Zc– Lindelof Spaces in General Topology

1RM. Sivagama Sundari. 2A.P.Dhana Balan

1,2Department of Mathematics; Alagappa Govt. Arts College, Karaikudi-630003; Tamil Nadu; India.

Abstract

In this paper, we introduce the concept of Zc–Lindelof using \(\omega Zc\)-open sets. Some properties and theorems using \(\omega Zc\)-open sets through Zc–open sets are also discussed.

Keywords: Z–open, Zc–open sets, \(\omega Zc\)-open, Zc–closed, Zc–open cover, Zc–Lindelof.

1. Introduction:

A topological space \(X\) is said to be Lindelof, or have the Lindelof property, if every open cover of \(X\) has a countable subcover. The Lindelof property was introduced by Alexandroff and Urysohn in 1929, the term ‘Lindelof ’ referring back to Lindelof ’s result that any family of open subsets of Euclidean space has a countable sub-family with the same union. The Lindelof property is a weakening of the more commonly used notion of compactness which requires the existence of a finite subcover.

2. Preliminaries:

Throughout this paper \((X,\tau)\) represents a topological space on which no separation axiom is assumed unless otherwise stated. Let \(A \subseteq (X,\tau)\), then \(cl(A)\) and \(int(A)\) denotes the closure of \(A\) and the interior of \(A\) respectively.

Definition 2.1 [1]: A subset \(A\) of a space \(X\) is said to be

i) Z-open set if \(A \subseteq cl(\delta-int(A)) \cup int(cl(A))\),

ii) Z-closed set if \(int(\delta-cl(A)) \cap cl(int(A)) \subseteq A\). The family of all Z-open (resp. Z-closed sets) subsets of a space \((X,\tau)\) will be denoted by \(ZO(X)\) (resp., \(ZC(X)\)).

Definition 2.2 [2]: A subset \(A\) of a space \(X\) is \(Zc\)-open if for each \(x \in A \subseteq ZO(X), \)there exists a closed set \(F\) such that \(x \in F \subseteq A\). A family of all \(Zc\)-open subsets of a topological space \((X,\tau)\) is denoted by \(ZcO(X,\tau)\) or \(ZcC(X,\tau)\).

Example 2.3 [2]: Consider \(X=\{a,b,c,d\}\) with \(\tau = \{\emptyset,X,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}\).

Then the family of closed sets are \(\{\emptyset,X,\{b,c,d\},\{a,b,d\},\{a,c,d\},\{c,d\},\{a,d\},\{b,d\},\{d\}\}\).

The family of \(Zc\)-open sets are:

\(ZO(X) = \{\emptyset,X,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{b,d\},\{c,d\},\{a,b,c\}\}\)

The family of \(Zc\)-closed sets are:

\(ZC(X) = \{\emptyset,X,\{b,c\},\{a,c\},\{a,b\},\{c\},\{a\},\{b\}\}\).

Proposition 2.4 [2]: (i) A subset \(A\) of a space \(X\) is \(Zc\)-open if and only if \(A\) is \(Z\)-open and it is the union of closed sets. That is where \(A\) is \(Z\)-open and \(F_\alpha\) is closed sets for each \(\alpha\).

(ii) A subset \(A\) of a space \(X\) is \(Zc\)-closed if and only if \(A\) is \(Z\)-closed and it is an intersection of open sets.

Remark 2.5 [2]: It is clear from the definition of \(Zc\)-open(resp. \(Zc\)-closed) sets, that every \(Z\)-open(resp. \(Z\)-closed) subset of a space \(X\) is \(Z\)-open, but the converse is not true in general as shown in example 2.3 where \(\{a\}\) belongs to \(ZO(X)\) whereas \(\{a\}\) does not belongs to \(ZO(X)\) and \(\{a,d\}\) belongs to \(ZcO(X)\) whereas \(\{a\}\) does not belongs to \(ZcO(X)\).

Definition 2.6 [3]: Let \((X,\tau)\) be a topological space. Then

(i) \(Zc\)-interior of \(A\) is union of all \(Zc\)-open sets contained in \(A\) and is denoted by \(Zc-Int(A)\).

(ii) \(Zc\)-closure of \(A\) is the intersection of all \(Zc\)-closed sets containing \(A\) and is denoted by \(Zc-Cl(A)\).

3. \(\omega Zc\)-open sets

Definition 3.1: A subset \(W\) of a space is \(\omega\)-open iff for each \(x \in W\), there exists \(U \in \tau\) such that \(x \in U\) and \(U - W\) is countable.

Definition 3.2: A point \(x \in X\) is called a condensation point of \(A\) if for each \(U \in \tau\) with \(x \in U\), the set \(U \cap A\) is countable \(A\) is said to be \(\omega\)–closed if it contains all its condensation points. The complement of an \(\omega\)–closed set is \(\omega\)–open.
Definition 3.3: Let $(X, \tau)$ be a topological space. Then

(i) the union of all open sets contained in $A$ is called the $\omega$-interior of $A$ and is denoted by $\text{int}_\omega(A)$.

