Numerical Solutions of Nonlinear Fractional Fornberg-Whitham Equation by an Accurate Technique

Mohamed S. Mohamed 1,2

1Mathematics Department, Faculty of Science, Taif University, Taif, Saudi Arabia, 2Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt.

Abstract

The pivotal aim of this article is to propose an efficient computational technique namely optimal q-homotopy analysis transform method (Oq-HATM) to solve the nonlinear Fornberg–Whitham equation with fractional time derivative. Comparison of optimal q-homotopy analysis transform method (Oq-HATM) with the optimal q-homotopy analysis method (Oq-HAM), homotopy analysis method (HAM) and the homotopy perturbation method (HPM) are made. The results reveal that the Oq-HATM has more accuracy than the others. Finally, numerical example is given to illustrate the accuracy and stability of this method. We show that the proposed method is very efficient and computationally attractive. The numerical results are presented graphically. The results show that the analytical scheme is very effective and user-friendly for solving nonlinear fractional differential equations describing physical problems.

Keywords: Fractional Fornberg–Whitham equation, Caputo fractional derivative, Homotopy analysis transform method, q-homotopy analysis method.

INTRODUCTION

Nonlinear fractional partial differential equations have many applications in various fields of science and engineering such as fluid mechanics, thermodynamics, mass and heat transfer, and micro-electro mechanics system. Fractional order ordinary and partial differential equations, as generalization of classical integer order differential equations, are increasingly used to model problems in fluid mechanics, viscoelasticity, biology, physics and engineering, and others applications [1]. Several methods have been suggested to solve nonlinear equations. These methods include the homotopy perturbation method (HPM) [2], Luapanov’s artificial small parameter method [3], Adomian decomposition method [4, 5], variation iterative method [6] and so on. Homotopy analysis method (HAM), first proposed by Liao in his Ph.D dissertation[7], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [8-12].

The homotopy analysis transform method (HATM) basically illustrates how the Laplace transform can be used to approximate the solutions of the linear and nonlinear fractional differential equations by manipulating the homotopy analysis method. The proposed method is coupling of the q-homotopy analysis method and Laplace transform method. The main advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series for fractional partial differential equations [13]. In 2005 Liao [14] has pointed out that the HPM is only a special case of the HAM (The case of ℎ=−1). El-Tawil and Huseen [15] proposed a method namely q-homotopy analysis method (q-HAM) which is more general method of homotopy analysis method (HAM) . The q-HAM contains an auxiliary parameter 𝑛 as well as ℎ such that the cases of (q-HAM ; 𝑛=1 ) the standard homotopy analysis method (HAM) can be reached. The q-HAM has been successfully applied to solve many types of nonlinear problems [16-21].

In this paper, we have to solve the nonlinear time-fractional Fornberg-Whitham equation by optimal q-homotopy analysis transform method (Oq-HATM). This equation can be written in operator form as,

\[ u_t^\alpha - u_{xx} + u_x = uu_{xx} - uu_x + 3u_x u_{xx}, \quad 0 < \alpha \leq 1, \quad t > 0, \]

subject to the initial condition

\[ u(x,0) = e^{x^2}. \]

where \( u(x,t) \) is the fluid velocity, \( x \) and \( t \) represent the spatial coordinate and the time respectively. To obtain the approximate or numerical solution of fractional Fornberg–Whitham equation, many effective methods have been developed, such as homotopy perturbation method (HPM) [22], variational iteration method (VIM) [23], combination Laplace transform and HPM [24], homotopy analysis method (HAM) [25], differential transform method (DTM) [26] and fractional homotopy analysis transform method (FHATM) [27].

In this work, we analyze the nonlinear fractional Fornberg-Whitham equation by using q-homotopy analysis transform method (q-HATM). The q-HATM is an innovative
amalgamation of the Laplace transform scheme and the \( q \)-HATM. The supremacy of this technique is its potential of combining two robust computational techniques for solving fractional problems.

1. Basic Tools

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this paper.

Definition 2.1. A real function \( h(t), t > 0 \), is said to be in the space \( C_\mu, \mu \in \mathbb{R} \) if there exists a real number \( p > \mu \), such that \( h(t) = t^p h_1(t) \), where \( h_1(t) \in C[0, \infty) \), and it is said to be in the space \( C^n_\mu \) if and only if \( h^{(n)}(t) \in C_\mu, n \in \mathbb{N} \).

Definition 2.2. The Riemann-Liouville fractional integral operator \( (J^\alpha) \) of order \( \alpha \geq 0 \), of a function \( h \in C_\mu, \mu \geq -1 \), is defined as

\[
J^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau \quad (\alpha > 0),
\]

\[
J^0 h(t) = h(t),
\]

\[
\Gamma(\alpha) \quad \text{is the well-known Gamma function.}
\]

Definition 2.3. The fractional derivative \( (D^\alpha) \) of \( h(t) \) in the Caputo’s sense is defined as \[12\]

\[
D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau) d\tau ,
\]

for \( n-1 < \alpha \leq n, \ n \in \mathbb{N}, \ t > 0, \ h \in C^n_\alpha \).

