Numerical Ways for Solving Fuzzy Differential Equations

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Abstract

Fuzzy differential equations are suggested as a way of modeling uncertain and incompletely specified systems.

Runge-Kutta algorithms for solving fuzzy ordinary differential equations are considered. A theorem of convergence for the solution is stated and proved.

Keywords: fuzzy ordinary differential equation, fuzzy numbers, Runge-Kutta.

INTRODUCTION

In this paper we consider the first–order initial value problem

\[ \dot{x}(t) = g(t, x), \quad t \in [0, T] \]

\[ x(t_0) = x_0, \quad t_0 \in [0, T], \text{ for some } T > 0, \]  \hfill (1)

Where, \( x_0 \) is a fuzzy number, \( x \) is a fuzzy function of \( t \), \( g(t, x) \) is a fuzzy function of the crisp variable \( t \) and fuzzy variable, and \( \dot{x} \) is the fuzzy derivative of \( x \).

Sufficient conditions for the existence of a unique solution to Equation (1) are that \( g \) is continuous and satisfy Lipschitz condition [2]

\[ |g(t,x) - g(t,z)| \leq L|x - z|, \quad L > 0 \]  \hfill (2)

Kaleva’s definition for fuzzy numbers [1] will be adopted here.

Definition 1 A fuzzy number \( u \) is a pair of functions \( (u_1, u_2) \) of functions \( u_1(\alpha), u_2(\alpha); \alpha \in [0, 1] \), which satisfy:

1. \( u_1(\alpha) \) is a bounded monotonic increasing left continuous function.

2. \( u_2(\alpha) \) is a bounded monotonic decreasing left continuous function.

3. \( u_1(\alpha) \leq u_2(\alpha), \alpha \in [0, 1] \)

The set of all fuzzy numbers is denoted by \( E \). The fuzzy number space \( E \) as in [7] can be embedded into the Banach space \( B = \mathcal{C}[0, 1] \times \mathcal{C}[0, 1] \) where the metric is usually defined as,

\[ \|u, v\| = \max\left\{ \sup_{0 \leq \alpha \leq 1} |u(\alpha)|, \sup_{0 \leq \alpha \leq 1} |v(\alpha)| \right\} \]  \hfill (3)

By [2] we may replace Equation (1) by the equivalent system

\[ \dot{x}_1(t) = \min\{g(t, w), w \in [x_1, x_2]\} = G_1(t, x_1, x_2), \quad x_1(t_0) = x_0 \]

\[ \dot{x}_2(t) = \max\{g(t, w), w \in [x_1, x_2]\} = G_2(t, x_1, x_2), \quad x_2(t_0) = x_0 \]  \hfill (4)

which has a unique solution \( (x_1, x_2) \in B \), which is a fuzzy function, where \( [x_1(t; \alpha), x_2(t; \alpha)] \in E \). The parametric form of Equation (4) following [3] is given by

\[ \dot{x}_1(t; \alpha) = G_1(t, x_1(t; \alpha), x_2(t; \alpha)), \quad x_1(t_0; \alpha) = x_{01}(\alpha) \]

\[ \dot{x}_2(t; \alpha) = G_2(t, x_1(t; \alpha), x_2(t; \alpha)), \quad x_2(t_0; \alpha) = x_{02}(\alpha) \]  \hfill (5)

For \( \alpha \in [0, 1] \). A solution to Equation (4), since equality between two fuzzy numbers in \( B \) yields a pointwise equality because we use the sup norm.

In very few cases fuzzy initial value problems are solved analytically; however in general, numerical algorithms are needed and some of these algorithms have been developed by using the standard Euler method [4, 5] and the Taylor method [6]. In the following we will develop an algorithm based on Runge-Kutta Methods.

RUNGE-KUTTA METHOD

The Runge-Kutta class of numerical solutions is one-step method which can be constructed of any order of accuracy and without the need of evaluating higher order derivatives. We will approximate the exact solution \((Y_1(t; \alpha), Y_2(t; \alpha))\) by \([y_1(t; \alpha), y_2(t; \alpha)]\). The \( t \)-axis is discretized over the finite interval \([t_0, T]\). The subdivision points \( t_n, n=0, \ldots, N \), are often chosen equally spaced; that is \( t_n = t_0 + nh \), where the step size \( h \) is \( h = \frac{T - t_0}{N} \). The exact and approximate solutions at \( t_n, 0 \leq n \leq N \), are denoted by \([Y_{1,n}(\alpha), Y_{2,n}(\alpha)]\) and \([y_{1,n}(\alpha), y_{2,n}(\alpha)]\), respectively.