(ii) the intersection of all $\omega$-closed sets containing $A$ is called the $\omega$-closure of $A$ and is denoted by $\text{cl}_\omega(A)$. The family of all $\omega$-open subsets of $(X, \tau)$ will be denoted by $\tau_\omega$ or $\omega(O(X))$ forms a topology on $X$ finer than $\tau$.

Definition 3.4: A subset $A$ of a space $X$ is said to be $\omega Z$-open if for every $x \in A$, there exists a $Z$-open subset $U_x \subseteq X$ containing $x$ such that $U_x \setminus A$ is countable. The complement of an subset is said to be $\omega Z$-closed.

Example 3.5: Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$. Then $\{c\}$ is $\omega Z$-open since $X$ is a countable set and it is not $Z$-open.

Lemma 3.6: A subset of a space $X$ is $\omega Z$-open iff for every $x \in A$, there exists a $Z$-open subset $U$ containing $x$ and a countable subset $C$ such that $U \cap C \subseteq A$.

Proof: Let $A$ be $\omega Z$-open and $x \in A$. By definition, there exists an $Z$-open subset $U_x$ containing $x$ such that $(U_x \setminus A)$ is countable. Let $C = U_x \setminus A = U_x \cap (X \setminus A)$. Then $U_x \cap C \subseteq A$. Conversely, let $x \in A$. Then there exists a $Z$-open subset $U_x$ containing $x$ and a countable subset $C$ such that $U_x \cap C \subseteq A$ is countable.

Theorem 3.7: Let $X$ be a space and $C \subseteq X$. If $C$ is $\omega Z$-closed then $C \subseteq A \cup B$ for some $Z$-closed subset $A$ and a countable subset $B$.

Proof: If $C$ is $\omega Z$-closed, then $X - C$ is $\omega Z$-open and hence for every $x \in X - C$, there exists a $Z$-open set $U$ containing $x$ and a countable set $B$ such that $U \setminus B \subseteq X - C$. Thus $C \subseteq X - (U \setminus B) = X - (U \setminus (X - B)) = (X - U) \cup B$. Let $A = X - U$. Then $A$ is $Z$-closed such that $C \subseteq A \cup B$.

4. $\omega Z$-open sets

Definition 4.1: A subset $A$ of a space $X$ is said to be $\omega Z$-open if for every $x \in A$, there exists a $Z$-open subset $U_x \subseteq X$ containing $x$ such that $U_x \setminus A$ is countable. The complement of an $\omega Z$-open subset is said to be $\omega Z$-closed.

Example 4.2: Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$. Then $(a), \{b\}, \{c\}$ is $\omega Z$-open since $X$ is countable but they are not $Z$-open sets.

Lemma 4.3: Let $(X, \tau)$ be a topological space. The intersection of an open set and a $Z$-open set is $Z$-open.

Proposition 4.4: The intersection of an $\omega Z$-open set and an $\omega Z$-open set is $\omega Z$-open.

Proof: Let $A$ be an $\omega Z$-open set and $B$ an $\omega Z$-open set in space $X$. Let $x \in A \cap B$. Since $A$ is $\omega Z$-open, there exists a $Z$-open set $U_A$ containing $x$ such that $(U_A \setminus A)$ is countable. Since $B$ is $\omega Z$-open, there exists an open set $U_B$ containing $x$ such that $(U_B \setminus B)$ is countable. By Lemma 3.3, $U_A \cap U_B$ is a $Z$-open set containing $x$. Also, $(U_A \cap U_B) \cap A = (U_A \setminus A) \cap (U_B \setminus B) \subseteq (U_A \setminus A) \cap (U_B \setminus B) = (U_A \setminus A) \cup (U_B \setminus B)$. Since $(U_A \setminus A) \cup (U_B \setminus B)$ is countable, we get $(U_A \setminus A) \cup (U_B \setminus B)$, as a countable set. Thus $A \cap B$ is $\omega Z$-open.

Result 4.5: The intersection of two $\omega Z$-open sets need not be $\omega Z$-open.

Proposition 4.6: The union of any family of $\omega Z$-open sets is $\omega Z$-open.

Proof: If $\{A_\alpha : \alpha \in \Lambda \}$ is a collection of $\omega Z$-open subsets of $X$, then for every $x \in \bigcup_{\alpha \in \Lambda} A_\alpha$, $x \in A_\beta$ for some $\beta \in \Lambda$. Hence there exists a $Z$-open subset $U$ of $X$ containing $x$ such that $U \setminus A_\beta$ is countable. Now as $U \setminus (\bigcup_{\alpha \in \Lambda} A_\alpha) \subseteq U \setminus A_\beta$ and thus $U \setminus ((\bigcup_{\alpha \in \Lambda} A_\alpha))$ is countable. Therefore $\bigcup_{\alpha \in \Lambda} A_\alpha$ is $\omega Z$-open.