Definition 2.4. Some of the useful Laplace transforms which are applied in this paper are as follows:

\[
L[D^\alpha f(x)] = \frac{s^\alpha f(x) - s^{\alpha-1} f(0) - s^{\alpha-2} f'(0) - \cdots - f^{(n-1)}(0)}{s^\alpha - \lambda \phi}.
\]

\[
(6)
\]

BASIC IDEA OF THE OPTIMAL \( q \)-HOMOTOPY ANALYSIS METHOD (Oq-HATM)

To describe the basic ideas of the optimal Oq-HATM for nonlinear partial differential equations, Let us consider the following nonlinear partial differential equation:

\[
N[D^\alpha u(x,t)] - f(x,t) = 0,
\]

\[
(7)
\]

where \( N \) is linear and nonlinear operator for this problem, \( x \) and \( t \) denote the independent variables, \( D^\alpha u(x,t) \) denotes the Caputo fractional derivative, \( u(x,t) \) is an unknown function and \( f \) is a known function. We first construct the zero-order deformation equation as follows:-

\[
(1 - nq)L[\phi(x,t;q) - u_0(x,t)] = qH(x,t)N[D^\alpha \phi(x,t;q) - f(x,t)],
\]

\[
(8)
\]

where \( n>1, q \in [0, \frac{1}{n}] \) is the embedding parameter, \( h \neq 0 \) is an auxiliary parameter, \( H(x,t) \neq 0 \) is an auxiliary function, \( L \) is an auxiliary linear operator and \( u_0(x,t) \) is an initial guess. Clearly, when \( q = 0 \) and \( q = \frac{1}{n} \), equation \( (4) \) becomes:

\[
\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;\frac{1}{n}) = u(x,t).
\]

\[
(9)
\]

respectively, so, as \( q \) increases from 0 to \( \frac{1}{n} \) the solution \( \phi(x,t,q) \) varies from the initial guess \( u_0(x,t) \) to the solution \( u(x,t) \). If \( u_0(x,t) \), \( L, h, H(x,t) \) are chosen appropriately, solution of equation \( (5) \) exists for \( q \in [0, \frac{1}{n}] \).

Taylor series expression of \( \phi(x,t,q) \) with respect to \( q \) in the form

\[
\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m,
\]

\[
(10)
\]

where

\[
\phi_m(x,t) = \left. \frac{\partial^m \phi(x,t;q)}{\partial q^m} \right|_{q=0}.
\]

\[
(11)
\]

We assume that the auxiliary linear operator, the initial guess, the auxiliary parameter \( h \) and the auxiliary function \( H(x,t) \) is selected such that the series \( (11) \) is convergent.
when $q \rightarrow \frac{1}{n}$, then the approximate solution (6) takes the form:

$$u(x,t) = \phi(x,t) \frac{1}{n} = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \left( \frac{1}{n} \right)^m.$$  \quad (12)

Let us define the vector

$$u_n^m(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\}.$$  

Differentiating (8) $m$ times with respect to $q$, then setting $q = 0$ and dividing then by $m!$, we have the $m^{th}$-order deformation equation (Lioa [7-8]) as

$$L[u_n^m(x,t) - \chi_m^m u_{m-1}^m(x,t)] = hH(x,t) R_m(u_{m-1}^m(x,t)),$$  \quad (13)

with initial conditions

$$u_m^{(k)}(x,t) = 0, \quad k = 0, 1, 2, 3, \ldots, m - 1$$

where

$$R_m(u_{m-1}^m(x,t)) = \frac{1}{(m-1)!} \frac{\partial}{\partial q^{m-1}} \left[ D^\alpha \phi(x,t;q) - f(x,t) \right] \bigg|_{q=0},$$  \quad (14)

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ n & m > 1. \end{cases}$$  \quad (15)

It should be emphasized that $u_m(x,t)$ for $m \geq 1$ is governed by the linear equation (13) with linear boundary conditions that come from the original problem. Due to the existence of the factor $\left( \frac{1}{n} \right)^m$, more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the cases of $n=1$ in equation (8), standard HAM can be reached.

The $h$-curves cannot tell us the best convergence-control parameter, which corresponds to the fastest convergent series. In 2007, Yabushita et al. [28] applied the HAM to solve two coupled nonlinear ODEs. They suggested the so-called optimization method to find out the two optimal convergence-control parameters by means of the minimum of the squared residual error of governing equations. In 2008, Akyildiz and Vajravelu [29] gained optimal convergence-control parameter by the minimum of squared residual of governing equation, and found that the corresponding homotopy-series solution converges very quickly.

