Abundant Solutions with Distinct Physical Structure for Nonlinear Integro and Partial Differential Equations

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Abstract

In this article, we use two direct methods namely the generalized Kudryashov method and the generalized (G'/G)-expansion method to discuss the traveling wave solutions to the nonlinear integro- partial differential equations. In the generalized (G'/G)-expansion method, we suppose the trial equation for \( G \) satisfies the nonlinear second order differential equation \( A G G'^* (\xi) - B G G' - E G^2 - C G'^2 = 0 \) while \( Q \) in the generalized Kudryashov method satisfies Bernoulli first order differential equation \( Q' = AQ^2 + BQ \)

We construct the exact solutions for some nonlinear integro - partial differential equations in mathematical physics via (3+1)- dimensional Gardner type integro- differential equation and (2+1) dimensional Sawada- Kotera nonlinear integro partial differential equation. We obtain the traveling wave solutions as a rational formula in the hyperbolic functions, trigonometric functions and rational function, when \( G \) satisfies a nonlinear second order ordinary differential equation and \( Q \) satisfies the Bernoulli first order differential equation. When the parameters are taken some special values, the solitary wave are derived from the traveling waves. This method is reliable, simple and gives many new exact solutions.

Keywords: Generalized (G'/G)- expansion method, Generalized Kudryashov method, Traveling wave solutions, Gardner type Kudryashov- differential equation, Sawada- Kotera nonlinear integro partial differential equation

INTRODUCTION

The study of partial differential equations has a significant role in identifying some of the physical and natural phenomena surrounding us and through its knowledge of predicting some natural problems that may be induced in the near future. Many natural and physical problems can be visualized in many nonlinear partial differential equations and by analyzing their analytical solutions, physicists and engineers can interpret those. There are many methods for obtaining exact solutions to nonlinear partial differential equations such as the inverse scattering method [1], Hirota’s bilinear method [2], Backlund transformation [3], the first integral method [4], Painlevé expansion [5], sine–cosine method [6], homogenous balance method [7], extended trial equation method [8,9], perturbation method [10,11], variation method [12], tanh - function method [13,14], Jacobi elliptic function expansion method [15,16], Exp-function method [17,18] and F-expansion method [19,20]. Wang et al. [21] suggested a direct method called the (G'/G) expansion method to find the traveling wave solutions for nonlinear partial differential equations (NPDEs). Zayed et al. [22,23] have used the (G'/G) expansion method and modified (G'/G) expansion method to obtain more than traveling wave solutions for some nonlinear partial differential equations. Shehata [24] have successfully obtained more traveling wave solutions for some important NPDEs when \( G \) satisfies a linear differential equations \( G^* - \mu G = 0 \).

There are many authors have successively applied the (G'/G) expansion method to study the exact solutions for nonlinear evolution equations see [25-28]. In this paper we use the generalized (G'/G) - expansion function method when \( G \) satisfies a nonlinear differential equations \( A G G'^* (\xi) - B G G' - E G^2 - C G'^2 = 0 \), where \( A, B, C, E \) are real arbitrary constants to find the traveling wave solutions for some nonlinear integro- partial differential equations in mathematical physics. Also we use the generalized Kudryashov method [29,30] to discuss the rational traveling wave solutions for some nonlinear integro- partial differential equations. The solitary wave solutions are deduced form the traveling wave solutions when the parameter are taking some special values.

DESCRIPTION OF THE GENERALIZED (G'/G) EXPANSION FUNCTION METHOD FOR NPDEs

In this part of the manuscript, the generalized (G'/G) expansion method will be given. In order to apply this method to nonlinear partial differential equations we consider the
following steps[27,28]

**Step 1.** We consider the nonlinear partial differential equation, say in two independent variables \( x \) and \( t \) is given by

\[
P(u,u_x,u_{xx},u_{xt},u_{tt},u_{xx},u_{xx}) = 0,
\]

(1)

where \( u = u(x,t) \) is an unknown function, \( P \) is a polynomial in \( u \) and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

**Step 2.** We use the following travelling wave transformation:

\[
u = U(\xi), \quad \xi = k_x t - wt,
\]

(2)

where \( k_x, w \) is a nonzero constant. We can rewrite Eq.(1) in the following form:

\[
P(U, U', U'', \ldots) = 0
\]

(3)

**Step 3.** We assume that the solutions of Eq. (3) can be expressed in the following form:

\[
U(\xi) = \sum_{i=-m}^{m} a_i \left( G'(\xi) / G(\xi) \right)^i,
\]

(4)

where \( a_i (i = 0, \pm 1, \ldots, \pm m) \) are arbitrary constants , \( \rho \) is nonzero constant to be determined later, \( m \) is a positive integer and \( G(\xi) \) satisfies a nonlinear second order differential equation

\[
AGG''(\xi) - BGG' - EG^2 - CG^2 = 0
\]

(5)

where \( A, B, C, E \) are real nonzero constants.

**Step 4.** Determine the positive integer \( m \) by balancing the highest order nonlinear term(s) and the highest order derivative in Eq (3).

**Step 5.** Substituting Eq. (4) into (3) along with (5), cleaning the denominator and then setting each coefficient of \((G'(\xi) / G(\xi))^i, i = 0, \pm 1, \pm 2, \ldots\) to be zero, yield a set of algebraic equations for \( a_i (i = 0, \pm 1, \ldots, \pm m) \), \( m \) and \( \rho \).

**Step 6.** Solving these over-determined system of algebraic equations with the help of Maple software package to determine \( a_i (i = 0, \pm 1, \ldots, \pm m) \), \( m \) and \( \rho \).