Lemma 4.7: A subset $A$ of a space $X$ is $\omega Z$-open iff for every $x \in A$ there exists an $Z$-open subset $U$ containing $x$ and a countable subset $C$ such that $U \setminus C \subseteq A$.

Proof: It follows from Lemma 3.6.

Theorem 4.8: Let $X$ be a space and $C \subseteq X$. If $C$ is $\omega Z$-closed, then $C \subseteq K \cup B$ for some $Z$-closed subset $K$ and a countable subset $B$.

Proof: Follows from Theorem 4.7.

Theorem 4.9: If each non-empty $Z$-open set of a space $X$ is countable, then $Zccl(A) = \omega Zccl(A)$ for each clopen set $A$ of $X$.

Proof: Clearly $\omega Zccl(A) \subseteq Zccl(A)$. On the other hand, let $x \in Zccl(A)$ and $B$ be an $\omega Z$-open subset containing $x$. Then by Lemma 4.7, there exists a $Z$-open set $V$ containing $x$ and a countable set $C$ such that $V \setminus C \subseteq B$. Thus $(V \setminus C) \cap A \subseteq B \cap A$ and so $(V \cap A) \setminus C \subseteq B \cap A$. Since $x \in V$ and $x \in Zccl(A)$, $V \cap A \neq \emptyset$ and $V \cap A$ is $Z$-open. Since $V$ is $Z$-open and $A$ is open. By hypothesis, each non-empty $Z$-open set of a space is countable and hence $(V \cap A) \setminus C$ is also countable. Thus $B \cap A$ is countable. Therefore, $B \cap A \neq \emptyset$ which means that $x \in \omega Zccl(A)$.

Corollary 4.10: If each non-empty $Z$-open set of a space $X$ is countable, then $Zccl(A) = \omega Zccl(A)$ for each closed set $A$ of $X$.
**Theorem 4.11:** Every \(\omega Zc\)-open is \(\omega Z\)-open .

**Proof:** Let \(A\) be an \(\omega Zc\)-open, then for each \(x \in A\), there exists \(Zc\)-open set \(U_x\) containing \(x\) such that \(U_x - A\) is countable. Since every \(Zc\)-open set is \(Z\)-open, \(A\) is \(\omega Z\)-open.

**Lemma 4.12:** Every \(Zc\)-open is \(\omega Zc\)-open .

5. \(Zc\)-Lindelöf Space

**Definition 5.1:** A space \(X\) is said to be \(Zc\)-Lindelöf if every \(Zc\)-open cover of \(X\) has a countable subcover.

**Definition 5.2:** A subset \(A\) of \(X\) is said to be \(Zc\)-Lindelöf relative to \(X\) if every cover of \(A\) by \(Zc\)-open sets of \(X\) has a countable subcover.

**Theorem 5.3:** For any space \(X\), the following are equivalent.

(a) \(X\) is \(Zc\)-Lindelöf

(b) Every \(\omega Zc\)-open cover of \(X\) has a countable subcover.

**Proof:** (a)\(\Rightarrow\) (b) Let \(\{G_\alpha; \alpha \in \Lambda\}\) be any two \(\omega Zc\)-open cover of \(X\). For each \(x \in X\), there exists \(\alpha(x) \in \Lambda\) such that \(x \in G_{\alpha(x)}\). Since \(G_{\alpha(x)}\) is \(\omega Zc\)-open, there exists \(Zc\)-open set \(V_{\alpha(x)}\) such that \(x \in V_{\alpha(x)}\) and \(V_{\alpha(x)} - G_{\alpha(x)}\) is countable. The family of \(\{V_{\alpha(x)}; x \in X\}\) is a \(Zc\)-open cover of \(X\). Since \(X\) is \(Zc\)-Lindelöf there exists a countable subset \(\{\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n), \ldots\}\) such that \(X = \{\bigcup_{i \in \mathbb{N}} V_{\alpha(x_i)}\} = \{\bigcup_{i \in \mathbb{N}} G_{\alpha(x_i)}\}\). For each \(\alpha(x_i), \beta(x_i)\) \(\in G_{\alpha(x_i)}\) is a countable set and there exists a countable subset \(\Lambda_{\alpha(x_i)}\) of \(\Lambda\) such that \(V_{\alpha(x_i)} = G_{\alpha(x_i)} \subseteq \bigcup \{G_{\alpha}; \alpha \in \Lambda_{\alpha(x_i)}\}\). Hence we observe that \(X = \{\bigcup_{i \in \mathbb{N}} (\bigcup_{\alpha \in \Lambda_{\alpha(x_i)}} G_{\alpha})\} = \bigcup_{i \in \mathbb{N}} G_{\alpha(x_i)}\). Thus (a)\(\Rightarrow\) (b).

