1}(-1)^j[\mathcal{G}_i(\alpha\mathcal{I} + \mathcal{D})^{-1}]^j.\end{aligned}$$

Now,

$$\begin{aligned}&(\alpha\mathcal{I} - \mathcal{T}_i)(\alpha\mathcal{I} + \mathcal{T}_i)^{-1} \\ &\quad = (\alpha\mathcal{I} - \mathcal{D} - \mathcal{G}_i)(\alpha\mathcal{I} + \mathcal{T}_i)^{-1}, \\ &\quad = (\alpha\mathcal{I} - \mathcal{D} - \mathcal{G}_i)(\alpha\mathcal{I} + \mathcal{D})^{-1}\sum_{j=0}^{n-1}(-1)^j[\mathcal{G}_i(\alpha\mathcal{I} + \mathcal{D})^{-1}]^j, \\ &\quad \approx (\alpha\mathcal{I} - \mathcal{D})(\alpha\mathcal{I} + \mathcal{D})^{-1}[\mathcal{I} - \mathcal{G}_i(\alpha\mathcal{I} + \mathcal{D})^{-1}],\end{aligned}$$

by evaluating the above expression, and after ignoring higher order approximations and taking norm on both sides, we get

$$\begin{aligned}&\|(\alpha\mathcal{I} - \mathcal{T}_i)(\alpha\mathcal{I} + \mathcal{T}_i)^{-1}\|_2 \approx \|(\alpha\mathcal{I} - \mathcal{D}_i)(\alpha\mathcal{I} + \mathcal{D}_i)^{-1}\|_2, \\ &\|\mathcal{W}(\alpha)\|_2 = \max_{1\le j\le n}\{(\alpha - d_{jj})(\alpha + d_{jj})^{-1}\}.\end{aligned}$$

That is,

$$\|\mathcal{W}(\alpha)\|_2 = \{(\alpha - \sigma_1)(\alpha + \sigma_1)^{-1}\}, \qquad (8)$$

$$\Rightarrow \|\mathcal{W}(\alpha)\|_2 = \sqrt{\sigma_1\sigma_1} = \sigma_1. \qquad (9)$$

similarly

$$\|\mathcal{W}^{'}(\alpha)\|_2 = \sqrt{\frac{\epsilon}{2}\frac{\epsilon}{2}} = \frac{\epsilon}{2}.$$

$$\Rightarrow \|\mathcal{M}(\alpha)\|_2 = |\mathcal{W}(\alpha)\|_2.|\mathcal{W}^{'}(\alpha)\|_2 = \sigma_1\frac{\epsilon}{2} < 1 \quad \text{if } \sigma_1 < 1.$$

NUMERICAL RESULTS

In this section, we find the optimal parameter α of the circulant stochastic rate matrix. We examine the effectiveness of the MTTS method for the numerical solution in the case of circulant Markov process and compare it with thatof the Jacobi, TTS iteration methods. Consider the 4×4 circulant stochastic rate matrix

$$\mathcal{Q} = \begin{bmatrix} 0.3 & -0.1 & -0.15 & -0.05 \\ -0.05 & 0.3 & -0.1 & -0.15 \\ -0.15 & -0.05 & 0.3 & -0.1 \\ -0.1 & -0.15 & -0.05 & 0.4 \end{bmatrix}.$$

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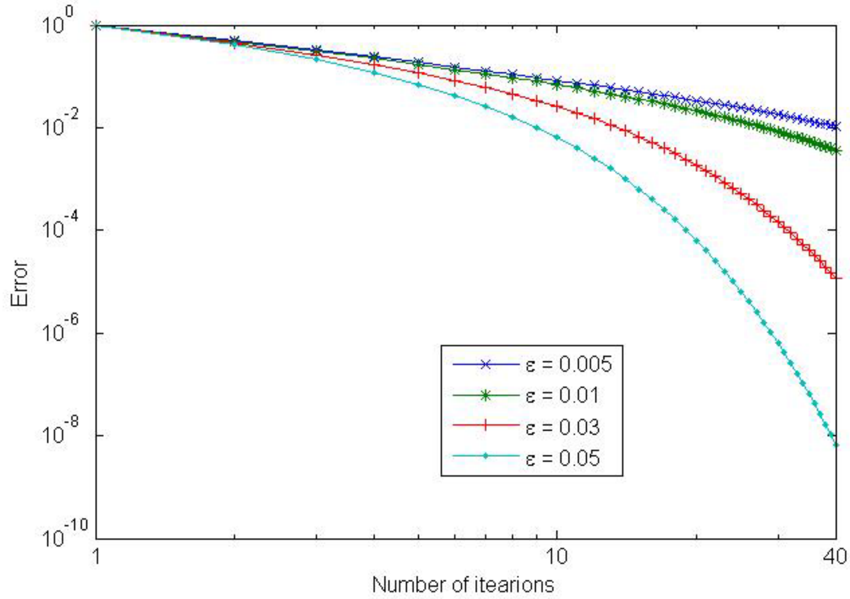

FIGURE 1. Relative error of the MTTS method for the contraction factor $\alpha = 0.3$ for different values of ϵ.