**Step 7.** The general solution of Eq. (5), takes the following cases:

(i) When \( B \neq 0 \), \( \Omega = B^2 + 4E(A - C) \geq 0 \),

\[
\Gamma = A - C
\]

we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

\[
G(z) = \frac{\frac{\sqrt{\Omega}}{2 \Gamma} \left[ C_1 \cosh(\frac{\sqrt{\Omega}}{2 \Gamma} z) + C_2 \sinh(\frac{\sqrt{\Omega}}{2 \Gamma} z) \right]}{\left[ \sqrt{\Omega} \right]^{2 \Gamma} A}
\]

(6)

where \( C_1 \) and \( C_2 \) are arbitrary constants. In this case the ratio between \( G' \) and \( G \) takes the form

\[
\frac{G'}{G} = \frac{B}{2 \Gamma} + \frac{\sqrt{\Omega}}{2 \Gamma} \left[ \frac{C_1 \sinh(\frac{\sqrt{\Omega}}{2 \Gamma} z) + C_2 \cosh(\frac{\sqrt{\Omega}}{2 \Gamma} z)}{C_1 \cosh(\frac{\sqrt{\Omega}}{2 \Gamma} z) + C_2 \sinh(\frac{\sqrt{\Omega}}{2 \Gamma} z)} \right]
\]

(7)

(ii) When \( B \neq 0 \), \( \Omega = B^2 + 4E(A - C) < 0 \), \( \Gamma = A - C \), we obtain the trigonometric exact solution of Eq.(5) takes the form

\[
\frac{G'}{G} = \frac{B}{2 \Gamma} + \frac{\sqrt{\Omega}}{2 \Gamma} \left[ \frac{-C_1 \sin(\frac{\sqrt{\Omega}}{2 \Gamma} z) + C_2 \cos(\frac{\sqrt{\Omega}}{2 \Gamma} z)}{C_1 \cos(\frac{\sqrt{\Omega}}{2 \Gamma} z) + C_2 \sin(\frac{\sqrt{\Omega}}{2 \Gamma} z)} \right]
\]

(8)

(iii) When \( B = 0 \), \( \Delta = E \Gamma > 0 \), we obtain the rational exact solution of Eq.(5) takes the form

\[
\frac{G'}{G} = \frac{1}{C_1} + \frac{C_2 \xi}{C_2 \xi - C_1}
\]

(9)

(iv) When \( B = 0 \), \( \Delta = E \Gamma < 0 \), we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

\[
\frac{G'}{G} = \sqrt{-\Delta} \frac{C_1 \sinh(\frac{\sqrt{\Delta}}{\Gamma} z) + C_2 \cosh(\frac{\sqrt{\Delta}}{\Gamma} z)}{C_1 \cosh(\frac{\sqrt{\Delta}}{\Gamma} z) + C_2 \sinh(\frac{\sqrt{\Delta}}{\Gamma} z)}
\]

(10)

(v) When \( B = 0 \), \( \Delta = E \Gamma < 0 \), we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

\[
\frac{G'}{G} = \sqrt{-\Delta} \frac{-C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma} z) + C_2 \cos(\frac{\sqrt{\Delta}}{\Gamma} z)}{C_1 \cos(\frac{\sqrt{\Delta}}{\Gamma} z) + C_2 \sin(\frac{\sqrt{\Delta}}{\Gamma} z)}
\]

(11)

**Step 8.** Substituting the constants \( a_i (i = 0, \pm 1, \ldots , \pm m) \), \( m \) and \( \rho \) which obtained by solving the algebraic equations in Step 5, and the general solutions of Eq.(5) in step 6 into
Eq. (4), we obtain new exact solutions of Eq. (1) immediately.

**DESCRIPTION OF THE GENERALIZED KUDRYASHOV METHOD FOR NPDE**

The basic steps in the application of the GKM detailed in the following [29,30]:

**Step 1.** We suppose the exact solution of Eq. (3) to be in the following rational form:

$$V(\xi) = \sum_{i=0}^{N} a_i Q^i(\xi)$$

$$\sum_{j=0}^{M} a_j Q^j(\xi)$$

(12)

Where $a_i, b_j$ are constants to be determined later such that $a_{N} \neq 0, b_0 \neq 0$. We suppose the trial equation for $Q$ satisfies the first order Bernoulli differential equation:

$$Q' = AQ^2 + BQ$$

(13)

**Step 3.** Determine the positive integer numbers $N$ and $M$ in Eq. (12) by balancing the highest order derivatives and the nonlinear terms in Eq. (3).

**Step 4.** Substituting Eqs. (12) and (13) into Eq. (3), we obtain a polynomial in $Q^{i+j}(\xi), (i, j = 0, 1, 2, \ldots)$. Setting all coefficients of this polynomial to be zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica software package to get the unknown parameters $a_i (i = 0, 1, 2, \ldots, N)$ and $b_j (j = 0, 1, 2, \ldots, M)$. Consequently, we obtain the exact solutions of Eq. (1).

**TRAVELING WAVE SOLUTIONS FOR THE FIRST EQUATION TO THE GARDNER TYPE INTEGRO-DIFFERENTIAL EQUATION**

In this section, we use two different methods namely the generalized $(G'/G)$ expansion method and the generalized kudryashov method to discuss the exact solutions for the nonlinear evolution equations in mathematical physics via the $(3+1)$ dimensional Gardner type integro-differential equations which are very important in the mathematical science and have been paid attention by many researchers in physics and engineering. The $(3+1)$ dimensional Gardner type integro-differential equation takes the following form:

$$u_t + 6\beta u_x u + u_{xxx} - \frac{3}{2} \alpha^2 u_x^2 u_{xx} + 3\sigma^2 \frac{u_x}{u_{yy}} dx' = 0$$

