Abstract
An analysis is carried out to study the steady MHD flow and heat transfer characteristics in a visco-elastic fluid flow over a stretching sheet with internal heat generation or absorption. The governing partial differential equations are converted into ordinary differential equations by a similarity transformation. These equations are solved by a numerical method, quasilinearization technique. Heat transfer analysis is carried out for two types of thermal boundary conditions namely, (i) Prescribed Surface temperature (PST) and (ii) Prescribed wall Heat Flux (PHF). The effects of various physical parameters on flow and heat transfer are presented through graphs and discussed.

Keywords: visco-elastic, stretching sheet, Magneto Hydrodynamic, boundary layer, heat transfer, internal heat generation.

INTRODUCTION
The field of boundary layer flow problem over a stretching sheet has many industrial applications such as polymer sheet or filament extrusion from a dye or long thread between feed roll or wind-up roll, glass fiber and paper production, drawing of plastic films, liquid films in condensation process. Due to its high applicability in industrial phenomena, it has attracted the attention of many researchers and one of the pioneering studies that has been performed by Sakiadis [1]. For producing thin plastic film a cautious heat exchange with cooling media should be applied. In these processes, it is very important to control the drag and heat flux at the stretching surface in order to obtain good product quality. The success of the whole process depends on rheological properties of the fluid above the sheet, as it is the fluid viscosity which determines the (drag) force required to pull the sheet. The water and air are amongst the most-widely used fluids as the cooling medium. However, the rate of heat exchange achievable by above fluids is realized to be not suitable for certain sheet materials. To have a better control on the rate of heat exchange, it was proposed to employ the fluids which are more viscoelastic in nature than viscous such as water with polymeric additives [2, 3]. Normally increment of such additives to the fluids leads to increasing of the fluid viscosity to alter flow kinematics in such a way that it leads to a slower rate of solidification compared to water.

In reality most liquids are non-Newtonian in nature, which are abundantly used in many industrial applications. Non-Newtonian fluids have gained considerable importance, because the power required in stretching a sheet and the heat transfer rate for a viscoelastic fluid is found to be less in Non–Newtonian fluids when compared to Newtonian fluids. In addition, the inadequacy of the classical Navier–Stokes theory to describe rheological complex fluids such as polymer solution, blood, paints, certain oils and greases, has led to the development of several theories of non-Newtonian fluids [4–7].

A series of studies on heat transfer effects on viscoelastic fluid have been made by many authors under different physical situations. Vajravelu and Hadjinicolaou [8] have studied the heat transfer characteristics in a viscous fluid over a stretching sheet with viscous dissipation and internal heat generation. Vajravelu and Roper [9] analyzed heat transfer characteristics in a second grade fluid over stretching sheet with viscous dissipation, internal heat generation, and work due to deformation. Rollins and Vajravelu [10] studied flow and heat transfer characteristics in a second order fluid over a stretching sheet with internal heat generation.

Motivated by these studies, this paper extends the work of Rollins and Vajravelu [10] to MHD visco elastic fluid (Walter’s liquid B model) past a stretching sheet. A numerical approach, Quasilinearization technique is used to study flow and heat transfer characteristics of the fluid. This approach is easily adoptable and it is observed that results are in good agreement with the available literature. The effects of various parameters on flow and heat transfer are analyzed through numerical calculations.

MATHEMATICAL FORMULATION
Following the postulates of gradually fading memory, Coleman and Noll [11] derived the constitutive equation of second-order fluid flow in the form

\[ T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \]  \hspace{1cm} (1)

where \( T \) is the Cauchy stress tensor, \(-pI\) is the spherical stress due to constraint of incompressibility, \( \mu \) is the dynamics viscosity, \( \alpha_1, \alpha_2 \) are the material constants and \( A_1 \) and \( A_2 \) are the first two Rivlin–Ericksen tensors [12] defined as

\[ A_1 = (\text{grad} \ v) + (\text{grad} \ v)^T \] \hspace{1cm} (2)
\[ A_2 = \frac{dA_1}{dt} + A_1(\text{grad } v) + (\text{grad } v)^T A_1 \]  

(3)