S. J. Liao [30] and M. S. Mohamed et al. [19, 31] have discussed the optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their approach is based on the square residual error. Let $\Delta(h)$ denote the square residual error of the governing equation (8) and express as:

$$\Delta(h) = \int_{\Omega} \left( N[u_n(t)] \right)^2 \, d\Omega,$$  \quad (16)

where

$$u_m(t) = u_0(t) + \sum_{k=1}^{m} u_k(t).$$  \quad (17)

The optimal value of the auxiliary parameter $h$ is given by solving the following nonlinear algebraic equation

$$\frac{d\Delta(h)}{dh} = 0.$$  \quad (18)

**Numerical Analysis of Fractional Differential Equations**

In this section, we demonstrate the efficiency and applicability of newly $q$-HATM to derive the approximate solution of linear and nonlinear partial differential equations of fractional order, we use auxiliary function $H(x, t) = 1$. We first consider the following time-fractional Fornberg-Whitham equation:

$$u_t^\alpha - u_{xx} + u_x = uu_{xx} - uu_x + 3u_x u_{xx}, \quad 0 < \alpha \leq 1, \quad t > 0,$$  \quad (19)

with the initial condition

$$u(x,0) = Ae^{\frac{x}{2}},$$

with the exact solution at $\alpha = 1$

$$u(x,t) = Ae^{-(\frac{x}{2}+\frac{2}{3}t)},$$  \quad (20)

where $A$ is an arbitrary constant. We first apply the Laplace transform $\mathcal{L}$ on (19) to get the following equation:
We define a nonlinear operator as
\[ N[\phi(x,t;q)] = \mathcal{A}[\phi(x,t;q)] - \frac{1}{s} u_0(x,t) + \frac{1}{s^\alpha} \mathcal{A}[u_0(x,t)] + \mathcal{A}[u_0(x,t)] + \phi(x,t;q)\phi'_x(x,t;q) + \phi(x,t;q)\phi''_x(x,t;q). \]  

(21)

We construct the zero order deformation equation
\[ (1-nq)\mathcal{A}[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[D^q_x \phi(x,t;q)]. \]

For \( q = 0 \) and \( q = 1 \), we can write
\[ \phi(x,t;0) = u_0(x,t), \]
\[ \phi(x,t;1) = u(x,t). \]

The \( m^{th} \)-order deformation equation is
\[ \mathcal{A}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hR_m(u_{m-1}'(x,t)), \]

(22)

with the initial conditions for \( m \geq 1 \)
\[ u_m(x,0) = 0, \]

(23)

where \( \chi_m \) as defined by (15) and
\[ R_m(u_{m-1}') = \mathcal{A}[u_{m-1}'] - \frac{1}{n} \mathcal{A}[u_0(x,t)] + \frac{1}{s^\alpha} \mathcal{A}[u_0(x,t)] + \sum_{i=0}^{m-1} u_i(u_{m-1-i})_{xx} + \frac{1}{s^\alpha} \sum_{i=0}^{m-1} u_i(u_{m-1-i})_{x} - 3 \sum_{i=0}^{m-1} u_i(u_{m-1-i})_{x} \]

(24)

Taking the inverse Laplace transform \( \mathcal{A}^{-1} \) on (22). Now the solution of the \( m^{th} \)-order deformation equations for \( m \geq 1 \) becomes
\[ u_m(x,t) = \chi_m u_{m-1}(x,t) + h \mathcal{A}^{-1}(R_m(u_{m-1}')) + c_1 \]

(25)

where the constant of integration \( c_1 \) is determined by the initial conditions (23). Then, the components of the solution using Oq-HATM are
\[ u(x,0) = e^{\frac{x}{2}}, \]
\[ u_i(x,t) = \frac{e^{\frac{x}{2}}}{2\alpha^\Gamma(\alpha)} h t^\alpha, \]
\[ u_2(x,t) = \frac{e^{\frac{x}{2}}}{2\alpha^\Gamma(\alpha)} h n t^\alpha + h^2 \frac{e^{\frac{x}{2}}}{8\alpha^2 \Gamma^2(\alpha)} (t^{\alpha-1}(2t - \alpha) + \frac{t^\alpha}{\alpha^\Gamma(\alpha)}), \]
\[ ... \]
\[ ... \]

According to the optimal q-homotopy analysis transform method, we can conclude that
\[ u(x,t;n,h) \approx U_m(x,t;n,h) = \sum_{i=0}^{M} u_i(x,t;n,h)(\frac{1}{n})^i. \]

(27)
Equation (27) is an approximate solution to the problem (19) in terms of convergence parameter $h$ and $n$. Then we have at $\alpha = 1$,

$$u_{app} = u_0(x,t) + \left(\frac{1}{n}\right)u_1(x,t) + \left(\frac{1}{n}\right)^2u_2(x,t) + \left(\frac{1}{n}\right)^3u_3(x,t) + \left(\frac{1}{n}\right)^4u_4(x,t) + \left(\frac{1}{n}\right)^5u_5(x,t) + ...$$

(28)

The result is complete agreement with qH-HAM [31]. As special case if $n=1$ and $h=-1$, then we obtain the same result in [22-27].

