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Solving Singularly Perturbed Differential-Difference Equations using Numerical Integration Method

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Abstract

In this paper, we introduce a numerical integration method for solving singularly perturbed differential-difference equations with delay and advanced parameters. Initially, the given second order differential-difference equation is replaced by an asymptotically equivalent delay differential equation. Then, numerical integration method is employed to obtain a tridiagonal system which is solved efficiently by Thomas algorithm. We have presented maximum absolute errors for standard examples chosen from literature. Numerical results are presented to illustrate the efficiency of the method.

Keywords: differential- difference equations, delay differential equation, boundary layer, integrating factor.

INTRODUCTION

The boundary value problems for singularly perturbed differential difference equations with delay as well as advance are universal in the mathematical modelling of various practical phenomena in biology and physics, such as in variational problems in control theory, in describing the human pupil-light reflex, in a variety of models for physiological processes or diseases and first exit time problems in the modelling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites. These biological applications motivate the study of boundary value problems for singularly perturbed differential difference equations with delay as well as advance. For further study of neurophysiological and mathematical aspects of the above class of models, readers can refer to Stein [1], Tuckwell [2], Lange and Miura [3], Derstine and et al [4], Longtin and Milton [5], Wazewska-Czyzewska and Lasota [6], Mackey and Glass [7], etc. The solution changes rapidly and form boundary or transition layers in these narrow regions. Attributable to this, numerous numerical techniques have been developed to solve singularly perturbed ODEs with delay and advance. Lange and Miura gave a series of papers [8,9] investigating different classes of BVPs of singularly perturbed differential difference equations by extending the method of matched asymptotic expansions developed for ODEs. First-order numerical algorithms based on finite difference schemes are found in Sharma [10],Applications of Differential Equations with Deviating Arguments were discussed in El'sgol'ts and Norkin [11], Exponential fitting of the delayed recruitment/renewal equation was extended by Brian J. McCartin[12],Due to the singular behaviour of the solution of the problem in the inner regions, the classical numerical schemes are found to be inadequate to approximate the solution of the singularly perturbed problems.

In this paper, we present the numerical integration method to solve singularly perturbed differential-difference equations with delay and advanced parameters. Initially, the given second order differential-difference equation is replaced by an asymptotically equivalent delay differential equation. Then, numerical integration method is employed to obtain a tridiagonal system which is solved efficiently by Thomas algorithm.

DESCRIPTION OF THE METHOD

Consider singularly perturbed differential-difference equation of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-\delta) + c(x)y(x) + d(x)y(x+\eta) = f(x) \tag{1}$$

$\forall x \in (0,1)$ and subject to the interval and boundary conditions

$$y(x) = \varphi(x), \ on -\delta \le x \le 0 \tag{2}$$

$$y(x) = \gamma(x), on \ \ 1 \le x \le 1+\eta \tag{3}$$

When $a(x), b(x), c(x), d(x), \varphi(x)\ and\ \gamma(x)$ are bounded and continuously differentiable functions on $(0,1)$,$0 < \varepsilon << 1$ is the singular perturbation parameter; and $0 < \delta = o(\varepsilon)$and $0 < \eta = o(\varepsilon)$ are the delay and the advance parameters respectively. In general, the solution of Eqs.(1)-(3) shows boundary layer behaviour at one end of the interval [0,1] depending upon the sign.

NUMERICAL SCHEME

By using Taylor series expansion in the neighborhood of the point x and the small positive deviating argument$\sqrt{\varepsilon}$, we have

$$y(x-\delta) \approx y(x) - \delta y'(x) \tag{4}$$

$$y(x+\eta) \approx y(x) + \eta y'(x) \tag{5}$$

$$y\left(x-\sqrt{\varepsilon}\right) = y(x) - \sqrt{\varepsilon}y'(x) + \frac{\varepsilon}{2}y''(x)$$

$$y''(x) = \frac{2y\left(x-\sqrt{\varepsilon}\right) - 2y(x) + 2\sqrt{\varepsilon}y'(x)}{\varepsilon} \tag{6}$$