(b) \(\Rightarrow\) (a)

Let \(\{G_\alpha; \alpha \in \Lambda\}\) be any \(Zc\)-open cover of \(X\). We claim: \(X\) is \(Zc\)-Lindelöf. Every \(Zc\)-open is \(\omega Zc\)-open, by lemma 4.12. Also by theorem 5.3 \(\{G_\alpha; \alpha \in \Lambda\}\) is \(\omega Zc\)-open cover of \(X\) has a countable subcover. Hence \(X\) is \(Zc\)-Lindelöf.

**Proposition 5.4:** Every \(\omega Z\)-closed subset of a \(Zc\)-Lindelöf space of \(X\) is \(Zc\)-Lindelöf relative to \(X\).

**Proof:** Let \(A\) be an \(\omega Z\)-closed subset of \(X\). Let \(\{G_\alpha; \alpha \in \Lambda\}\) be a cover of \(A\) by \(Zc\)-open set of \(X\). Now for each \(x \in \mathbb{R}^d\), there is a \(Zc\)-open set \(V_x\) such that \(V_x \cap A\) is countable. Since \(X\) is \(Zc\)-Lindelöf and the collection \(\{G_\alpha; \alpha \in \Lambda\}\) is a \(Zc\)-open cover of \(X\), there exists a countable subcover \(\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}\) of \(X\), such that \(x \in \bigcup_{i \in \mathbb{N}} (V_{\alpha_i} \cap A)\). Hence \(\{G_{\alpha_i}; i \in \mathbb{N}\}\) is a countable subcover of \(\{G_\alpha; \alpha \in \Lambda\}\) and it covers \(A\). Hence \(A\) is \(Zc\)-Lindelöf relative to \(X\).

**Proposition 5.5:** If \(X\) is a space such that every \(Zc\)-open subset of \(X\) is a \(Zc\)-Lindelöf relative to \(X\), then every subset is \(Zc\)-Lindelöf relative to \(X\).

**Proof:** Let \(A\) be an arbitrary subset of \(X\) and let \(\{U_i; i \in \mathbb{N}\}\) be a cover of \(A\) by \(Zc\)-open set. Then the family \(\{U_i; i \in \mathbb{N}\}\) is a \(Zc\)-open cover of the \(Zc\)-open set \(U \cup \{U_i; i \in \mathbb{N}\}\). By our assumption there is a countable subfamily \(\{U_{i_1}; i_1 \in \mathbb{N}\}\) which covers \(U \cup \{U_i; i \in \mathbb{N}\}\). This subfamily is also a cover of the set \(A\).

**Theorem 5.6:** A space \(X\) is \(Zc\)-Lindelöf if and only if for every collection of \(Zc\)-closed sets with countable intersection property \(\bigcap_{\alpha \in \Lambda} F_\alpha \neq \emptyset\).

**Proof:** Let \(X\) be \(Zc\)-Lindelöf. \(\{F_\alpha; \alpha \in \Lambda\}\) be a collection of \(Zc\)-closed sets with countable intersection property, \(\bigcap_{\alpha \in \Lambda} F_\alpha \neq \emptyset\). Suppose that \(\bigcap_{\alpha \in \Lambda} F_\alpha = \emptyset\). Then \(X = \bigcup_{\alpha \in \Lambda} F_\alpha\), where \(F_\alpha\) is \(Zc\)-open for each \(\alpha \in \Lambda\). Hence \(\bigcap_{\alpha \in \Lambda} (\bigcup_{\alpha \in \Lambda} F_\alpha) = \emptyset\). Conversely, let every collection of \(Zc\)-closed subsets of \(X\) with the countable intersection property, \(\bigcap_{\alpha \in \Lambda} F_\alpha \neq \emptyset\). Suppose that \(X\) is not \(Zc\)-Lindelöf, then there exists \(Zc\)-open cover \(\{G_\alpha; \alpha \in \Lambda\}\) of \(X\) has no countable subcover \(\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}\), thus \(X = G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}\). Then \(\bigcap_{\alpha \in \Lambda} G_\alpha = \emptyset\). But \(\{G_\alpha; \alpha \in \Lambda\}\) be a collection of \(Zc\)-closed set of \(X\) with countable intersection property by assumption. Then \(\bigcap_{\alpha \in \Lambda} G_\alpha = \emptyset\), \(\bigcap_{\alpha \in \Lambda} G_\alpha \neq \emptyset\) which is a contradiction that \(G\) is a \(Zc\)-open cover of \(X\), which have a countable subcover and hence \(X\) is \(Zc\)-Lindelöf.

References


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