Norm Inequalities for Positive Semidefinite Matrices

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Abstract
In this paper, we introduce and study some inequalities involving symmetric norms and positive semidefinite matrices. Then we will use the properties of symmetric norms to generalize the inequality obtained by Bhatia and Kittaneh[3].

Keywords: Symmetric norms, positive semidefinite matrices, normal matrices, singular values.

INTRODUCTION
The class of Hermitian matrices is a generalization of real numbers and the class of positive semidefinite matrices is a generalization of nonnegative real numbers. This observation often provides insight into the properties and applications of positive semidefinite matrices. In this paper we will use the inequality that said if a, b are positive real number then for any complex number z

\[ |a - z|b| \leq |a + z|b| \]  \hspace{1cm} (1)

we discuss several matrix norm inequalities for positive semidefinite matrices. All through this paper \( M_{m,n} \) and \( M_n \) will stand, respectively, for spaces of all \( m \times n \) and \( n \times n \) complex matrices. Let \( ||.|| \) denote any symmetric norm (or unitarily invariant norm) on \( M_n \). Therefore, \( ||UAV|| = ||A|| \) for all \( A \in M_n \) and for all unitary matrices \( U, V \in M_n \).

PRELIMINARIES

Definition 3.1:
Let \( A \in M_n \). Then:
1. \( A \) is called Hermitian (or, self adjoint) if \( A^* = A \).
2. \( A \) is called normal if \( A^*A = AA^* \).
3. \( A \) is called unitary if \( A^*A = AA^* = I_n \), where \( I_n \) is the identity matrix of order \( n \).
4. \( A \) is called positive semidefinite or nonnegative definite (written as \( A \succeq 0 \)) if \( A \) is Hermitian and \( \langle Ax, x \rangle \geq 0 \), for all \( x \in \mathbb{C}^n \).

Definition 3.2:
For any matrix \( A \in M_n \), we define the absolute value \( |A| \) of \( A \) to be the positive semidefinite matrix square root of \( A^*A \). Then the singular values of \( A \), \( s_1(A), \ldots, s_n(A) \) are defined to be the eigenvalues of \( |A| \) which ordered from largest to smallest

\( s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A) \).

In this paper, we will apply Ky Fan’s maximum principle to the absolute value,

\[ |A|, \text{ of } A \in M_n \text{, we get for each } k = 1, 2, \ldots, n \text{ that } \]

\[ \sum_{j=1}^{k} s_j(A) = \max \sum_{j=1}^{k} |\langle Ax_j, y_j \rangle| \]

where the maximum is taken over all choices of orthonormal \( k \)-tuples \( x_1, \ldots, x_k \) and \( y_1, \ldots, y_k \).

Also Fan Dominance Theorem saying that if \( A, B \in M_n \), then:

\[ \|A\| \leq \|B\| \text{ for all symmetric norms on } M_n \text{ if and only if } \]

\[ \sum_{j=1}^{k} s_j(A) \leq \sum_{j=1}^{k} s_j(B) \text{ for all } k = 1, 2, \ldots, n. \]

SEVERAL NORM INEQUALITIES FOR MATRICES

Proposition 4.1
If \( A, B \in M_n \) are normal matrices, then \( \|A + B\| \leq \|A\| + \|B\| \).

This satisfies a triangle inequality for normal matrices.

A stronger triangle inequality holds if we use Hermitian matrices:

Proposition 4.2
If \( X, Y \) are Hermitian matrices, then for some unitaries \( U, V \)

\[ |X + Y| \leq \frac{1}{2} (U(|X| + |Y|U^*) + (V(|X| + |Y|V^*)) \]

Proposition 4.2 implies Proposition 4.1. by substituting

\[ X = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix} \]

Now Proposition 4.1, shows that if \( A, B \succeq 0 \), and any complex number \( z \),

\[ \|A + zB\| \leq \|A\| + |z|\|B\| \]

Also Bhatia and Kittaneh [3] generalized inequality (1) by using positive semidefinite matrices \( A, B \succeq 0 \), and any complex number \( z \),

\[ \sum_{j=1}^{k} s_j(A - |z|B) \leq \sum_{j=1}^{k} s_j(A + zB) \]

\[ \leq \sum_{j=1}^{k} s_j(A + |z|B) \quad \text{for all } k = 1, 2, \ldots, n. \]
Which equivalent to

\[ ||A - |z|B|| \leq ||A + zB|| \leq ||A + |z|B|| \]  

(2)

By using Fan Dominance Theorem.