and consequently, equation (1) is replaced by the following first order delay differential equation:

$$y'(x) = p(x)y(x-\sqrt{\varepsilon}) + q(x)y(x) + r(x),$$

$$for\ \sqrt{\varepsilon} \le x \le 1 \qquad (7)$$

with $y'(\sqrt{\varepsilon}) = c_1\varphi_0 + c_2 y(\sqrt{\varepsilon}) + c_3$

where $c_1=p(\sqrt{\varepsilon})$, $c_2=q(\sqrt{\varepsilon})$ and $c_3=r(\sqrt{\varepsilon})$

$$y(1) = \gamma(1) = \gamma_1$$

$$p(x) = \frac{-2}{2\sqrt{\varepsilon} + a(x) + d(x)\eta - b(x)\delta}$$

$$q(x) = \frac{2 - b(x) - c(x) - d(x)}{2\sqrt{\varepsilon} + a(x) + d(x)\eta - b(x)\delta}$$

$$r(x) = \frac{f(x)}{2\sqrt{\varepsilon} + a(x) + d(x)\eta - b(x)\delta}$$

The transition from Eq.(1) to Eq.(7) is allowed, because of the condition that $\sqrt{\varepsilon}$, $0<\delta \ll 1$ and $0<\eta \ll 1$ are sufficiently small. This substitution is significant from the computational point of view. Further details on the validity of this transition is found in El'sgol'ts and Norkin[11]. Thus, the solution of Eq.(7) will give a good approximation to the solution of Eq.(1). Further, we assume that $[a(x)+d(x)\eta-b(x)\delta] \ge M > 0$, $[b(x)+c(x)+d(x)] \le 0$Throughout the interval [0, 1], where M is some positive constant. Under these assumption, Eq.(7) has a unique solution $y(x)$ which shows a boundary layer of width$O(\varepsilon)$ on the left side ($x = 0$) of the underlying interval.

Now we divide the interval [0, 1] into N equal subintervals of mesh size $h = 1/N$ so that $x_i = ih,\ i = 0, 1, 2, \ldots, N$.

Here, for consolidation of our ideas of the method, we assume that $a(x), b(x), c(x)$ and $d(x)$ are constants. Hence, here $p(x)$ and $q(x)$ are only the constants.

Rewriting the equation (7) as

$$y'(x) - qy(x) = py(x-\sqrt{\varepsilon}) + r(x)$$

We then apply an integrating factor e^{-qx}, producing (as in Brian J. McCartin [12])

$$\frac{d}{dx}[e^{-qx}y(x)] = e^{-qx}[py(x-\sqrt{\varepsilon}) + r(x)]$$

Next, integrating from x_i to x_{i+1}, we get

$$e^{-qx_{i+1}}y_{i+1} - e^{-qx_i}y_i = \int_{x_i}^{x_{i+1}} e^{-qx}py(x-\sqrt{\varepsilon})dx + \int_{x_i}^{x_{i+1}} e^{-qx}r(x)dx$$

Using the Trapezoidal rule to evaluate the integrals and simplifying, we get

$$y_{i+1} = e^{qh}y_i + \frac{hp}{2}\left(e^{qh}y(x_i-\sqrt{\varepsilon}) + y(x_{i+1}-\sqrt{\varepsilon})\right) + \frac{h}{2}(e^{qh}r_i + r_{i+1}) \qquad (8)$$

Approximating $y'(x)$ by linear interpolation, we get

$$y(x_i-\sqrt{\varepsilon}) \approx \left(1-\frac{\sqrt{\varepsilon}}{h}\right)y_i + \frac{\sqrt{\varepsilon}}{h}y_{i-1} \qquad (9)$$

$$y(x_{i+1}-\sqrt{\varepsilon}) \approx \left(1-\frac{\sqrt{\varepsilon}}{h}\right)y_{i+1} + \frac{\sqrt{\varepsilon}}{h}y_i \qquad (10)$$

