Common Coupled Fixed Point in C*-algebras Valued Metric Spaces

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Abstract
In this paper we introduced a Common coupled fixed-point theorem in the C*-algebras valued metric spaces with certain contraction condition.

INTRODUCTION
We begin with the some basic definitions and facts about structures of C*-algebra and Fixed Point Theory. We also give some facts which play a central role in the C*-algebra valued metric spaces.

A C*-algebra is a complex Banach algebra with a conjugate-linear involution*: \( A \rightarrow A \), such that
\[
(x^*)^* = x, (xy)^* = y^*x^*, (x + y)^* = x^* + y^* \text{, } ||x^*|| = ||x||^2
\]
for all \( x, y \) in \( A \). The C*-condition \( ||x*: = ||x||^2 \) implies that the involution is an isometry in the sense that \( ||x^*|| = ||x|| \) for all \( x \) in \( A \).

A C*-algebra is called unital if it possesses a unit. It follows easily that \( ||1|| = 1 \).

In general C*-algebra is non-commutative, for the commutative C*-algebra its completely determined by Gelfand, as in the following

Theorem 1.1 [Gelfand,3]. If \( A \) is a non-zero commutative C*-algebra, then the Gelfand representation
\[
\varphi: A \rightarrow C_0(\Omega(A)), \ a \mapsto \hat{a} \text{ is an isometric } * \text{-isomorphism.}
\]

Theorem 1.2 [2] Every C*-algebra has a faithful representation on some Hilbert space.

This theorem was proved in [2] and means that every C*-algebra is isometrically isomorphic to a normclosed *-algebra in \( B(H) \), for some Hilbert space \( H \). This is one of the most important results in the theory of C*-algebras. We now introduce some basics of Positive C*-algebras, we refer to [2] and [3] for more details and proofs. An element \( a \) in a C*-algebras is called self adjoint if \( a = a^* \), denote \( A_{sa} \) the set of all self adjoint elements in \( A \), \( a \in A \) is called positive element if \( a \in A_{sa} \) and \( \text{Sp}(a) \subset \mathbb{R}^+ \). We write \( a \geq 0 \) if \( a \) is positive. And denote by \( A_+ \) the set of all positive elements in \( A \). The set \( A_+ \) is a closed cone in the sense that
\[
(a + b \in A_+, if a, b \in A_+ and A + (−A_+) = [0])
\]

Lemma 1.3 [3]. Let \( A \) be a unital C*-algebra and let \( a \in A \). Then the following are equivalent:

(i) \( a \geq 0 \),
(ii) \( a = b^2 \) for some \( b \in A_{sa} \),
(iii) \( a = bb^* \) for some \( b \in A \).

For a given \( a, b \in A_+ \), we denote \( a \leq b \) if \( b - a \geq 0 \), \( A_+ \) becomes a partially ordered vector space.

Theorem 1.4 [3]. Suppose that \( A \) is unital C*-algebra with a unit 1.

1. If \( a \geq 0 \) and \( a = a^* \), then \( 1 - a \) is invertible and
\[ ||a(1 - a)^{-1}|| < 1 \]
2. Suppose that \( a, b \in A \) with \( a, b \geq \theta \) and \( ab = ba \), then \( \theta \leq \theta \).
3. Suppose that \( a, b \in A \) with \( a \leq b \), then \( ||a|| \leq ||b|| \).
4. Suppose that \( c \geq 0 \) and \( a \in A \) then \( a^*c^*a \geq 0 \).

C*-ALGEBRAS VALUED METRIC SPACE
In the next we introduced the definition of the C*-algebras valued metric spaces and give some examples. Moreover we introduced the meaning of Cauchy sequence and convergent.

The main reference in this section is [6] and there is a generalization for these results was introduced in [4]

Definition 2.1. Let \( X \) be a nonempty set. Suppose the mapping \( d: X \times X \rightarrow A \) satisfies:

(1) \( 0 \leq d(x, y) \) for all \( x, y \in X \) and \( 0 = d(x, y) \iff x = y \)
(2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \)
(3) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \)

Then \( d \) is called a a C*-algebra valued metric on \( X \) and \( (X, A, d) \) is called C*-algebra valued metric space.