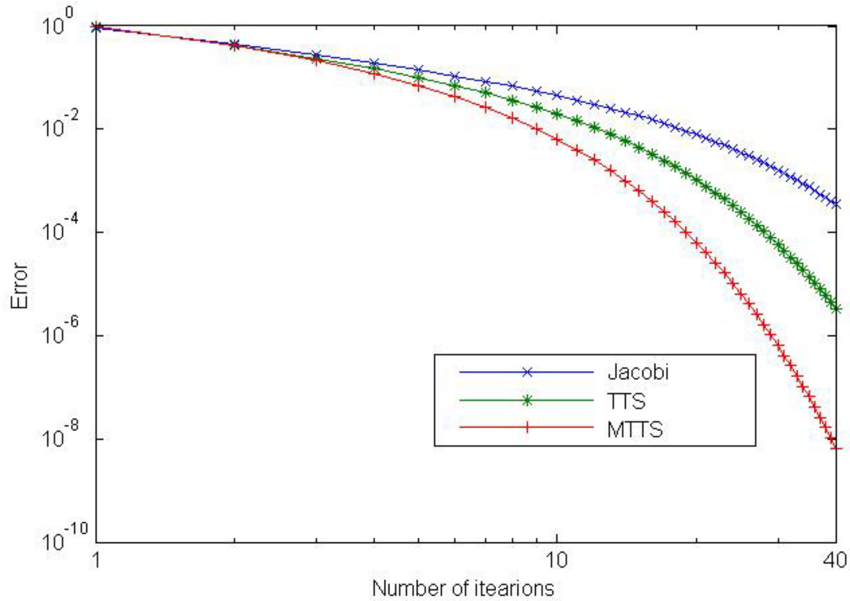

FIGURE 2. Relative errors of the Jacobi, TTS, and MTTS methods for the contraction factor $\alpha = 0.3$, and $\epsilon = 0.05$.

We consider only one case $\mathcal{A} = [\pounds + \mathcal{D}] + [\mathcal{U}] = \mathcal{T}_1 + \mathcal{T}_1^{'}$ of MTTS splitting method and other methods would follow. The initial distribution $x^{(0)} = [0, 0, 0, 1]^T$ has been taken for the scheme (5) to compute relative error where absolute value lessthan or equal to 10^{-10}. Finally, the steady state probability distribution vector x of the preconditioned linear system (2) obtained, and results are presented in table 1. The error Vs number of iterations are presented in the Figs. 1–2.

Figure 1 illustrates the results for the case of contraction factor $\alpha = 0.3$ choosen based on the analysis given in the above section, and different arbitrary values of ϵ. From this figure, we conclude that as ϵ value increases, the relative error decrease in three methods. Figure 2 depicts the results of contraction factor $\alpha = 0.3$, and $\epsilon = 0.05$, in the cases of MTTS and the existing methods. From the table 1 and figures, we conclude that the MTTS iterative solution converges rapidly than the TTS and Jacobi's methods.

Table 1. Comparison of relative error between Jacobi, TTS, and MTTS

Number of Iterations	Jacobi	TTS	MTTS
5	0.1402	0.1008	0.0698
10	0.0447	0.0195	0.0064
15	0.0181	0.0044	6.3548e-004
20	0.0079	0.0010	6.4030e-005
25	0.0036	2.4746e-004	6.4572e-006
30	0.0016	5.8806e-005	6.5124e-007
35	7.5024e-004	1.3981e-005	6.5681e-008
40	3.4627e-004	3.3241e-006	6.6244e-009

CONCLUSIONS

In this paper, we present MTTS splitting iterative method for the regularized matrix of the system of circulant stochastic matrix. We proved that the regularized matrix is positive definite under specific condition. Moreover, we proved that the regularized matrix is diagonally dominant and signature of the determinent of said matrix is positive. We compared the relative error of TTS method, and Jacobi method with proposed MTTS method. We conclude that this method unconditionally converges to a unique solution and the convergence rate is rapid when compared to the existing methods. Numerical results show that the MTSS method performs very well than the existing Jacobi, and TTS methods.

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REFERENCES

[1] A. Andersen, B. Nielsen, A Markovian approach for modeling packet traffic with long-range dependence, IEEE Journal on Selected Areas in Communications, 16(1998) 719–732.

[2] Z.-Z. Bai, G.H. Golub, L.-Z. Lu, J.F. Yin, Block triangular and skew-Hermitian splitting methods for positive-definite linear systems, SIAM J. Sci. Comput., 20 (2005) 844–863.

[3] Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl., 24(2003) 603–626.

[4] G. Barker, R.J. Plemmons, Convergent iterations for computing stationary distributions of Markov chains, SIAM J. Algebr. Disc. Meth., 7 (1986) 390–398.

[5] M. Benzi, B. Uar, Product preconditioning for Markov chain problems, in: A.N. Langville, W.J. Stewart (Eds.), Proceedings of the 2006 Markov Anniversary Meeting, Raleigh, NC, Boson Books, Charleston, SC, 2006, 239–256.

[6] A. Berman, R. Plemmons, Nonnegative Matrices in Mathematical Sciences, SIAM, Philadelphia, 1994.