Substituting Eq.(9), Eq.(10) in Eq. (8) and rearranging the terms we get the following three term relation

$$\left(\frac{-e^{qh}\sqrt{\varepsilon}p}{2}\right)y_{i-1} - \left(e^{qh} + \frac{e^{qh}ph}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right) + \frac{\sqrt{\varepsilon}p}{2}\right)y_i + \left(1 - \frac{ph}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)\right)y_{i+1} = \frac{h}{2}(e^{qh}r_i + r_{i+1})$$

for $i = 1,2,\ldots,N\text{-}1$

The above relation can be written as

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i\ , for\ i = 1,2,\ldots\ldots, N-1 \qquad (11)$$

where

$$E_i = \frac{-e^{q\square}\sqrt{\varepsilon}p}{2}\ ,\quad F_i = e^{qh} + \frac{e^{qh}ph}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right) + \frac{\sqrt{\varepsilon}p}{2}\ ,$$

$$G_i = 1 - \frac{ph}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right)\ ,\quad H_i = \frac{h}{2}(e^{qh}r_i + r_{i+1})$$

Equation (11) represents a tridiagonal system which can be solved by Thomas algorithm.

CONVERGENCE ANALYSIS

Writing the tridiagonal system Eq.(11) in matrix-vector form, we get

$$AY = C \qquad (12)$$

in which $A = (m_{i\,j}),\ 1 \le i, j \le n\text{-}1$ is a tridiagonal matrix of order N-1 , with

$$m_{i\,i+1} = 1 - \frac{ph}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right),$$

$$m_{i\,i} = e^{qh} + \frac{e^{qh}ph}{2}\left(1-\frac{\sqrt{\varepsilon}}{h}\right) + \frac{\sqrt{\varepsilon}p}{2},$$

$$m_{i\,i-1} = \frac{-e^{qh}\sqrt{\varepsilon}p}{2}$$

and $C = (d_i)$ is a column vector with

$d_i = \frac{h}{2}(e^{qh}r_i + r_{i+1})$, where $i = 1(1)N-1$

with local truncation error $T_i(h_i) = h\left[\frac{-\varepsilon y_i''}{2\sqrt{\varepsilon}+a}\right] + O(h^2)$ (13)

we also have $A\bar{Y} - T(h) = C$ (14)

where $\bar{Y} = (\overline{y_0}, \overline{y_1}, \overline{y_2}, - - - - \overline{y_N})^T$ denote the actual solution and

$T(h) = \left(T_1(h_0), T_1(h_1), - - - - - T_N(h_N)\right)^T$ is the local truncation error.

From Eq.(12) and Eq.(14), we get

$$A(\bar{Y} - Y) = T(h) \quad (15)$$

Thus the error equation is $AE = T(h)$ (16)

where $E = \bar{Y} - Y = (e_0, e_1, e_2, - - - - - - - e_N)^T$

Clearly, we have

$$S_1 = \sum_{j=1}^{N-1} m_{1j} = \left(1 + \frac{p\sqrt{\varepsilon}}{2}\right) - h\left(\frac{pq\sqrt{\varepsilon}}{2}\right) + \frac{h^2}{2}\left(pq - pq^2\sqrt{\varepsilon}\right) + O(h^3)$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = 2 + h\left(q\left(1 - p\sqrt{\varepsilon}\right)\right) + O(h^2) = 2 + O(h) = B_0$$

$$where\ B_0 = 2,\ i = 2,3, - - -, N-2$$

$$S_{N-1} = \sum_{j=1}^{N-1} m_{N-1\,j} = (1 - p\sqrt{\varepsilon}) + h\left(q\left(1 - p\sqrt{\varepsilon}\right) + \frac{p}{2}\right) + O(h^2)$$

We can choose h sufficiently small so that the matrix A is irreducible and monotone. It follows that A^{-1} exists and its elements are non negative.