To obtain a \( p \)-stage Runge-Kutta Method (\( p \) function evaluation per step) for the fuzzy initial problem (1) we let

\[ y_{1,n+1}(\alpha) = y_{1,n}(\alpha) + h\psi_1(\alpha), \]

\[ y_{2,n+1}(\alpha) = y_{2,n}(\alpha) + h\psi_2(\alpha), \]  \hfill (6)

Where,

\[ \psi_1(\alpha, h) = \psi(t_n, y_{1,n}, y_{2,n}; h) = \sum_{i=1}^{p} w_i k_i(\alpha) \]  \hfill (7)

\[ \psi_2(\alpha, h) = \psi(t_n, y_{1,n}, y_{2,n}; h) = \sum_{i=1}^{p} u_i k_i(\alpha) \]  \hfill (8)

\[ k_i(\alpha) = G_1(t_n + h\alpha_i, y_{1,n} + h \sum_{j=1}^{i-1} \eta_{ij} k_j, y_{2,n} + h \sum_{j=1}^{i-1} \eta_{ij} k_j) \]  \hfill (9)
Proof. We have, 
\[ |W_{n+1}| + |V_{n+1}| \leq |W_n| + |V_n| + 2A|W_n| + 2B = (1 + 2A)(|W_n| + |V_n|) + 2B \]
By applying lemma \ref{lem:2} for \( U_n \), \( 0 \leq n \leq N \). We conclude the result.

Now we will introduce the convergence theorem for the Runge-Kutta method.

The domain of \( \mathcal{O}(u,v) \), \( \psi(t,u,v) \), \( G_1(t,u,v) \) and \( G_2(t,u,v) \) is \( K = \{ (t,u,v) : t_0 \leq t \leq T, -\infty < u < \infty, -\infty < v < \infty \} \).

**Theorem 1** Let \( \mathcal{O}(t,u,v) \) and \( \psi(t,u,v) \) belong to \( C^1(K) \) and each satisfies a Lipschitz condition in \( u \) and \( v \). Assume \( \Omega(a,h) \), \( \psi(a,h) \) satisfy the consistency condition
\[ G_1(t, u_n, a, h) = 0, \quad G_2(t, u_0, a, h) = \psi(a, h), \]
Then the Runge-Kutta approximates of (11) converge to the exact solutions
\[ Y_1(t); \quad Y_2(t); \quad \]

**Proof.** It is sufficient to prove that
\[ \lim_{n \to \infty} Y_{1,n}(t; h) = Y_1(t,a), \quad \lim_{n \to \infty} Y_{2,n}(t; h) = Y_2(t,a). \]

Let 
\[ W_{n+1} = Y_{1,n+1}(t; a) - Y_{1,n+1}(t; a), \quad V_{n+1} = Y_{2,n+1}(t; a) - Y_{2,n+1}(t; a), \]
\[ W_{n+1} = Y_{1,n+1}(t; a) - Y_{1,n+1}(t; a) \]
\[ = Y_{1,n}(t; a) + h \left( G_1(t + h Y_1(t + h), Y_2(t + h); a; h) \right) \]
\[ - Y_{1,n}(t; a) - h \Omega(t_n, Y_{1,n}, Y_{2,n}; a; h). \]

Then
\[ |W_{n+1}| \leq |W_n| + h(|G_1(t_n + h Y_1(t_n + h), Y_2(t_n + h); a; h)) - \Omega(t_n, Y_{1,n}, Y_{2,n}; a; h)| \]
We can write the part in the parentheses as:
\[ G_1(t_n + h Y_1(t_n + h), Y_2(t_n + h); a; h) = G_1(t, Y_{1,2n}; a) + \Omega(t_n, Y_{1,n}, Y_{2,n}; a; h) \]

\[ \leq A + B^{n-1} \]
where \( A^* = 1 + 2A \) and \( B^* = 2B \)
Let
\[
\chi_1(h) = \max_{\theta \in [0, 1]} \left| G_1 \left( \frac{t + \theta h, Y_1(t_n + \theta h)}{Y_2(t + \theta h); \alpha; h} \right) - G_1(t, Y_1, Y_2; \alpha) \right|
\]

and
\[
\xi_1(h) = \max_{t \in [t_0, t]} \left| \emptyset(t, Y_1, Y_2; \alpha; 0) - \emptyset(t, Y_1, Y_2; \alpha; h) \right|.
\]

Since \( \emptyset(t, Y_1, Y_2; \alpha; h) \) is continuous and satisfies a Lipschitz condition, then
\[
\left| \emptyset(t_n, Y_1, Y_2; \alpha; h) - \emptyset(t_n, Y_1, Y_2; \alpha) \right| \leq 2L \max \{|W_n|, |V_n|\}.
\]

Thus
\[
|W_{n+1}| \leq |W_n| + h(\chi_1(h) + \xi_1(h)) + 2hL \max \{|W_n|, |V_n|\}. \tag{14}
\]

Similarly
\[
V_{n+1} = Y_{2,n+1}(t; \alpha) - y_{2,n+1}(t; \alpha)
\]
\[
= y_2(t; \alpha) + h \left( G_2 \left( \frac{t_n + \theta h, Y_1(t_n + \theta h)}{Y_{2,n}((t_n + \theta h); \alpha; h)} \right) \right)
\]
\[
- y_2(t; \alpha) - h\psi(t_n, y_{1,n}, y_{2,n}; \alpha; h). \tag{15}
\]