Definition 2.2. Let \( (X, A, d) \) be a C*-algebra valued metric space and \( \{x_n\} \subset X \) is a sequence in \( X \). If \( x \in X \) and \( \varepsilon > 0 \) there is \( N \) such that for all \( n > N, \|d(x, x_n)\| \leq \varepsilon \), then \( \{x_n\} \) is called a convergent sequence in \( X \) to \( x \) and denote it by \( \lim_{n \rightarrow 10} x_n = x \).
Moreover, if for any \( \varepsilon > 0 \) there is \( N \) such that for all \( n, m > N, \|x_n - x_m\| \leq \varepsilon \), then \( (x_n) \) is called a cauchy sequence in \( X \).

**Definition 2.3.** The tripled \((X, \mathcal{A}, d)\) is a completed \( C^* \)-algebras valued metric space if every cauchy sequence is convergent.

**Example 2.4.** If \( X \) is a Banach space, then \((X, \mathcal{A}, d)\) is a completed \( C^* \)-algebras valued metric space with the metric
\[
d(x,y) = \|x - y\|, x, y \in X.
\]

**Example 2.5.** Let \( X = \mathbb{C} \) and \( \mathcal{A} = M_{2 \times 2}(\mathbb{C}) \). Then \((X, \mathcal{A}, d)\) is a \( C^* \)-algebras valued metric space, where
\[
d(x,y) = \begin{bmatrix} |x - y| & 0 \\ 0 & \alpha |x - y| \end{bmatrix}
\]

**Theorem 3.2:** Let \((X, A, d)\) be a complete \( C^* \)-valued metric space, and let \( F, G : X \rightarrow X \) be mappings such that
\[
d(F(x, y), G(u, v)) \leq w(x, y, u, v) a^* \quad \text{for all } x, y, u, v \in X,
\]
where \( w(x, y, u, v) \in [d(x, u), d(y, v), \frac{1}{2}(d(F(x, y), x) + d(G(u, v), u))] a^* \). Then, \( F, G \) have a unique common coupled fixed point.

**Proof:** Let \( x_0, y_0 \) be two arbitrary elements in \( X \), choose \( x_1, y_1 \in X \) such that \( x_1 = F(x_0, y_0) \) and \( y_1 = F(x_0, y_0) \). Again choose \( x_2, y_2 \in X \) such that \( x_2 = G(x_1, y_1) \) and \( y_2 = G(x_1, y_1) \). Continuing this process, we can construct two sequences \((x_n)\) and \((y_n)\) in \( X \) such that \( x_{n+1} = F(x_n, y_n) \), \( y_{n+1} = F(y_n, x_n) \), \( x_{n+2} = G(x_{n+1}, y_{n+1}) \), \( y_{n+2} = G(y_{n+1}, x_{n+1}) \) for \( n = 0, 1, 2, \ldots \). Then we have the following cases:

**Case 1:** \( w(x, y, u, v) = d(x, u) \).

From
\[
d(x_{2n+1}, x_{2n+2}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \leq a \cdot d(x_{2n}, x_{2n+1}) a^*.
\]
and
\[
d(y_{2n+1}, y_{2n+2}) = d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1})) \leq a \cdot d(y_{2n}, y_{2n+1}) a^*.
\]
Then from (1) and (2) we have
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq a \cdot d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}) a^*.
\]

**Case 2:** \( w(x, y, u, v) = d(y, v) \).

By similar arguments to case 1 we get
\[
d(x_{2n+1}, x_{2n+2}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \leq a \cdot d(x_{2n}, x_{2n+1}) a^*.
\]
and
\[
d(y_{2n+1}, y_{2n+2}) = d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1})) \leq a \cdot d(y_{2n}, y_{2n+1}) a^*.
\]
Then from (1) and (2) we have
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq a \cdot d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}) a^*.
\]

**Case 3:** \( w(x, y, u, v) = \frac{1}{2}(d(F(x, y), x) + d(G(u, v), u)) \).

We get
\[
d(x_{2n+1}, x_{2n+2}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \leq \frac{a}{2} \cdot d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) a^*.
\]
Similarly we have
\[
d(y_{2n+1}, y_{2n+2}) = d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1})) \leq \frac{a}{2} (d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}))a^* \\
\]
(8)

From (7), (8)
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq \frac{a}{2} (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}))a^* \\
\]
(9)

Case 4: \(w(x, y, u, v) = \frac{1}{2}(d(F(x, y), u) + d(G(u, v), x))\). Similarly we obtain
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq \frac{a}{2} (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}))a^* \\
\]
(9)