Hence from Eq.(16), we get $E = A^{-1}T(h)$ (17)

also, from the theory of matrices we have

$\sum_{i=1}^{N-1} \overline{m_{k,i}} S_i = 1,\ \ k = 1(1)N-1$ (18)

where $\overline{m_{k,i}}$ is *(k,i)* element of the matrix A^{-1},

therefore

$\sum_{i=1}^{N-1} \overline{m_{k,i}} S_i \leq \frac{1}{\min_{1 \leq i \leq N-1} S_i} = \frac{1}{B_o} \leq \frac{1}{2}$ (19)

From Eq.(13), Eq.(17) and Eq.(19), we get

$$e_j = \sum_{i=1}^{N-1} \overline{m_{k,i}} T_i(h),\ \ j = 1(1)N-1$$

which implies

$e_j \leq \frac{kh}{2}$, (20)

where k is a constant independent of h, that is $k = \frac{-\varepsilon y''}{2\sqrt{\varepsilon}+a}$

therefore,

$$\|E_i\| = O(h)$$

i.e., our method gives a first order convergent for uniform mesh.

NUMERICAL EXAMPLES

The applicability of the method is validated on model examples of the type given by Eqs.(1)-(3) with left-end boundary layer. The exact solution of singularly perturbed differential-difference equation:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-\delta) + c(x)y(x) + d(x)y(x+\eta) = f(x)$$

$\forall x \in (0,1)$ and subject to the interval and boundary conditions

$$y(x) = \varphi(x), \quad on\ -\delta \leq x \leq 0$$

$$y(x) = \gamma(x), \quad on\ \ 1 \leq x \leq 1+\eta$$

with constant coefficients(i.e,

$a(x) = a, b(x) = b, c(x) = c, d(x) = d, f(x) = f, \varphi(x) = \varphi$ *and* $\gamma(x) = \gamma$) is given by $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + f/c_3$

where

$$m_1 = \frac{-(a+d\eta-b\delta)+\sqrt{(a+d\eta-b\delta)^2-4\varepsilon c_3}}{2\varepsilon},$$

$$m_2 = \frac{-(a+d\eta-b\delta)-\sqrt{(a+d\eta-b\delta)^2-4\varepsilon c_3}}{2\varepsilon}$$

$$c_1 = \frac{-f+\gamma c_3+e^{m_2}(f-\varphi c_3)}{(e^{m_1}-e^{m_2})c_3}, \quad c_2 = \frac{f-\gamma c_3+e^{m_1}(-f+\varphi c_3)}{(e^{m_1}-e^{m_2})c_3}$$

$$c_3 = b + c + d$$

Example-1. Consider the singularly perturbed differential-difference equation with left end boundary layer:

$$\varepsilon y''(x) + y'(x) + 2y(x-\delta) - 3y(x) = 0, \quad \varphi(x) = 1, \quad \gamma(x) = 1$$

The numerical results are presented in Table-1 and Figure-1.

Example-2. Consider the singularly perturbed differential-difference equation with left end boundary layer:

$$\varepsilon y''(x) + y'(x) - 2y(x-\delta) - 5y(x) + y(x+\eta) = 0, \varphi(x) = 1, \gamma(x) = 1$$

The numerical results are presented in Table-2 and Figure-2.

Example-3. Consider the singularly perturbed differential-difference equation with left end boundary layer:

$$\varepsilon y''(x) + y'(x) - 2y(x-\delta) + y(x) - y(x+\eta) = -1, \varphi(x) = 1, \gamma(x) = 1$$

The numerical results are presented in Table-3 and Figure-3.

Example-4. Consider the singularly perturbed differential-difference equation with left end boundary layer:

$$\varepsilon y''(x) + 0.5y'(x) - 3y(x-\delta) - 2y(x) + 2y(x+\eta) = 1, \varphi(x) = 1, \gamma(x) = 0$$

The numerical results are presented in Table-4 and Figure-4.