Here, \( V \) denotes the velocity field and \( \frac{d}{dt} \) is the material time derivative. If the fluid of second grade modeled by (1) is to be compatible with thermodynamics and is to satisfy the Clausius-Duhem inequality for all motions and the assumption that the specific Helmholtz free energy of the fluid is a minimum when it is locally at rest, Dunn and Fosdick [13] found that the material moduli must satisfy

\[ \mu \geq 0, \alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0 \]

(4)

But later on Fosdick and Rajagopal [14] have reported, by using the data reduction from experiments, that in the case of a second order fluid the material constants \( \mu, \alpha_1, \alpha_2 \) should satisfy the relation

\[ \mu \geq 0, \alpha_1 \leq 0, \alpha_1 + \alpha_2 \neq 0 \]

(5)

They also reported that the fluids modeled by (1) with the relationship (5) exhibit some anomalous behavior. A critical review on this controversial issue can be found in the work of Dunn and Rajagopal [15]. Second-order fluid, obeying model equation (1) with \( \alpha_1 < \alpha_2, \alpha_1 < 0 \) although exhibits some undesirable instability characteristics, its approximations are valid at low shear rate. Now in literature the fluid satisfying the model equation (1) with \( \alpha_1 < 0 \) is termed as second-order fluid and with \( \alpha_1 > 0 \) is termed as second grade fluid.

A laminar steady flow of an incompressible electrically conducting viscoelastic (Walters’ liquid B model) fluid over a wall coinciding with the plane \( y = 0 \) is considered, the flow being confined to \( y > 0 \). Two equal and opposite forces are applied along the x-axis, so that a sheet is stretched with a velocity proportional to the distance from the origin. The flow satisfies the rheological equation of state derived by Beard and Walters [16]. Further, this flow field is exposed under the influence of uniform transverse magnetic field. Hence modified viscoelastic boundary layer MHD equation takes the form

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

(6)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \]

\[ -k_0 \left\{ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right\} + \alpha_1 \left\{ \frac{\partial^2 u}{\partial x \partial y} \right\} + \beta_1 \left\{ \frac{\partial^2 u}{\partial y \partial x} \right\} - \frac{\sigma B_0^2}{\rho} \]

(7)

Where \( u, v \) are the velocity components along the x and y directions respectively, \( \nu \) is the kinematic viscosity \( k_0 = -\alpha_1 / \rho \) is the co-efficient of elasticity, and \( \rho \) is the density. Hence, in the case of second order fluid flow, \( k_0 \) takes positive value, as \( \alpha_1 \) takes negative value and other quantities have their usual meanings. In deriving the equation (7), it is assumed that the normal stress is of the same order of magnitude as that of the shear stress, in addition to usual boundary layer approximations.

We assume that the flow is subjected to suction on the boundary sheet with velocity \( v_0 \). The boundary conditions for the velocity field are

\[ u = u_w = bx, \quad v = -v_0 \quad \text{at} \quad y = 0, b > 0 \]

(8)

\[ u \to 0, \frac{\partial u}{\partial y} \to 0 \quad \text{as} \quad y \to \infty \]

The condition \( \frac{\partial u}{\partial y} \to 0 \quad \text{as} \quad y \to \infty \) is the augmented condition, since the flow is in an unbounded domain, which has been discussed by Rajgopal [17]. In this case, the flow is caused solely by the stretching of the sheet, since the free stream velocity is zero.

Defining new variables:

\[ u = bx f_\eta(\eta), \quad v = -\sqrt{b \nu} f_\eta(\eta), \quad \eta = \sqrt{b \nu} y \]

(9)

where \( f_\eta(\eta) \) denotes differentiation with respect to \( \eta \). Clearly \( u \) and \( v \) defined above satisfy the continuity equation (6), and the equation (7) reduces to

\[ f_\eta^2 - f_\eta = f_\eta - k_1 (2 f_\eta f_\eta - f_\eta^2) - M n f_\eta \]

(10)

where \( k_1 = \frac{k_0 b}{\nu} \) is the viscoelastic parameter, \( M n = \frac{\sigma B_0^2}{\rho b} \) is magnetic parameter.