So,
\[
|V_{n+1}| \leq |V_n| + h \left( G_2 \left( \frac{t_n + \theta h, Y_1(t_n + \theta h)}{Y_{2,n}((t_n + \theta h); \alpha; h)} \right) - \psi(t_n, y_{1,n}, y_{2,n}; \alpha; h) \right). \tag{16}
\]

We can write the part in the parentheses as:
\[
\left| G_2 \left( \frac{t_n + \theta h, Y_1(t_n + \theta h)}{Y_{2,n}((t_n + \theta h); \alpha; h)} \right) - \psi(t_n, y_{1,n}, y_{2,n}; \alpha; h) \right|
\]
\[
= | G_2 \left( \frac{t_n + \theta h, Y_1(t_n + \theta h)}{Y_{2,n}((t_n + \theta h); \alpha; h)} \right) - G_2(t, Y_{1,n}, Y_{2,n}; \alpha) + \psi(t_n, y_{1,n}, y_{2,n}; \alpha; 0) - \psi(t_n, y_{1,n}, y_{2,n}; \alpha; h) + \psi(t_n, y_{1,n}, y_{2,n}; h) - \psi(t_n, y_{1,n}, y_{2,n}; \alpha; h) |. \tag{17}
\]

And
\[
\xi_2(h) = \max_{t \in [t_0, t]} |\psi(t, Y_1, Y_2; \alpha; 0) - \psi(t, Y_1, Y_2; \alpha; h)|.
\]

Since \( \psi(t, Y_{1,n}, Y_{2,n}; \alpha; h) \) is continuous and satisfies a Lipschitz condition, then
\[
|\psi(t_n, Y_{1,n}, Y_{2,n}; \alpha; h) - \psi(t_n, y_{1,n}, y_{2,n}; \alpha; h)| \leq 2L \max \{|W_n|, |V_n|\}.
\]

Thus
\[
|V_{n+1}| \leq |V_n| + h(\chi_2(h) + \xi_2(h)) + 2hL \max \{|W_n|, |V_n|\}. \tag{18}
\]

Applying Lemma 2 on (14) and (18) we have
\[
|W_n| \leq (1 + 4hL)^n|U_0| + 2h \left( \frac{(1 + 4hL)^n - 1}{4hL} \right) (x_1(h) + \xi_1(h)),
\]
\[
|V_n| \leq (1 + 4hL)^n|U_0| + 2h \left( \frac{(1 + 4hL)^n - 1}{4hL} \right) (x_2(h) + \xi_2(h)),
\]
where \( |U_0| = |W_0| + |V_0| \).

In particular
\[
|W_n| \leq (1 + 4hL)^n|U_0| + 2h \left( \frac{(1 + 4hL)^n - 1}{4hL} \right) (x_1(h) + \xi_3(h)),
\]
\[
|V_n| \leq (1 + 4hL)^n|U_0| + 2h \left( \frac{(1 + 4hL)^n - 1}{4hL} \right) (x_2(h) + \xi_3(h)),
\]

Since \( W_0 = V_0 = 0 \), we have
\[
|W_n| \leq 2 \left( \frac{e^{4h(T-t_0)} - 1}{4L} \right) (x_1(h) + \xi_1(h)),
\]
\[
|V_n| \leq 2 \left( \frac{e^{4h(T-t_0)} - 1}{4L} \right) (x_2(h) + \xi_2(h)),
\]
so taking the limit as \( h \to 0 \), we have \( W_N \to 0, V_N \to 0 \), and the proof is completed.
Illustration: Consider the fuzzy differential equation
\[ \dot{x}(t) = x(t), \]
\[ x(0) = (0.75 + 0.25\alpha, 1.125 - 0.125\alpha). \quad (19) \]

Using Equation (5) this is equivalent to the system
\[ \dot{x}_1(t; \alpha) = x_1(t; \alpha), \quad x_1(0; \alpha) = 0.75 + 0.25\alpha \]
\[ \dot{x}_2(t; \alpha) = x_2(t; \alpha), \quad x_2(0; \alpha) = 1.125 - 0.125\alpha \quad (20) \]

where the exact solution at \( t = 1 \), is given by
\[ Y = e^{[0.75 + 0.25\alpha, 1.125 - 0.125\alpha]} \]
The approximate solution at \( t = 1 \) for the Classical 4\(^{th}\) Order Runge-Kutta Method using \( h = 0.10 \) is
\[ y_1 = (0.75 + 0.25\alpha) \left[ 1 + 0.10 + \frac{0.10^2}{2} + \frac{0.10^3}{6} + \frac{0.10^4}{24} \right]^{10}, \]
\[ y_2 = (1.125 - 0.125\alpha) \left[ 1 + 0.10 + \frac{0.10^2}{2} + \frac{0.10^3}{6} + \frac{0.10^4}{24} \right]^{10}, \]
That is
\[ y = 2.71818746 \left[ 0.75 + 0.25\alpha, 1.125 - 0.125\alpha \right], \]
which is more accurate than the Euler solution given by
\[ y = 2.59374246 \left[ 0.75 + 0.25\alpha, 1.125 - 0.125\alpha \right]. \]

REFERENCES