(14)

where $\alpha, \beta, \delta$ and $\sigma$ are arbitrary constants. Gardner type integro-differential equations have many applications in different branches of physics such as plasma physics, fluid physics, and quantum field theory [1–7]. We take the transformation:

$$u = v_x$$

(15)

to convert the Gardner type integro-differential equations to the nonlinear partial differential equation:

$$v_{xx} + 6\beta v_x v_{xx} + 3\sigma^2 v_{yy} = 0$$

(16)

Traveling wave transformation

$$v = \phi(\xi), \quad \xi = k_1 x + k_2 y + k_3 z - \omega t$$

(17)

permits us to convert the nonlinear partial differential equation (16) to the following ordinary differential equation

$$-k_1 w \phi'' + 6 \beta k_3^2 \phi \phi'' + k_1^4 \phi^{(4)} - \frac{3}{2} \alpha^2 k_4^2 \phi^2 = 0$$

(18)

By using the integration equation (18) can be written in the following form:

$$\frac{1}{2} (3\sigma^2 k_2^2 + 3\sigma^2 k_3^2 - k_1 w) \phi'' + (\beta k_3^2 - \frac{1}{2} \alpha \sigma k_4^2 k_2) \phi^3$$

$$+ \frac{1}{2} k_1^4 \phi'' - \frac{1}{8} \alpha^2 k_4^2 \phi^4 + c_1 \phi' + c_2 = 0$$

(19)

where $c_1$ and $c_2$ are the integration constants. If , we take $\psi(\xi) = \phi'(\xi)$ equation (19) can be reduced to the following ODE's:

$$\frac{1}{2} (3\sigma^2 k_2^2 + 3\sigma^2 k_3^2 - k_1 w) \psi'' + (\beta k_3^2 - \frac{1}{2} \alpha \sigma k_4^2 k_2) \psi^3$$

$$+ \frac{1}{2} k_1^4 \psi'' - \frac{1}{8} \alpha^2 k_4^2 \psi^4 + c_1 \psi' + c_2 = 0$$

(20)

Generalized $(G'/G)$ expansion method to the $(3+1)$ dimensional Gardner type integro-differential equation:

In this subsection we discuss the solution of Eq.(20) by using generalized $(G'/G)$ expansion method. Balancing the highest order derivative $\psi''$ with the nonlinear term $\psi^4$, we get the solution formula of Eq.(20) has the following form:

$$\psi(\xi) = a_0 + \frac{a_1 G'(\xi)/G(\xi)}{[1 + \rho G'(\xi)/G(\xi) + b_1 [1 + \rho G'(\xi)/G(\xi)]/G'(\xi)/G(\xi)]}$$

(21)

where $a_0, a_1, b_1$ and $\rho$ are constants to be determined.
later. Substituting Eq. (21) along with (5) into Eq. (20) and cleaning the denominator and collecting all terms with the same order of \( (G'(\xi)/G(\xi)) \) together, the left hand side of Eq. (20) are converted into polynomial in \( (G'(\xi)/G(\xi)) \). Setting each coefficient of these polynomials to be zero, we derive a set of algebraic equations for \( a_0, a_1, b_1, k_1, k_2, k_3, w, \sigma \). Solving the set of algebraic equations by using Maple or Mathematica, software package to get the following results:

**Case 1.**

\[
a_0 = \frac{2b_1k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2}, \quad a_1 = \frac{B^2 - 4E(C - A)}{2k_1^2 EA\alpha}, \quad b_1 = -\frac{2E}{A\alpha k_1}, \quad \rho = \frac{B}{2E}, \quad w = \frac{1}{2\alpha^2 k_1 A^2}[4\alpha^2(B^2 - 4E(C - A)) + 12A^2k_1^2\beta^2

-12A^2k_1k_2\beta\alpha\sigma + 3A^2\sigma^2\alpha^2k_2^2 + 6A^2\sigma^2\alpha^2(k_2^2 + k_3^2)]
\]

\[
c_1 = \frac{1}{2A^2\sigma^2 k_1^4}\{4\sigma k_2\alpha^3[B^2 - 4E(C - A)] - 8\alpha^2\sigma^2[B^2 - 4E(C - A)] - A^2\sigma^2\alpha^3k_3^2

-12A^2\sigma k_2\alpha k_1^2\beta^2 + 6A^2\sigma^2k_2^2k_1\alpha^2\beta + 8A^2k_1^3\beta^3}\]
\]

\[
\psi(\xi) = \frac{2b_1k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2} + \frac{\Omega[(BC_1 + C_2\sqrt{\Omega})\cosh(\sqrt{\Omega} 2\Gamma)\xi + (BC_2 + C_1\sqrt{\Omega})\sinh(\sqrt{\Omega} 2\Gamma)\xi]}{k_1^2 A\alpha\{((4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega})\cosh(\sqrt{\Omega} 2\Gamma)\xi + ((4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega})\sinh(\sqrt{\Omega} 2\Gamma)\xi\}]

+ \frac{k_1^2 A\alpha\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega})\cosh(\sqrt{\Omega} 2\Gamma)\xi + ((4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega})\sinh(\sqrt{\Omega} 2\Gamma)\xi\}}{A\alpha k_1^2[(BC_1 + C_2\sqrt{\Omega})\cosh(\sqrt{\Omega} 2\Gamma)\xi + (BC_2 + C_1\sqrt{\Omega})\sinh(\sqrt{\Omega} 2\Gamma)\xi]}\].
\]

\[
c_2 = \frac{1}{8\alpha^6 k_1^4 A^4} \{32A^2k_1^2\beta^2\alpha^2[B^2 - 4E(C - A)]