Now we will give the generalization of inequality (2)

**Theorem 4.1:**

If A, B ∈ M_n are positive semidefinite matrices, and z_1, z_2 are complex numbers, then

\[ ||z_1 A - |z_2|B|| \leq ||z_1 A + z_2 B|| \leq ||z_1 A + |z_2|B|| \]  

(3)

**Proof:**

If z_1 = 0, its obvious holds.

But if z_1 ≠ 0, let z = \( \frac{z_2}{z_1} \) in inequality (2), then

\[ ||A - |\frac{z_2}{z_1}|B|| \leq ||A + |\frac{z_2}{z_1}|B|| \leq ||A + zB|| \], so

\[ ||z_1 A - |\frac{z_2}{z_1}|B|| \leq ||z_1 A + z_2 B|| \leq ||z_1 A + |\frac{z_2}{z_1}|B|| \].

**Theorem 4.2:**

If A, B_1, ..., B_n ∈ M_n are positive semidefinite matrices, and z_1, z_2, ..., z_n be complex numbers, then

\[ ||A + z_1 B_1 + ... + z_n B_n|| \leq ||A + |z_1| B_1 + ... + |z_n| B_n|| \]

**Proof:**

\[ ||A + z_1 B_1 + ... + z_n B_n|| = \sum_{j=1}^{k} S_j(A + z_1 B_1 + ... + z_n B_n) \]

= max \( \sum_{j=1}^{k} |\langle A + z_1 B_1 + ... + z_n B_n, x_j, y_j \rangle| \) (using Ky Fan’s maximum principle over all choices of orthonormal k-tuples \( x_1, ..., x_k \) and \( y_1, ..., y_k \).

\[ = \max \sum_{j=1}^{k} |\langle A, x_j, y_j \rangle + z_1 |\langle B_1, x_j, y_j \rangle + ... + z_n |\langle B_n, x_j, y_j \rangle| \]

\[ \leq \max \sum_{j=1}^{k} (|\langle A, x_j, y_j \rangle| + |\langle B_1, x_j, y_j \rangle| + ... + |\langle B_n, x_j, y_j \rangle|)

\[ \leq \max \sum_{j=1}^{k} (|\langle A, x_j, y_j \rangle| + |\langle B_1, x_j, y_j \rangle| + ... + |\langle B_n, x_j, y_j \rangle|)

\[ + |\langle A, x_j, y_j \rangle| + |\langle B_1, x_j, y_j \rangle| + ... + |\langle B_n, x_j, y_j \rangle|)

\]

\[ \leq \frac{1}{2} \sum_{j=1}^{k} (\langle A + |z_1| B_1 + ... + |z_n| B_n, x_j, y_j \rangle + \langle A + |z_1| B_1 + ... + |z_n| B_n, y_j, x_j \rangle)

\]

\[ \langle A + |z_1| B_1 + ... + |z_n| B_n, y_j, x_j \rangle \]

Now using Ky Fan’s maximum principle, we get

\[ \sum_{j=1}^{k} S_j(A + z_1 B_1 + ... + z_n B_n) \leq \sum_{j=1}^{k} S_j(A + |z_1| B_1 + ... + |z_n| B_n) \]

Finally, by Fan Dominance Theorem

\[ ||A + z_1 B_1 + ... + z_n B_n|| \leq ||A + |z_1| B_1 + ... + |z_n| B_n||. \]

If n = 2 and \( A = B_1 = B_2 = \begin{pmatrix} 1 & \frac{1}{i} \\ \frac{1}{i} & 1 \end{pmatrix} \), z_1 = i and z_2 = -1 - i.

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**REFERENCES**