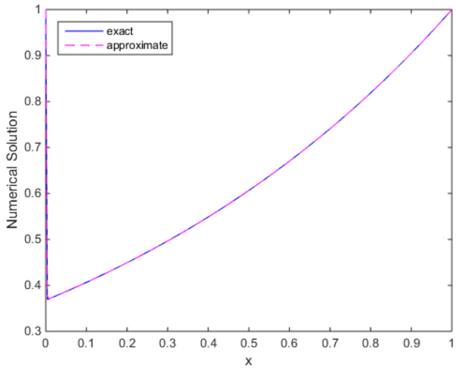

Figure-1. Numerical solution of Example-1 with $N = 256,\ \varepsilon = 10^{-4}$

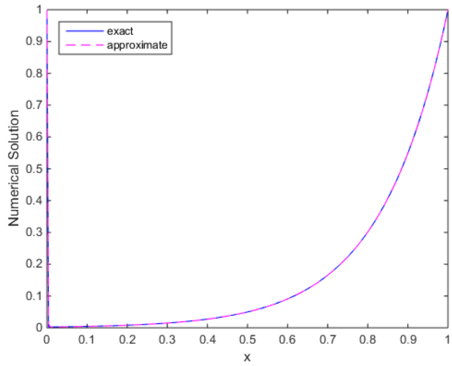

Figure-2. Numerical solution of Example-2 with $N = 256,\ \varepsilon = 10^{-4}$

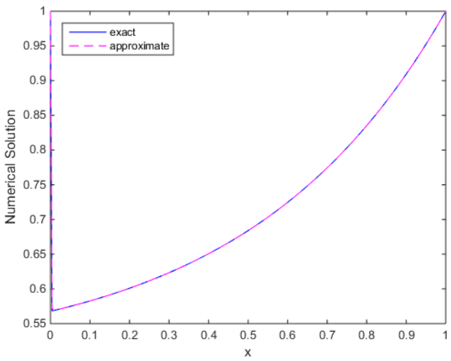

Figure-3. Numerical solution of Example-3 with $N = 256,\ \varepsilon = 10^{-4}$

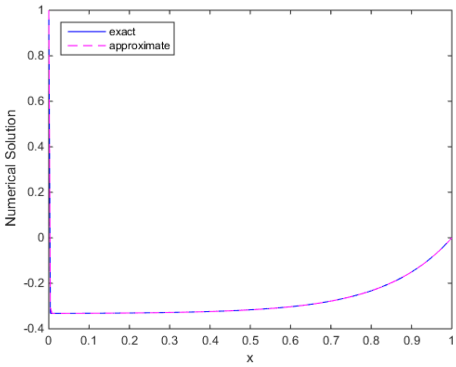

Figure-4. Numerical solution of Example-4 with $N = 256,\ \varepsilon = 10^{-4}$

Table-1. The maximum absolute errors in solution of Example-1 for $\delta = 0.1\varepsilon$ and $\eta = 0.1\varepsilon$

$\varepsilon \backslash N$	64	128	256	512	1024	2048
10^{-4}	6.4284e-03	6.3111e-03	6.2630e-03	6.2417e-03	6.1954e-03	1.4401e-03
10^{-5}	2.0965e-03	2.0280e-03	2.0048e-03	1.9960e-03	1.9922e-03	1.9905e-03
10^{-6}	7.0621e-04	6.5378e-04	6.3873e-04	6.3400e-04	6.3233e-04	6.3167e-04
10^{-7}	2.6450e-04	2.1720e-04	2.0476e-04	2.0134e-04	2.0034e-04	2.0001e-04
10^{-8}	1.2461e-04	7.8942e-05	6.7331e-05	6.4331e-05	6.3532e-05	6.3308e-05
10^{-9}	8.0344e-05	3.5192e-05	2.3843e-05	2.0975e-05	2.0243e-05	2.0052e-05
10^{-10}	6.6387e-05	2.1398e-05	1.0133e-05	7.3066e-06	6.5952e-06	6.4149e-06

Table-2. The maximum absolute errors in solution of Example-2 for $\delta = 0.1\varepsilon$ and $\eta = 0.1\varepsilon$