- 8A^2\sigma^2k_1^2\alpha^4[B^2 - 4E(C - A)] + 32A^2\alpha^3k_1k_2B\sigma[B^2

- 4E(C - A)] + 16\alpha^4[B^2 - 4E(C - A)]^2

+ A^4(\sigma k_2\alpha - 2k_1\beta)^4\}
\]

where \( C, B, E, A, \sigma, \alpha, \beta, k_1, k_2, k_3 \) are arbitrary constants. There are many other cases which are omitted here for convenience to the reader. In this case the traveling wave solution of Eq.(20) takes the following form:

\[
\psi(\xi) = \frac{2b_1k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2} + \frac{(B^2 - 4E(C - A))G'(\xi)/G(\xi)}{A\alpha k_1^2 G'(\xi)/G(\xi)}\]

(23)

There are many families to discuss the types of the traveling wave solutions of Eq.(20) as follows:

**Family 1.** When \( B \neq 0, \Omega = B^2 + 4E(A - C) > 0 \), we obtain the hyperbolic exact solution of Eq.(20) takes the following:

\[
\]
Consequently the hyperbolic traveling wave solution of Eq.(14) has the following form:

\[
\psi(\xi) = \frac{2\beta_1 - \alpha \sigma \xi_2}{\alpha^2 k_1^2} + \frac{\Omega((BC_1 + C_2 \sqrt{\Omega}) \cosh(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + (BC_2 - C_1 \sqrt{-\Omega}) \sinh(\frac{\sqrt{-\Omega}}{2\Gamma} \xi))}{k_1 A \alpha \left\{ ((4\Gamma + B^2)C_1 + BC_2 \sqrt{-\Omega}) \cosh(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + ((4\Gamma + B^2)C_2 + BC_1 \sqrt{-\Omega}) \sinh(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) \right\}}
\]

(25)

where

\[
\xi = k_1 x + k_2 y + k_3 z - \frac{t}{2\alpha^2 k_1 A^2} [-4\alpha^2 (B^2 - 4E(C - A)) + 12A^2 k_1^2 \beta^2 - 12A^2 k_1 k_2 \beta \alpha \sigma + 3A^2 \sigma^2 \alpha^2 k_2^2 + 6A^2 \sigma^2 \alpha^2 (k_2^2 + k_3^2)]
\]

Family 2. When \( B \neq 0 \), \( \Omega = B^2 + 4E(A - C) < 0 \), we obtain the trigonometric exact solution of Eq.(20) takes the following form:

\[
u_1(x, y, z, t) = \frac{2\beta_1 - \alpha \sigma \xi_2}{\alpha^2 k_1^2}
\] + \[
\frac{\Omega((BC_1 + C_2 \sqrt{-\Omega}) \cos(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + (BC_2 - C_1 \sqrt{-\Omega}) \sin(\frac{\sqrt{-\Omega}}{2\Gamma} \xi))}{k_1 A \alpha \left\{ ((4\Gamma + B^2)C_1 + BC_2 \sqrt{-\Omega}) \cos(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + ((4\Gamma + B^2)C_2 + BC_1 \sqrt{-\Omega}) \sin(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) \right\}}
\]\n
(26)

Consequently the periodic trigonometric traveling wave solution of Eq.(14) has the following form:

\[
u_2(x, y, z, t) = \frac{2\beta_1 - \alpha \sigma \xi_2}{\alpha^2 k_1^2}
\] + \[
\frac{\Omega((BC_1 + C_2 \sqrt{-\Omega}) \cos(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + (BC_2 - C_1 \sqrt{-\Omega}) \sin(\frac{\sqrt{-\Omega}}{2\Gamma} \xi))}{k_1 A \alpha \left\{ ((4\Gamma + B^2)C_1 + BC_2 \sqrt{-\Omega}) \cos(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + ((4\Gamma + B^2)C_2 + BC_1 \sqrt{-\Omega}) \sin(\frac{\sqrt{-\Omega}}{2\Gamma} \xi) \right\}}
\]\n
(26)
Family 3. When $B \neq 0$, $\Omega = 0$, we obtain the rational exact solution of Eq. (20) takes the following form:
\[
\psi(x) = \frac{2\beta k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2} - \frac{1}{A \alpha k_1 B(C_1 + C_2 \xi) + 2BC_2} \bigg[ \xi \bigg( (4k_1 + B^2)(C_1 + C_2 \xi) + 2BC_2 \bigg) \bigg].
\] (27)

Consequently the rational traveling wave solution of Eq. (14) has the following form:
\[
u_3(x, y, z, t) = \frac{2\beta k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2} - \frac{1}{A \alpha k_1 B(C_1 + C_2 \xi) + 2BC_2} \bigg[ \xi \bigg( (4k_1 + B^2)(C_1 + C_2 \xi) + 2BC_2 \bigg) \bigg].
\] (28)

Family 4. When $B = 0$, $\Delta = E \Gamma > 0$, we obtain the hyperbolic exact solution of Eq. (15) takes the following form:
\[
\psi(x) = \frac{2\beta k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2} - \frac{2\sqrt{\Delta}}{A \alpha k_1^2} \bigg[ C_1 \sinh\bigg( \frac{\sqrt{\Delta}}{\Gamma} \xi \bigg) + C_2 \cosh\bigg( \frac{\sqrt{\Delta}}{\Gamma} \xi \bigg) \bigg].
\] (29)

Consequently the rational traveling wave solution of Eq. (14) has the following form:
\[
u_4(x, y, z, t) = \frac{2\beta k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2} - \frac{2\sqrt{\Delta}}{A \alpha k_1^2} \bigg[ C_1 \sinh\bigg( \frac{\sqrt{\Delta}}{\Gamma} \xi \bigg) + C_2 \cosh\bigg( \frac{\sqrt{\Delta}}{\Gamma} \xi \bigg) \bigg].
\] (30)

where
\[
\xi = k_1 x + k_2 y + k_3 z.
\]

\[ - \frac{t}{2\alpha^2 k_1^2} \bigg[ 16\alpha^2 E(C - A) \bigg] + 12\alpha^2 \bigg( k_1 \beta \alpha \sigma + 3\alpha^2 \sigma^2 \alpha^2 k_1^2 + 6\alpha^2 \sigma^2 \alpha^2 \bigg( k_1^2 + k_3^2 \bigg) \bigg].
\] (31)