The boundary conditions (8) become

\[ f(0) = R, f_\eta(0) = 1 \]

(11a)

\[ f_\eta(\eta) \to 0, f_\eta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \]

(11b)

HEAT TRANSFER ANALYSIS

The governing boundary layer equation with internal heat generation or absorption is given by

\[ \rho C_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} + Q(T - T_\infty) \]

(12)

Where \( k \) is the thermal diffusivity and \( C_p \) is the specific heat of a fluid at constant pressure and \( Q \) is the volumetric rate of heat generation.

Since there is an appreciable temperature difference between the surface and the ambient fluid, temperature dependent heat sources or sinks exert strong influence on the heat transfer.
characteristics, so above equation (12) includes internal heat generation or absorption. Here the thermal boundary condition depends on the type of heating process under consideration. Heat transfer analysis is carried out for two types of thermal boundary conditions namely, (i) Prescribed Surface temperature (PST) and (ii) Prescribed wall Heat Flux (PHF), which are given below.

**Prescribed Surface Temperature (PST case)**

For this circumstance, the boundary conditions are

\[
T = T_w = T_\infty + A \left( \frac{x}{l} \right)^2 \text{ at } y = 0 \tag{13a}
\]

\[
T \to T_\infty \text{ as } y \to \infty \tag{13b}
\]

Define non-dimensional temperature as

\[
\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{14}
\]

Using (14), equation (12) reduces to

\[
f^{iv} + Pr f^{\theta} - (2 Pr f' - \alpha) \theta = 0 \tag{15}
\]

where \( Pr = \mu c_p / k \), Prandtl number

\[
\alpha = Q/(b \rho c_p), \text{ heat source/sink parameter}
\]

with boundary conditions

\[
\theta(0) = 1, \theta(\infty) \to 0 \tag{16}
\]

**Prescribed surface Heat Flux (PHF Case)**

Prescribed power law surface heat flux (PHF), where surface is subjected to a power law heat flux \( q_w \) on the wall surface is considered to be a quadratic power of \( x \) in the form

\[
-f'' = \frac{1}{k_c} \left[ (f')^2 - f'' - f''' + M b f' + 2k_c f'''' - k_1 (f'')^2 \right] \tag{21}
\]

\[
\theta' = - Pr f' + Pr(2 f' - \alpha) \theta \quad \text{(PST case)} \tag{22a}
\]

\[
g' = -Pr f' + (2 Pr f' - \alpha) g \quad \text{(PHF case)} \tag{22b}
\]

In order to implement the quasilinearization method, the equations (21) and (22) are converted to a system of first order differential equations by substituting

\[
(f, f', f'', f''' \theta, \theta') = (x_1, x_2, x_3, x_4, x_5, x_6) \quad \text{(PST case)} \tag{23}
\]

\[
(f, f', f'', f''' g, g') = (x_1, x_2, x_3, x_4, x_5, x_6) \quad \text{(PHF case)}
\]

Then equations (21) and (22) reduce to:

\[
\frac{dx_1}{d\eta} = x_2
\]

\[
\frac{dx_2}{d\eta} = x_3
\]

\[
\frac{dx_3}{d\eta} = x_4
\]

\[
\frac{dx_4}{d\eta} = \frac{1}{k_1} \left[ x_2^2 - x_3 x_5 + M b x_2 + 2 k_c x_2 x_4 - k_1 x_3^2 \right]
\]

Using (18), equation (12) reduces to

\[
g_{\eta\eta} + Pr f_{\eta} g - (2 Pr f_\eta - \alpha) g = 0 \tag{19}
\]

Boundary conditions are:

\[
g(\eta) = -1 \text{ at } \eta = 0
\]

\[
g(\eta) \to 0 \text{ as } \eta \to \infty \tag{20}
\]

