On Some Properties of $\beta$-open sets

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Abstract

In this paper we introduce and study the concepts of $\beta$-open set, $\beta$-continuous functions, then we also study the concepts of $\beta$-compact subsets and study some new characterizations of $\beta$-separation axioms such as $\beta$-$T_2$. Then we discuss the relations between the $\beta$-continuous functions and these concepts.

Keywords: $\beta$-open set, $\beta$-compact, $\beta$-open cover, $\beta$-closed sets, $\beta$-continuous

INTRODUCTION

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Levine [7] introduced the notion of semi-open sets and semi-continuity in topological spaces. Andrijevic [2] introduced a class of generalized open sets in topological spaces. Mashhour [9] introduced pre open sets in topological spaces. Monsief et al. [1] initiated the study of $\beta$-open sets and $\beta$-continuity in a topological space. The class of $\beta$-open sets is contained in the class of semi-open and pre-open sets. In this paper we discuss the covering properties of $\beta$-sets and $\beta$-continuous functions. All through this paper $(X, \tau)$ and $(Y, \sigma)$ stand for topological spaces with no separation assumed.

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PRELIMINARIES

Definition 3.1 A subset $A$ of a space $X$ is said to be [2],[10]:
1. Semi-open if $A \subseteq \text{Cl} (\text{Int}(A))$
2. Pre-open if $A \subseteq \text{Int} (\text{Cl}(A))$
3. $\alpha$-open if $A \subseteq \text{Int} (\text{Cl} (\text{Int}(A)))$
4. $b$-open if $A \subseteq \text{Cl} (\text{Int}(A)) \cup \text{Int} (\text{Cl}(A))$
5. $\beta$-open if $A \subseteq \text{Cl} (\text{Int} (\text{Cl}(A)))$

Definition 3.2 A function $f : X \to Y$ is called [1], [9]:
1. Semi-continuous if $f^{-1}(V)$ is semi open in $X$ for each open set $V$ of $Y$.
2. Pre-continuous iff $f^{-1}(V)$ is pre open in $X$ for each open set $V$ of $Y$.
3. $\alpha$-continuous if $f^{-1}(V)$ is $\alpha$-open in $X$ for each open set $V$ of $Y$.
4. $b$-continuous if $f^{-1}(V)$ is $b$-open in $X$ for each open set $V$ of $Y$.
5. $\beta$-continuous if $f^{-1}(V)$ is $\beta$-open in $X$ for each open set $V$ of $Y$.

Definition 3.3 [10] A space $X$ is a $\beta$-$T_2$ space iff for each $x, y \in X$ such that $x \neq y$ there are $\beta$-open sets $U, V \subseteq X$ so that $x \in U, y \in V$ and $U \cap V = \emptyset$.

COVERING PROPERTIES

Definition 4.1 Let $\{G_\alpha : \alpha \in \Delta \}$ be a family of $\beta$-open sets of the space $X$. The family $\{G_\alpha : \alpha \in \Delta \}$ covers $X$ if $X \subseteq \bigcup_{\alpha \in \Delta} G_\alpha$.

Definition 4.2 A space $X$ is called a $\beta$-compact space if each $\beta$-open cover of $X$ has a finite subcover for $X$.

Theorem 4.3 Let $A$ be a $\beta$-compact subset of the $\beta$-$T_2$ space $X$ and $\not\in A$, then there exist two disjoint $\beta$-open sets $U$ and $V$ containing $x$ and $A$, respectively.

Proof: Let $y \in A$, since $X$ is $\beta$-$T_2$ space there exist two $\beta$-open sets $U_x, V_y \in X$ such that $x \in U_x, y \in V_y, U_x \cap V_y = \emptyset$, the family $\{A \cap V_y : y \in A\}$ is $\beta$-open cover of $A$ has a finite $\beta$-subcover $\{A \cap V_{y_1}, A \cap V_{y_2}, \ldots, A \cap V_{y_n}\}$, thus $U = U_{y_1} \cup U_{y_2} \cup \ldots \cup U_{y_n}$.

Theorem 4.4 If $X$ is $\beta$-$T_2$ space and $A$ is a $\beta$-open subset, if $A$ is $\beta$-compact then $A$ is $\beta$-closed.

Proof: Let $x \in X - A$ by the theorem 4.3 there exist two $\beta$-open sets $U$ and $V$ such that $x \in U, A \subseteq V, U \cap V = \emptyset$, thus $x \in U \subseteq X - V \subseteq X - A$, which implies $X - A$ is $\beta$-open so that $A$ is $\beta$-closed.
Theorem 4.5

Let $A$ and $B$ be a two $\beta$ - compact subsets of the $\beta$ - T2 space $X$, then there exist disjoint $\beta$ - open sets $U$ and $V$ containing $A$ and $C$, respectively.

**Proof:**

Let $b \in B$, since $A$ is a $\beta$ - compact subset and $\beta$ - open in $X$, there exist two $\beta$ - open sets $U_b$, $V_b$ such that $U_b \cap V_b = \phi$; $b \in V_b$, $A \subseteq U_b$, $F = \{b \cap V_b ; b \in B\}$ is a $\beta$ - open cover of $B$, since $B$ is $\beta$ - compact subset there exist finite subcover $\{B \cap V_b ; 1 \leq i \leq n\}$ from $F$.

Let $U = \bigcup_{i=1}^{n} U_b$, $V = \bigcup_{i=1}^{n} V_b$, thus $A \subseteq U$, $B \subseteq V$, $U \cap V = \phi$.

**Theorem 4.5**

Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a continuous surjection open function, if $X$ is a $\beta$ - compact then $Y$ is a $\beta$ -compact.

**Proof:**

Let $\beta = \{V_{\alpha} : \alpha \in \Delta\}$ be a $\beta$ - open cover of $Y$, then $L = \{f^{-1}(V) : \alpha \in \Delta\}$ is a $\beta$ - open cover of $X$ since $X$ is a $\beta$ - compact space, there exist a finite $\beta$ - subcover from $L$ to the space $X$, such that

$$X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$$

Thus

$$Y = f(X) \subseteq f\left( \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}) \right) = f\left( f^{-1}\left( \bigcup_{i=1}^{n} V_{\alpha_i} \right) \right) = \bigcup_{i=1}^{n} V_{\alpha_i}$$

Hence $Y \subseteq \bigcup_{i=1}^{n} (V_{\alpha_i})$, this shows $Y$ is a $\beta$-compact.

**Corollary 4.6**

$\beta$ - compactness is a topological property

**Proof:**

The proof from theorem Theorem 4.5.

**Definition 4.7:**

A family of sets $J$ has “finite intersection property” if every finite subfamily of $J$ has a nonempty intersection.

**Theorem 4.5**

A topological space is $\beta$-compact if and only if any collection of its $\beta$-closed sets having the finite intersection property has non-empty intersection.
REFERENCES