Family 5. When $B = 0$, $\Delta = E \Gamma < 0$, we obtain the hyperbolic exact solution of Eq. (20) takes the following form:
\[
\psi(x) = \frac{2\beta k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2} - \frac{2\sqrt{\Delta}}{A \alpha k_1^2} \bigg[ - C_1 \sin\bigg( \frac{\sqrt{\Delta}}{\Gamma} \xi \bigg) + C_2 \cos\bigg( \frac{\sqrt{\Delta}}{\Gamma} \xi \bigg) \bigg].
\] (32)

Consequently the rational traveling wave solution of Eq. (14) has the following form:
\[
u_5(x, y, z, t) = \frac{2\beta k_1 - \alpha \sigma k_2}{\alpha^2 k_1^2} - \frac{2\sqrt{\Delta}}{A \alpha k_1^2} \bigg[ - C_1 \sin\bigg( \frac{\sqrt{\Delta}}{\Gamma} \xi \bigg) + C_2 \cos\bigg( \frac{\sqrt{\Delta}}{\Gamma} \xi \bigg) \bigg].
\] (33)

where $\xi$ is defined as Eq. (31).

Generalized Kudryashov method to the (3+1) dimensional Gardner type integro-differential equation:

In this subsection we discuss the solution of Eq. (20) by using generalized Kudryashov method. Balancing the highest order derivative $\psi^4$ with the nonlinear term $\psi^4$, we have
\[ N - M = 1. \] (34)

Equation (34) has infinitely solutions, in the special if $M = 1$ then $N = 2$. Consequently the solution formula of
Eq.(20) has the following form:

$$\psi(\xi) = \frac{a_0 + a_1 Q(\xi) + a_2 Q^2(\xi)}{b_0 + b_1 Q(\xi)}$$

(35)

where $a_0, a_1, a_2, b_0, b_1$ are constants to be determined later and

$$Q'(\xi) = AQ^2(\xi) + BQ(\xi).$$

Substituting Eqs. (35) and (36) into Eq. (20), we obtain a polynomial in $Q^{i-j}$, $(i, j = 0, 1, 2, \ldots)$. Setting all coefficients of this polynomial to be zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica software package to get the unknown parameters $a_0, a_1, a_2, b_0, b_1, k_1, k_2, c_1$ and $w$.

$$a_0 = -\frac{b_0 (-2\beta k_1 + 2Bak^2 + \alpha \sigma k_2)}{\alpha^2 k_1^2},$$

$$a_1 = -\frac{2b_0 A (-2\beta k_1 + 2Bak^2 + \alpha \sigma k_2)}{B\alpha^2 k_1^2},$$

$$a_2 = -\frac{4b_0 A^2}{B\alpha}, \quad b_1 = \frac{2b_0 A}{B},$$

$$c_1 = -\frac{1}{2\alpha^4 k_1^2}\{8k_1^2 \beta \alpha^2 B^2 - 8\beta^3 k_1^3 + \sigma^3 k_2^3 \alpha \sigma + 12\alpha \beta^2 k_1^2 k_2 \sigma - 6\alpha^2 \beta k_1^2 \sigma^2 - 4\alpha^3 k_2 \alpha \sigma^2 k_1^2 B^2\},$$

$$c_2 = -\frac{1}{8\alpha^8 k_1^4}\{-32\alpha \beta^3 k_1^3 k_2 \sigma + 24\alpha^2 \beta^2 k_1^2 k_2 \sigma^2 - 8\alpha^3 \beta k_1^3 k_2 \sigma^3 + 8\sigma^2 k_2^3 \alpha^4 k_1^2 B^2 + 16\beta k_1^4 + 32\alpha^3 \beta k_1^3 B^2 k_2 \sigma + 4\alpha^3 \beta k_1^2 k_2 \sigma + 16\alpha^4 k_1^4 B^2 - 32\alpha^2 \beta^2 k_1^2 B^2\},$$

$$w = -\frac{1}{2\alpha^2 k_1}\{-12\beta^2 k_1^2 + 12\alpha \beta k_1 k_2 - 9\sigma^2 \alpha^2 k_2^2\} + 4k_1^4 \alpha^2 B^2 - 6\sigma^2 \alpha^2 k_2^2\}$$

(37)

where $b_0, k_1, k_2, k_3, \alpha, \beta, \sigma$, and $A, B$ are arbitrary constants. In this case the traveling wave solution takes the form:

$$\psi(\xi) = -\frac{B(-2\beta k_1 + 2Bak^2 + \alpha \sigma k_2) + 2A(-2\beta k_1 + 2Bak^2 + \alpha \sigma k_2) Q(\xi) + 4A^2 \alpha k_1^2 Q^2(\xi)}{\alpha^2 k_1^2 b_0 B + 2b_0 A \alpha^2 k_1^2 Q(\xi)}$$

(38)

Substituting by the general solutions of Eq.(13) into (38) we have the rational traveling wave solution:

$$\psi(\xi) = -\frac{1}{\alpha^2 k_1^2 b_0 (1 - A Ce^{B\xi})^2 + 2b_0 A \alpha^2 k_1^2 Ce^{B\xi} (1 - A Ce^{B\xi})}\{4A^2 \alpha k_1^2 B C^2 e^{2B\xi} + (-2\beta k_1 + 2Bak^2 + \alpha \sigma k_2)(1 - A Ce^{B\xi})^2 + 2AC(-2\beta k_1 + 2Bak^2 + \alpha \sigma k_2)e^{B\xi} (1 - A Ce^{B\xi})\}$$

(39)

where

$$\xi = k_1 x + k_2 y + k_3 z + \frac{t}{2\alpha^2 k_1}\{-12\beta^2 k_1^2 + 12\alpha \beta k_1 k_2 - 9\sigma^2 \alpha^2 k_2^2 + 4k_1^4 \alpha^2 B^2 - 6\sigma^2 \alpha^2 k_2^2\}$$

There are many other cases which omit for convenience to the reader.