[7] C. Blondia, The N/G/l finite capacity queue, Commum. Statist.-Stochastic Models, 5(1989) 273–294.

[8] P. Brown, H.F. Walker, GMRES on (nearly) singular systems, SIAM J. Matrix Anal. Appl., 18(1997) 37–51.

[9] R. Chan, W.K. Ching, Circulant preconditioners for stochastic automata networks, Numer. Math., 87(2000) 35–57.

[10] R. Bru, R. Cant, J.J. Climent, On M-multisplittings of singular M-matrices and Markov chains, Numer. Linear Algebra Appl., 5 (1998) 299–311.

[11] M. Crovella, A. Bestavros, Self-Similarity in World Wide Web traffic: evidence and possible causes, IEEE/ ACM Trans. Networking, 5(1997) 835–846.

[12] W. Fischer and K. Meier-Hellsterm, The Markov-modulated Poisson process (MMPP) Cookbook, Perf. Eval., 18(2)(1993) 149–171.

[13] H. Heffes, D. Lucantoni, A Markov modulated characterization of packetized voice and data traffic and related statistical multiplexer performance, IEEE J. Sel. Areas Commun., SAC-4(1986) 856–868.

[14] Kazarinoff, Nicholas D. Analytic inequalities. Courier Corporation, 2014.

[15] L.A. Krukier, T.S. Martynova, Z.-Z. Bai, Product-type skew-Hermitian triangular splitting iteration methods for strongly non-Hermitian positive definite linear systems, J. Comput. Appl. Math., 232(2009) 3–16.

[16] W.E. Leland, M.S. Taqqu, W. Willinger, and W.V. Wilson, On the Self-Similar Nature of Ethernet Traffic (Extended version), IEEE / ACM Trans. Networking., 2(1994) 1–15.

[17] P. Malla Reddy, Chih-How Chang, Shou-Kuo Shao, and Jingshown Wu, An Efficient Approximate Markovian Model for Optical Packet Switches Employing Partial Buffer Sharing Mechanism under Self-Similar Traffic Input, Proceedings of IEEE HPSR-2007 held at Polytechnic University, Brooklyn, New York, USA, May 30-June1,ISBN:1-4244-1206-4, IEEE Catalog number:

07EX1775C(2007).

[18] Marat Ibragimov, Conditions for Positive Definiteness of M-Matrices, Linear and Multi Linear Algebra, Vol. 48, pp. 93–106, 2000.

[19] H. Minc, Nonnegative Matrices, Wiyley, Newyork, 1988.

[20] V.Paxson, S. Floyd, Wide Area Traffic: The Failure of Poisson Modeling, IEEE/ACM Trans. Networking., 3(1995) 226–244.

[21] Rafael Bru, Maria T. Gass, Isabel Gimnez, and Jos A. Scott. The Hadamard Product of a Nonsingular General H-Matrix and Its Inverse Transpose Is Diagonally Dominant. Journal of Applied Mathematics, 2015.

[22] Rajaiah Dasari, Rajkumar L. P. and Malla Reddy Perati, Steady state probability vector of positive definite regularized linear systems of circulant stochastic matrices, vol. 65, no. 1, pp. 140–155, 2016.

[23] D. Rajaiah, P. Malla Reddy, "Loss Behavior of an Internet Router with Self-Similar Input Traffic via Fractal Point Process", Proceedings of IEEE ICON-2013 held at Singapore, December 11-13, 978-1-4799-2084-6/13(2013).

[24] D. Ranadheer, R. Ramesh, D. Rajaiah, and P. Malla Reddy, Study of Delay and Loss Behavior of Internet Router- Markovian, Modelling using Circulant Markov Modulated Poisson Process(CMMPP), Applied Mathematics, 5(2014), 512–519.

[25] Y. Saad, Iterative Methods for Sparse Linear Systems, 2nd ed. SIAM, Philadelphia, PA, 2003.

[26] S.K. Shao, P. Malla Reddy, M.G. Tsai, H.W. Tsao, and Jingshown Wu, Generalized variance-based Markovian fitting for self-similar traffic modeling, IEICE Trans. Commun., E88-B(4)(2005) 1493–1502.

[27] W. J. Stewart, Introduction to Numerical Solution of Markov Chains, Princeton University Press, Princeton, 1994.

[28] Te Wang, Hongbin Lv* and Haifeng Sang (2014). Estimations for spectral radius of nonnegative matrices and the smallest eigenvalue of M-matrices. Journal of Inequalities and Applications, 2014.

[29] Chun Wen, H. Ting-Zhu, W. Chao, Triangular and skew-symmetric splitting methods for numerical solutions of Markov chains, Comp. and Math. With App., 62(2011) 4039–4048.

[30] T. Yoshihara, S. Kasahara, Y. Takahashi, Practical time-scale fitting of self-similar traffic with Markov modulated Poisson process, Telecommun. Syst., 17(2001) 185–211.

[31] Zhang, C. Y., Yang, Y. Q., & Sun, Q. (2014). Convergence of TTS iterative method for non-Hermitian positive definite linear systems. Mathematical Problems in Engineering, 2014.