**Figure 1:** Three dimensional surfaces show the approximate solutions $u_{app}(x,t)$ at (a) $\alpha = 0.8$ and (b) $\alpha = 0.9$ at different values of $x$, $t$ and $h_{optimal} = -0.95$.

**Figure 2:** $h$ - curve for the (q-HATM; $n = 1$) approximation solution $U_5(x,t; 1)$ of problem (19) at different values of $x$, $t$ and $h_{optimal} = -0.95$.

**Figure 3:** $h$ - curve for the (q-HATM; $n = 50$) approximation solution $U_5(x,t; 50)$ of problem (19) at different values of $x$, $t$ and $h_{optimal} = -15.03$. 
Figure 4: \( h \) - curve for the (q-HATM; \( n = 1 \)) approximation solution \( U_{10} (x, t; 1) \) of problem (19) at different values of \( x \), \( t \) and \( h_{\text{optimal}} = -0.97 \).

Figure 5: \( h \) - curve for the (q-HATM; \( n = 2 \)) approximation solution \( U_{10} (x, t; 2) \) of problem (19) at different values of \( x \), \( t \) and \( h_{\text{optimal}} = -1.75 \).

Figure 6: \( h \) - curve for the (q-HATM; \( n = 5 \)) approximation solution \( U_{10} (x, t; 5) \) of problem (19) at different values of \( x \), \( t \) and \( h_{\text{optimal}} = -2.15 \).

Figure 7: \( h \) - curve for the (q-HATM; \( n = 10 \)) approximation solution \( U_{10} (x, t; 10) \) of problem (19) at different values of \( x \), \( t \) and \( h_{\text{optimal}} = -5.32 \).

Figure 8: \( h \) - curve for the (q-HATM; \( n = 20 \)) approximation solution \( U_{10} (x, t; 20) \) of problem (19) at different values of \( x \), \( t \) and \( h_{\text{optimal}} = -8.73 \).

Figure 9: \( h \) - curve for the (q-HATM; \( n = 50 \)) approximation solution \( U_{10} (x, t; 50) \) of problem (19) at different values of \( x \), \( t \) and \( h_{\text{optimal}} = -12.05 \).
based on these present results, we can say that oq-HATM is more effective than HAM and HPM.

Also in view of (16), we can find the optimal values of $h$ for the solution (28) for $\alpha$. But in this case, the double integration of sum square residual (16) is involved and we cannot evaluate it explicitly. So we use it's approximate sum form,

$$
\Delta_m(h) = \frac{1}{(M+1)(N+1)} \sum_{i=0}^{M} \sum_{j=0}^{N} (N(u_m(i\Delta x,j\Delta t)))^2.
$$

where $\Delta x = \frac{1.00}{M}$ and $\Delta t = \frac{0.10}{N}$. Equation (SRE) gives the square residual error at $m^{th}$ - order approximation. To obtain the optimal values $h$ of the convergent control parameter, we set $M = 20$ and $N = 20$.

In this paper, we apply oq-HATM and obtain the same result as HPM [22] and [24] for $h = -1$. When $h = -1$, $\alpha = 1$ and $n=1$, the result is complete agreement with HAM and ADM [25]. Therefore, the oq-HATM is rather general and contains the HPM, HAM and ADM [22-27].

**CONCLUSIONS**

In this work, the nonlinear fractional Fornberg-Whitham equation is successfully analyzed by using oq-HATM. The oq-HATM is a very powerful computational algorithm to solve nonlinear fractional order problems, whose fractional order mathematical models decode the real world problems in a better and more systematic manner. The oq-HATM provides a suitable way to adjust and control the convergence of the series solutions by properly choosing values of the auxiliary parameter $h$ and asymptotic parameter $n$. Thus, we can conclude that oq-HATM is an easier, more user-friendly and more powerful computational technique for investigating fractional problems. In this article, a numerical algorithm based on oq-HATM has been used to investigate the linear and nonlinear fractional problems.

From the results, the clear conclusion is that oq-HATM is generalized algorithm and provides the many more acceptable series solution which directly converges to HPM, VIM, ADM solution at $h = -1$ and $n=1$ and HAM at $n = 1$. The explanation and convergence speed of oq-HATM series solution in large admissible domain show the very high accuracy and performance of this approach to other existing methods.

The solution procedure and explanation show the flexible efficiency of $q$-HATM, compared to other existing classical techniques for solving three different kind of time-fractional partial differential equations.
REFERENCES