**Remark 1:** The general $G'/G$ expansion method is more effective than the generalized Kuderyshov method. The general $G'/G$ expansion is complicated than the generalized Kuderyshov method but determine many types of exact solutions such as the hyperbolic functions, trigonometric functions and rational function but the generalized Kuderyshov method determine only one type of solution.

**TRAVELING WAVE SOLUTIONS FOR (2+1) DIMENSIONAL SAWADA-KOTERA NONLINEAR INTEGRAL PARTIAL DIFFERENTIAL EQUATION**

In this section, we use generalized $(G'/G)$ expansion method to discuss the exact solutions for the nonlinear evolution equations in mathematical physics via (2+1) dimensional Sawada-Kotera nonlinear integro partial differential equation which are very important in the mathematical science and have been paid attention by many researchers in physics and engineering. The (2+1) dimensional Sawada-Kotera nonlinear integro partial
The differential equation takes the following form:
\[
\frac{1}{x} \frac{\partial}{\partial x} \left( x^2 \frac{\partial u}{\partial x} \right) + 5u \frac{\partial u}{\partial x} + \frac{5}{3} u^3 + \frac{\partial u}{\partial y} = 0
\]

The transformation (15) convert the (2+1) dimensional Sawada-Kotera nonlinear integro partial differential equation (40) to the following partial differential equation
\[
v_{st} = (v_{sx} + 5v_x v_{xxx} + 5 \frac{1}{3} v^3 + v_{xxy})_x - 5v_{yy} + 5v_x v_{xy} + 3v_{xx} v_y
\]

Traveling wave transformation (17) permits us to convert the nonlinear partial differential equation (41) to the following ordinary differential equation
\[
k w \phi'' + k_1 (k_1 \phi'^5) + 5k_1^2 \phi'^4 \phi'' + \frac{5}{3} k_1^3 \phi'^3
\]
\[
+ k_1^2 k_2 \phi'' - 5k_2^2 \phi'' + 10k_2^2 k_2 \phi'' = 0.
\]

By using the integration equation (42) can be written in the following form:
\[
(k_1 w - 5k_2^2) \phi' + k_1 (k_1 \phi'^5) + 5k_1^2 \phi'^4 \phi'' + \frac{5}{3} k_1^3 \phi'^3
\]
\[
+ k_1^2 k_2 \phi'' + 5k_2^2 k_2 \phi'' + c_1 = 0,
\]

where \( c_1 \) is the integration constant. If we take
\[
\psi'(x) = \phi'(x)
\]

equation (43) can be reduced to the following ODE’s:
\[
(k_1 w - 5k_2^2) \psi + k_1 (k_1 \psi^{(4)}) + 5k_1^2 \psi \psi'' + \frac{5}{3} k_1^3 \psi^3
\]
\[
+ k_1^2 k_2 \psi'' + 5k_2^2 k_2 \psi'' + c_1 = 0.
\]

We discuss the solution of Eq.(44) by using generalized rational \( (G'/G) \) expansion method. Balancing the highest order derivative \( \psi^{(4)} \) with the nonlinear term \( \psi^3 \), we get the solution formula of Eq.(44) has the following form:
\[
\psi(x) = a_0 + \frac{a_1 G'(\xi)/G(\xi)}{[1 + \rho G'(\xi)/G(\xi)]} + \frac{b_1 [1 + \rho G'(\xi)/G(\xi)]}{G'(\xi)/G(\xi)} + \frac{a_2 [G'(\xi)/G(\xi)]^2}{[1 + \rho G'(\xi)/G(\xi)]^2}
\]
\[
+ \frac{b_2 [1 + \rho G'(\xi)/G(\xi)]^2}{[G'(\xi)/G(\xi)]^2}
\]
\[
(45)
\]

where \( a_0, a_1, b_1, a_2, b_2 \) and \( \rho \) are constants to be determined later. Substituting Eq. (45) along with (5) into Eq. (44) and collecting the denominator and collecting all terms with the same order of \( (G'(\xi)/G(\xi)) \), together, the left hand side of Eq. (44) are converted into polynomial in \( (G'(\xi)/G(\xi)) \). Setting each coefficient of these polynomials to be zero, we derive a set of algebraic equations for \( a_0, a_1, b_1, a_2, b_2, k_1, k_2, k_3, w, c_1 \) and \( \rho \). Solving the set of algebraic equations by using Maple or Mathematica, software package to get the following results:

**Case 1.**
\[
a_0 = -\frac{1}{5A^2 k_1^2} [9k_2 A^2 - 10k_1^3 (B^2 - 4E(C - A))],
\]
\[
b_2 = \frac{12E^2 k_1}{A^2},
\]
\[
a_2 = -\frac{3k_1 [B^2 - 4E(C - A)]^2}{4A^2 E^2}, \quad \rho = \frac{B}{2E},
\]
\[
a_1 = b_1 = 0,
\]
\[
w = -\frac{2}{5k_1 A^4} \{ - 42240k_6^6 k_2 E^2 C^2 A^2 + 38400k_9 B^4 E
\]
\[
- 15360k_9 B^2 E^2 C^2 - 15360k_9 B^2 E^2 A^2 - 38400k_9 B^4 E
\]
\[
+ 30720k_9 B^4 C A^2 - 3200k_9 B^6 + 24k_2 A^6
\]
\[
- 20480k_9 E^3 A^3 + 20480k_9 E^3 C^3 + 21120k_6 B^2 E^2 E C k_2
\]
\[
- 6144k_9 E^2 C^2 A + 6144k_9 E^2 C^2 C - 264k_9 B^2 C A^2
\]
\[
- 42240k_6^4 E^2 A^2 k_2 - 21120k_9 B^2 E^2 A^2 k_2
\]
\[
+ 84480k_6^6 C A^3 k_2\}
\]

where \( C, B, E, A, \sigma, k_1, k_2 \) are arbitrary constants. There are many other cases which are omitted here for convenience to the reader. In this case the traveling wave solutions of Eq.(44) take the following form:
\[
\psi(x) = -\frac{1}{5A^2 k_1^2} [9k_2 A^2 - 10k_1^3 (B^2 - 4E(C - A))]
\]
\[
- \frac{3k_1 [B^2 - 4E(C - A)]^2 [G'(\xi)/G(\xi)]^2}{4A^2 E^2 [1 + \rho G'(\xi)/G(\xi)]^2}
\]
\[
- \frac{12E^2 k_1 [1 + \rho G'(\xi)/G(\xi)]^2}{A^2 [G'(\xi)/G(\xi)]^2}
\]

There are many families to discuss the types of the traveling wave solutions of Eq.(44) as follows:
When $B \neq 0$, $\Omega = B^2 + 4E(A - C) > 0$, $\Gamma = A - C$ we obtain the hyperbolic exact solution of Eq.(44) takes the following:

$$
\psi(\xi) = -\frac{1}{5A^2 k_1^2} [9k_2 A^2 - 10k_1^3 (B^2 - 4E(C - A))]
$$

$$
3k_1 \Omega^2 [(B_1 + C_2 \sqrt{\Omega}) \cosh (\frac{\sqrt{\Omega}}{2\Gamma} \xi) + (B_2 + C_1 \sqrt{\Omega}) \sinh (\frac{\sqrt{\Omega}}{2\Gamma} \xi)]^2
$$

$$
- \frac{3k_1}{A^2} \left\{ (4E + B^2)C_1 + BC_2 \sqrt{\Omega} \right\} \cosh (\frac{\sqrt{\Omega}}{2\Gamma} \xi) + (4E + B^2)C_2 + BC_1 \sqrt{\Omega} \sinh (\frac{\sqrt{\Omega}}{2\Gamma} \xi)]^2
$$

Consequently the hyperbolic traveling wave solution of Eq.(40) has the following form:

$$
u_1(x, y, t) = -\frac{1}{5A^2} [9k_2 A^2 - 10k_1^3 (B^2 - 4E(C - A))]
$$

$$
3k_1 \Omega^2 [(B_1 + C_2 \sqrt{\Omega}) \cosh (\frac{\sqrt{\Omega}}{2\Gamma} \xi) + (B_2 + C_1 \sqrt{\Omega}) \sinh (\frac{\sqrt{\Omega}}{2\Gamma} \xi)]^2
$$

$$
- \frac{3k_1}{A^2} \left\{ (4E + B^2)C_1 + BC_2 \sqrt{\Omega} \right\} \cosh (\frac{\sqrt{\Omega}}{2\Gamma} \xi) + (4E + B^2)C_2 + BC_1 \sqrt{\Omega} \sinh (\frac{\sqrt{\Omega}}{2\Gamma} \xi)]^2
$$

where

$$
\xi = k_1 x + k_2 y - \frac{2t}{5k_1 A^4} \left\{ -40k_1^5 (B^2 - 4E(C - A))^2 + 17k_2^2 A^4 \right\}
$$

When $B \neq 0$, $\Omega = B^2 + 4E(A - C) < 0$, we obtain the trigonometric exact solution of Eq.(44) takes the following form:

$$
\psi(\xi) = -\frac{1}{5A^2 k_1^2} [9k_2 A^2 - 10k_1^3 (B^2 - 4E(C - A))]
$$

$$
3k_1 \Omega^2 [(B_1 + C_2 \sqrt{-\Omega}) \cos (\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + (B_2 - C_1 \sqrt{-\Omega}) \sin (\frac{\sqrt{-\Omega}}{2\Gamma} \xi)]^2
$$

$$
- \frac{3k_1}{A^2} \left\{ (4E + B^2)C_1 + BC_2 \sqrt{-\Omega} \right\} \cos (\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + (4E + B^2)C_2 - BC_1 \sqrt{-\Omega} \sin (\frac{\sqrt{-\Omega}}{2\Gamma} \xi)]^2
$$

$$
\left\{ (BC_1 + C_2 \sqrt{-\Omega}) \cos (\frac{\sqrt{-\Omega}}{2\Gamma} \xi) + (BC_2 - C_1 \sqrt{-\Omega}) \sin (\frac{\sqrt{-\Omega}}{2\Gamma} \xi)]^2
$$
Consequently the periodic trigonometric traveling wave solution of Eq.(40) has the following form:

\[ u_2(x, y, t) = -\frac{1}{5A^2}[9k_2A^2 - 10k_1^3(B^2 - 4E(C - A))] + \]

\[ 3k_1^2\Omega^2[(BC_1 + C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}t) + (BC_2 - C_1\sqrt{-\Omega})\sin(\frac{\sqrt{-\Omega}}{2\Gamma}t)]^2 \]

\[ A^2\[(4\Delta + B^2)C_1 + BC_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}t) + (BC_2 - C_1\sqrt{-\Omega})\sin(\frac{\sqrt{-\Omega}}{2\Gamma}t)]^2 \]

(52)

where \( \xi \) is defined as Eq.(50).