Recall all above argument for each case we get

Case1:
\[
d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq a (d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}))a^* \\
\leq a^2 (d(x_{2n-1}, x_{2n}) + d(y_{2n-1}, y_{2n}))(a^*)^2 \leq \ldots \leq a^{2n+1} (d(x_0, x_1) + d(y_0, y_1))(a^*)^{2n+1} \\
\]
(10)

Put \(d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) = d_{n+1} \)

And \(d(x_0, x_1) + d(y_0, y_1) = d_{0,1} \)

We can rewrite eqn (9) in the following form
\[
d_{2n+1, 2n+2} \leq a^{2n+1} (d_{0,1})(a^*)^{2n+1} \text{ and then for each } n \in N_1 \text{ we obtain}
\]
\[
d_{n,n+1} \leq a^n (d_{0,1})(a^*)^n \text{ from lemma 1.4, we have}
\]
\[
\|d_{n,n+1}|| \leq ||a^n|| ||d_{0,1}|| (a^*)^n \\
\]
Since \(A\) is multiplicative, we get \(\|d_{n,n+1}|| \leq ||a^n|| ||d_{0,1}|| ||a^*||^n \) also \(||a|| = ||a^*|| \) so we get \(\|d_{n,n+1}|| \leq ||a||^{2n} ||d_{0,1}|| \)

choose \(||a||^2 = h < 1 \) se we have
\[
\|d_{n,n+1}|| \leq h^n ||d_{0,1}|| . \text{ for } m > n \text{ we get}
\]
\[
\|d_{n,m}|| \leq h^n(1 + h + h^2 + \ldots + h^n-m) ||d_{0,1}|| = \frac{h^n}{1-h} ||d_{0,1}|| \rightarrow 0 \text{ as } n, m \rightarrow \infty \\
\]
So \(\|d_{n,m}|| \leq 0, \text{ as } n, m \rightarrow \infty , \text{ therefore } d_{n,m} \text{ is Cauchy sequence in } X. \)

So \((x_n), (y_n)\), are Cauchy sequences in \(X. \) Since \(X\) is complete we get \(x\) and \(y\) such that \(x_m \rightarrow x, y_m \rightarrow y,\) as \(n, m \rightarrow \infty \) now, we prove that \(F(x, y) = G(x, y) = x\) and \(F(y, x) = G(y, x) = y\), for that \(d(F(x, y), x) \leq d(F(x, y), x_{2n+2}) + d(x_{2n+2}, x) = d(F(x, y), G(x_{2n+1}, y_{2n+1})) + d(x_{2n+2}, x) \leq a \cdot d(x, x_{2n+1})a^* + d(x_{2n+2}, x) \) since \((x_n)\) is Cauchy sequence \(d(F(x, y), x) \leq a \cdot d(x, x_{2n+1})a^* \) by using lemma 2.1
\[
\|d(F(x, y), x)|| \leq ||a|| ||d(x, x_{2n+1})|| ||a^*|| = ||a||^2 ||d(x, x_{2n+1})|| \rightarrow 0 \text{ as } n \rightarrow \infty \\
\]
Therefore \(d(F(x, y), x) = 0 \) and then \(F(x, y) = x. \)

Using similar arguments, we get
\[
d(G(x, y), x) \leq d(G(x, y), x_{2n+1}) + d(x_{2n+1}, x) = d(G(x, y), F(x_{2n}, y_{2n})) + d(x_{2n+1}, x) \leq a \cdot d(x, x_{2n})a^* + d(x_{2n+1}, x) \) since \((x_n)\) is Cauchy sequence \(d(G(x, y), x) \leq a \cdot d(x, x_{2n})a^* \) by using lemma 2.1
\[
\|d(G(x, y), x)|| \leq ||a|| ||d(x, x_{2n})|| ||a^*|| = ||a||^2 ||d(x, x_{2n})|| \rightarrow 0 \text{ as } n \rightarrow \infty \\
\]
Therefore \(d(G(x, y), x) = 0 \) and then \(G(x, y) = x. \)

Similarly we can show that \(F(y, x) = G(y, x) = y \) thus \((x, y)\) is a common coupled fixed point of a mappings \(F \) and \(G\)

To see \((x, y)\) is unique let \((x_0, y_0)\) be other common coupled fixed point of a mappings \(F \) and \(G\)

Let \(d(x, x_0) = d(F(x, x_0), G(x, x_0)) \leq a \cdot d(x, x_0)a^*. \)
\[ \|d(x, x_0)\| \leq \|a\|^2 \|d(x, x_0)\| \to 0 \text{ as } n \to \infty \]

So \( d(x, x_0) = 0 \) therefore \( x = x_0 \) and similarly \( y = y_0 \). 