**NUMERICAL SOLUTION OF THE PROBLEM**

The flow equation (10) coupled with energy equation (15) or (19) constitute a set of highly nonlinear differential equations in each thermal boundary condition. So obtaining closed form solution for this set is cumbersome and time consuming. Hence quasilinearization method, given by Bellman & Kalaba [18] is used to solve this system. This method is quadratically convergent, starting from the initial guess value and the solution is valid for a large range of parameters. Even when the required number of initial conditions is not given, this method converges very fast.
\[ \frac{dx_5}{d\eta} = x_6 \]

\[ \frac{dx_6}{d\eta} = -Pr x_4 x_6 + \left( 2Pr x_2 - \alpha \right) x_5 \quad \text{(PST & PHF cases)} \]

Using Quasilinearization technique, the system (23) can be linearized as

\[ \frac{dx_{r+1}}{d\eta} = x_{r+1} \]

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The above system of equations (23) is linear in \( x^{r+1}_i \) (i = 1, 2, ..., 6) and general solution can be obtained by using the principle of superposition.

The boundary conditions given by (11), (16) and (20) reduce to

\[ x^{1+1}_r(0) = R, \quad x^{1+1}_r(0) = 1, \quad x^{1+1}_r(0) = 1 \quad \text{for PST case} \]  

\[ x^{1+1}_r(0) = R, \quad x^{1+1}_r(0) = 1, \quad x^{1+1}_r(0) = -1 \quad \text{for PHF case} \]

\[ x^{1+1}_r(\eta) \rightarrow 0, x^{1+1}_r(\eta) \rightarrow 0, x^{1+1}_r(\eta) \rightarrow 0 \quad \text{as} \ \eta \rightarrow \infty \]

The initial values are chosen as follows:

For homogeneous solution:

\[ x^{h}_i(\eta) = [0 \ 0 \ 1 \ 0 \ 0 \ 0] \]

\[ x^{h}_i(\eta) = [0 \ 0 \ 0 \ 1 \ 0 \ 0] \quad \text{(PST case)} \]

\[ x^{h}_i(\eta) = [0 \ 0 \ 0 \ 0 \ 0 \ 0] \quad \text{(PHF case)} \]

For particular solution:

\[ x^{p}_i(\eta) = [R \ 1 \ 0 \ 0 \ 1 \ 0] \quad \text{(PST case)} \]

\[ x^{p}_i(\eta) = [R \ 1 \ 0 \ 0 \ 0 \ 0] \quad \text{(PHF case)} \]

The general solution for the system (23) is given by

\[ x^{r+1}_i(\eta) = C_1 x^{h}_i(\eta) + C_2 x^{h}_i(\eta) + C_3 x^{h}_i(\eta) + x^{p}_i(\eta) \]

where \( C_1, C_2, C_3 \) are the unknown constants, which are to be determined by considering the boundary conditions as \( \eta \rightarrow \infty \). This solution \( x^{r+1}_i, i = 1, 2, ..., 6 \) is then compared with solution at the previous step \( x^{r}_i, i = 1, 2, ..., 6 \) and next iteration is performed, if the convergence has not been achieved or greater accuracy is desired.

RESULTS AND DISCUSSIONS:

The flow and heat transfer characteristics of an incompressible second order fluid past stretching sheet are studied for two types of thermal boundary conditions i) Prescribed Surface Temperature (PST) and ii) Prescribed Heat Flux (PHF), in the presence of internal heat generation or absorption, have been examined. The governing boundary layer equations are solved using Quasilinearization Technique. The computational results of flow and heat transfer characteristics for various parameters are presented in graphs and discussed.

Fig 1 depicts the effect of magnetic field parameter (Mn) on the horizontal velocity profile \( (f_0(\eta)) \). Horizontal velocity profile decreases with increase in Magnetic field parameter, since increase of Magnetic field parameter signifies the increase of Lorentz force, which opposes the horizontal flow in the reverse direction.

Fig 2(a) and Fig 2(b) depict the effect of viscoelastic parameter \( k_1 \) on longitudinal and transverse velocity components. It can be seen, for a fixed value of \( \eta \), both \( f(\eta) \) and \( \tilde{f}(\eta) \) decrease with increasing values of viscoelastic parameter \( k_1 \). This can be explained by the fact that, as the viscoelastic parameter \( k_1 \) increases, the boundary layer adheres strongly to the surface.

\[ \frac{dx_5}{d\eta} = x_6 \]

\[ \frac{dx_6}{d\eta} = -Pr x_4 x_6 + \left( 2Pr x_2 - \alpha \right) x_5 \]

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which in turn retards the flow in longitudinal and transverse directions.