$\varepsilon \backslash N$	64	128	256	512	1024	2048
10^{-4}	1.0016e-02	9.9406e-03	9.9036e-03	9.8854e-03	9.8194e-03	2.3381e-03
10^{-5}	3.1962e-03	3.1695e-03	3.1565e-03	3.1500e-03	3.1468e-03	3.1452e-03
10^{-6}	1.0139e-03	1.0050e-03	1.0007e-03	9.9854e-04	9.9748e-04	9.9696e-04
10^{-7}	3.2121e-04	3.1814e-04	3.1670e-04	3.1601e-04	3.1567e-04	3.1550e-04
10^{-8}	1.0190e-04	1.0070e-04	1.0019e-04	9.9960e-05	9.9849e-05	9.9795e-05
10^{-9}	3.2526e-05	3.1919e-05	3.1704e-05	3.1617e-05	3.1578e-05	3.1560e-05
10^{-10}	1.0583e-05	1.0164e-05	1.0042e-05	1.0001e-05	9.9858e-06	9.9793e-06

Table-3. The maximum absolute errors in solution of Example-3 for $\delta = 0.1\varepsilon$ and $\eta = 0.1\varepsilon$

$\varepsilon \backslash N$	64	128	256	512	1024	2048
10^{-4}	4.6349e-03	4.3749e-03	4.2968e-03	4.2707e-03	4.2362e-03	9.8528e-04
10^{-5}	1.6812e-03	1.4491e-03	1.3867e-03	1.3689e-03	1.3634e-03	1.3615e-03
10^{-6}	7.3248e-04	5.0971e-04	4.5253e-04	4.3754e-04	4.3345e-04	4.3226e-04
10^{-7}	4.3099e-04	2.1124e-04	1.5575e-04	1.4164e-04	1.3800e-04	1.3704e-04
10^{-8}	3.3551e-04	1.1671e-04	6.1753e-05	4.7929e-05	4.4436e-05	4.3546e-05
10^{-9}	3.0529e-04	8.6802e-05	3.2013e-05	1.8278e-05	1.4831e-05	1.3964e-05
10^{-10}	2.9575e-04	7.7358e-05	2.2622e-05	8.9157e-06	5.4835e-06	4.6235e-06

Table-4. The maximum absolute errors in solution of Example-4 for $\delta = 0.1\varepsilon$ and $\eta = 0.1\varepsilon$

$\varepsilon \backslash N$	64	128	256	512	1024	2048
10^{-4}	2.5809e-02	2.6212e-02	2.6228e-02	2.6114e-02	1.6127e-02	8.9472e-02
10^{-5}	7.5403e-03	8.2280e-03	8.3686e-03	8.3882e-03	8.3853e-03	8.3806e-03
10^{-6}	1.6291e-03	2.4228e-03	2.6107e-03	2.6526e-03	2.6604e-03	2.6611e-03
10^{-7}	1.1231e-03	5.7492e-04	7.7870e-04	8.2799e-04	8.3948e-04	8.4194e-04
10^{-8}	1.1256e-03	2.8125e-04	1.9824e-04	2.4992e-04	2.6258e-04	2.6561e-04
10^{-9}	1.1264e-03	2.8148e-04	7.0363e-05	6.7010e-05	8.0038e-05	8.3253e-05
10^{-10}	1.1266e-03	2.8155e-04	7.0385e-05	1.7596e-05	2.2303e-05	2.5576e-05

DISCUSSIONS AND CONCLUSIONS

We have described a numerical integration methodfor solving a singularly perturbed differential difference equation with layer behaviour at one end point. We have considered few linear problems with left layer. This method controls the rapid changes that occur in the boundary layer region and it gives good results when $h \geq \varepsilon$. To discuss the applicability of the method we have solved few model examples by taking different values of $N, \varepsilon, \delta\ and\ \eta$. We have presented maximum absolute errors for the standard examples chosen from the literature. From results, it is observed that the present method approximates the exact solution very well.

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