Case 2: If \( w(x, y, u, v) = d(y, v) \), we follow the same arguments as in case 1, and get the Uniqueness and existence of the common coupled fixed points of the mappings \( F \) and \( G \).

Case 3: \( w(x, y, u, v) = \frac{1}{2} (d(F(x, y), x) + d(G(u, v), u)) \) from equation (9)

\[
d_{2n+1,2n+2} \leq \frac{a}{2} (d_{2n,2n} + d_{2n+1,2n+1}) \text{ from lemma 1.4} \\
d_{2n+1,2n+2} \leq \frac{1}{2} \|a\| (d_{2n,2n} + d_{2n+1,2n+1}) + \frac{1}{2} \|a\|^2 (d_{2n+1,2n+1} + d_{2n+2,2n+2}) \\
d_{2n+1,2n+2} \leq \frac{1}{2} \|a\|^2 (d_{2n,2n} + d_{2n+1,2n+1}) \text{ so for each } n \text{ we have} \\
\|a\|^2 (d_{n+1,2n+1}) \to 0 \text{ as } n \to \infty \\
\text{For } n > m \text{ one can find } \|a\|^2 (d_{n+1,2n+1}) \to 0 \text{ as } n, m \to \infty \\
\text{So \( (x_n, y_m) \), are Couchy sequences in } X \text{. Since } X \text{ is complete we get } x, y \text{ such that } x_n \to x, y_m \to y \text{ as } n, m \to \infty \text{.} \\
d(F(x, y), x) \leq d(F(x, y), x_{2n+2}) + d(x_{2n+2}, x) \leq d(F(x, y), x_{2n+2}) = d(F(x, y), G(x_{2n+1}, y_{2n+1})) \leq \frac{a}{2} (d(F(x, y), x) + d(G(x_{2n+1}, y_{2n+1}), x))a^* \\
\leq \frac{1}{2} (d(F(x, y), x) + d(x_{2n+2}, x))a^* \\
\|d(F(x, y), x)\| \leq \frac{\|a\|^2}{2} \|d(F(x, y), x)\| \\
\|d(F(x, y), x)\| (1 - \frac{\|a\|^2}{2}) \leq 0 \\
\|d(F(x, y), x)\| = 0 \text{ this implies that } d(F(x, y), x) = 0 \text{ and this gives } F(x, y) = x \text{ by a similar way } G(x, y) = x, F(y, x) = y \text{ and } G(y, x) = y. \\
\text{Thus } (x, y) \text{ is a common coupled fixed point of a mappings } F \text{ and } G \\
\text{To see } (x, y) \text{ is unique let } (x_0, y_0) \text{ be other common coupled fixed point of a mappings } F \text{ and } G \\
\text{Let } d(x, x_0) = d(F(x, x_0), G(x, x_0) \leq a \ d(x, x_0)a^*. \\
d(x, x_0) = d(F(x, x_0), G(x, x')) \leq \frac{a}{2} \ |d(F(x, x_0), x) + d(G(x, x_0), x) a \\
d(x, x_0) \leq \frac{a}{2} \ |d(F(x, x_0), x) + d(G(x, x_0), x) a \\
\|d(x, x_0)\| \leq \frac{\|a\|^2}{2} \|d(x, x_0)\| \to 0 \\
\|d(x, x_0)\| \to 0 \text{ then } x = x_0 \text{ similarly } y = y_0 \\
\text{Thus } (x, y) \text{ is a common coupled fixed point is a unique} \\
\text{Case 4: If } w(x, y, u, v) = \frac{1}{2} (d(F(x, y), u) + d(G(u, v), x)), \text{ we follow the same arguments as in case 3, and get the Uniqueness and existence of the common coupled fixed points of the mappings } F \text{ and } G.
CONCLUSIONS
We proved a main theorem in common coupled fixed point theorem in $C^*$-algebras valued metric space using suitable contraction condition that generalize the results obtained in case of real valued metric space.

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REFERENCES