In Fig 3(a) and 3(b), non-dimensional temperature $\theta(\eta)$ is plotted for various values internal heat source/sink parameter ($\alpha$). It shows that $\theta(\eta)$ increases with increasing values $\alpha$. This is due to the fact that heat is generated inside the boundary layer for increasing values of heat source/sink parameter ($\alpha$).

Fig 4(a) and 4(b) reveal the effect of Prandtl number (Pr) on non-dimensional temperature $\theta(\eta)$ profiles are shown. Temperature $\theta(\eta)$ decreases with increase in the Prandtl number Pr. This is consistent with the fact that the thermal boundary layer thickness decreases with increasing values Prandtl number Pr.

In Fig 5(a) and 5(b) display the values of temperature $\theta(\eta)$ for different values of viscoelastic parameter ($k_1$). It can be observed that, at a given point $\eta$, $\theta(\eta)$ increases with increasing values of $k_1$. This is due to the fact that viscoelastic normal stress gives rise to thickening of thermal boundary layer.

Fig 6(a) and 6(b) show the effect of Magnetic field parameter on temperature distribution in PST and PHF cases respectively. Temperature profile increases with increase in Magnetic field. Since increase of magnetic field increases the thermal boundary layer thickness. The increasing frictional drag due to Lorentz force is responsible for increasing the thermal boundary layer thickness.

Fig 7 depicts the effect of suction parameter (R) on the heat transfer. $\theta(\eta)$ decreases with increasing values of suction parameter (R). Due to suction parameter (R) there will be loss of fluid in the boundary layer region, hence there will be less scope for heat transfer from the sheet to the fluid. This causes the declination in the heat transfer for increasing values of suction parameter.

From our numerical results, it can be concluded that:

i. Horizontal velocity profile decreases with increase in viscoelastic parameter and it also decreases with increase in magnetic field parameter.

ii. Temperature profiles increases with increase in viscoelastic parameter as well as magnetic field parameter in both PST and PHF cases. Hence viscoelastic liquids having low viscous dissipation must be chosen and the strength of external magnetic field should be as mild as possible for effective cooling of the stretching sheet.

iii. Thermal boundary layer thickness decreases with increase in Prandtl number.

iv. The Power law Heat Flux boundary condition is better suited for effective cooling of the stretching sheet.

v. Temperature profiles increases with increasing values of heat source/sink parameter ($\alpha$) and decreases with increasing values of suction parameter (R).

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Figure 1: Plot of velocity ($f_\eta(\eta)$) vs $\eta$ for different values of Magnetic parameter (Mn)
Figure 2: Effect of viscoelasticity ($k_1$) on (a) transverse velocity component, (b) longitudinal component

Mn=1, Pr=3, $\alpha=0.5$

$k_1=0.3, 0.5, 0.7$
Figure 3: Effect of heat source/sink parameter ($\alpha$) on temperature distribution $\theta (\eta)$ in (a) PST case and (b) PHF case.

PST
- $M_n=1$, $Pr=3$, $k_1=0.3$
- $\alpha=-1.0, -0.5, 0.0, 0.5, 1.0$

PHF
- $M_n=1$, $Pr=3.0$, $k_1=0.3$
- $\alpha=-0.5, -0.2, 0.0, 0.2, 0.3, 0.5$
Figure 4: Effect of Prandtl number (Pr) on temperature distribution $\theta(\eta)$ in (a) PST case (b) PHF case

PST
$M_n=1, k_1=0.3, \alpha=0.5$

PHF
$M_n=1, k_1=0.5, \alpha=0.5$

Pr=0.7,1.0,1.5,3.0,6.0,10,
Figure 5. Effect of viscoelastic parameter ($k_1$) on $\theta (\eta)$ in a) PST case and b) PHF case
**Figure 6(a).** Effect of Magnetic field parameter (Mn) on temperature distribution $\theta(\eta)$

**Figure 6(b).** Effect of Magnetic field parameter (Mn) on temperature distribution
REFERENCES