**Family 3.** When \( B \neq 0, \ \Omega = 0 \), we obtain the rational exact solution of Eq.(44) takes the following form:

\[ \psi(\xi) = -\frac{9k_2}{5k_1^2} - \frac{3k_1}{A^2} \frac{[(4\Delta + B^2)(C_1 + C_2\xi) + 2BC_2\Gamma]^2}{[B(C_1 + C_2\xi) + 2\Gamma C_2]^2} \]

(53)

Consequently the rational traveling wave solution of Eq.(40) has the following form:

\[ u_4(x, y, t) = -\frac{1}{5A^2k_1}[9k_2A^2 - 40k_1^3\Delta] \]

\[ -\frac{48k_1^2\Delta}{4A^2}[C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2 \]

\[ 12\Delta k_1^2[C_1\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2 \]

\[ A^2[C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2 \]

(56)

where

\[ \xi = k_1x + k_2y - \frac{2t}{5k_1A^4}\{-640k_1^6\Delta^2 + 17k_1^2A^4\} \]

(57)

**Family 4.** When \( B = 0, \ \Delta = E\Gamma > 0 \), we obtain the hyperbolic exact solution of Eq.(44) takes the following form:

\[ \psi(\xi) = -\frac{1}{5A^2k_1^2}[9k_2A^2 - 40k_1^3\Delta] \]

\[ -\frac{48k_1^2\Delta}{4A^2}[C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2 \]

\[ 12\Delta k_1^2[C_1\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2 \]

\[ A^2[C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2 \]

(55)

**Family 5.** When \( B = 0, \ \Delta = E\Gamma < 0 \), we obtain the hyperbolic exact solution of Eq.(44) takes the following form:
\[
\psi(\xi) = -\frac{1}{5A^2 k_1^2} [9k_2 A^2 - 40k_1^3 \Delta]
\]

Consequently the rational traveling wave solution of Eq.(40)

\[
\begin{align*}
&+ \frac{48k_1 \Delta}{4A^2} \left[ -C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma} \xi) + C_2 \cos(\frac{\sqrt{\Delta}}{\Gamma} \xi) \right]^2 \\
&+ \frac{12k_1^2 \left[ C_1 \cos(\frac{\sqrt{\Delta}}{\Gamma} \xi) + C_2 \sin(\frac{\sqrt{\Delta}}{\Gamma} \xi) \right]^2}{A^2 \left[ -C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma} \xi) + C_2 \cos(\frac{\sqrt{\Delta}}{\Gamma} \xi) \right]^2}.
\end{align*}
\]

has the following form:

\[
\begin{align*}
u_5(x, y, t) &= -\frac{1}{5A^2 k_1} [9k_2 A^2 - 40k_1^3 \Delta] + \frac{48k_1 \Delta}{4A^2} \left[ -C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma} \xi) + C_2 \cos(\frac{\sqrt{\Delta}}{\Gamma} \xi) \right]^2 \\
&+ \frac{12k_1^2 \left[ C_1 \cos(\frac{\sqrt{\Delta}}{\Gamma} \xi) + C_2 \sin(\frac{\sqrt{\Delta}}{\Gamma} \xi) \right]^2}{A^2 \left[ -C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma} \xi) + C_2 \cos(\frac{\sqrt{\Delta}}{\Gamma} \xi) \right]^2}.
\end{align*}
\]

where \( \xi \) is defined as Eq.(51).

3.2. Numerical solutions for KdV equation

In this section we give some figures to illustrate some of our results which obtained in this section. To this end, we select some special values of the parameters to show the behavior of the extended rational \((G'/G)\) expansion method for the KdV equation.

\[\text{Figure 1. The exact extended (} G'/G \text{) expansion solution} U_1 \text{ in Eq. (24) and its projection at} t = 0 \text{when the parameters take special values } E = 1, C1 = 2, A = 2, B = 5, C = 1, k_1 = 5, k_2 = 3, k_3 = 7, \beta = 11, \sigma = 3, \alpha = 1, z = 3, y = 5.\]
Figure 2. The exact extended \((G'/G)\) expansion solution \(U_2\) in Eq. (26) and its projection at \(t = 0\) when the parameters take special values \(E = 1, C_1 = 2, C_2 = 3, A = 1, B = 1, C = 10, k_1 = 5, k_2 = 3, k_3 = 7, \beta = 11, \sigma = 3, \alpha = 1, z = 3, y = 5\).

Figure 3. The exact extended \((G'/G)\) expansion solution \(U_3\) in Eq. (29) and its projection at \(t = 0\) when the parameters take special values \(E = 1, C_1 = 2, C_2 = 3, A = 1, B = 2, C = 2, k_1 = 5, k_2 = 3, k_3 = 7, \beta = 11, \sigma = 3, \alpha = 1, z = 3, y = 5\).

Figure 4. The exact extended \((G'/G)\) expansion solution \(U_4\) in Eq. (30) and its projection at \(t = 0\) when the parameters take special values \(E = 1, C_1 = 2, C_2 = 3, A = 1, B = 0, C = 2, k_1 = 5, k_2 = 3, k_3 = 7, \beta = 11, \sigma = 3, \alpha = 1, z = 3, y = 5\).
CONCLUSION

In this paper we use the generalized \((G'/G)\) expansion method and generalized Kudryashov method to construct a series of some new traveling wave solutions for some nonlinear integro-partial differential equations in the mathematical physics. We constructed the rational exact solutions in many different functions such as hyperbolic function solutions, trigonometric function solutions and rational exact solution. The performance of this method reliable, effective and powerful for solving the nonlinear partial differential equations.

Conflict of Interests  The authors declare that there is no conflict of interests regarding the publication of this paper.

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