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M/G/1 Feedback Queue with Two Stage Heterogeneous Service and Deterministic Server Vacations

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Abstract: We analyze single server feedback queue with Poisson arrivals, two stages of heteroneous service with different (arbitrary) service time distributions subject to deterministic vacation of constant duration ($d > 0$). After fitst stage service, the server must provide the second stage service. However after the completion of second stage of service, if the customer is dissatisfied with his srvice, he can immediately join the tail of the queue as a feedback customer with prpbability p. Otherwise the customer may depart forever from the system with probability $q = 1 - p$. We assume that after completion of second stage service, the server may decide to take a vacation of fixed length $d(> 0)$ with probability θ or may continue to be available in the system for the next service with probabilty $1 - \theta$. The time dependent probability generating functions using supplemantary variable technique have been obtained in terms of their Laplace transforms and the corresponding steady state results are obtained explicitly.

AMS Subject classification: 60K25, 60K30.
Keywords: Poisson arrivals, steady state, deterministic vacation, supplementary variable technique.

1. INTRODUCTION

The single server queue with two phases of service and vacations have been the subject matter of current research mainly due to its applications in computer, production, manufacturing and communication systems. One of the most important result deals with such models is "Stochastic decomposition result" which was first established by Fuhrmann and Cooper (1985) for M/G/1 type queues with generalized vacations.

Doshi (1991) has analyzed two phase queueing systems with general service times. Selvam and Sivasankaran (1994) have studied a two - phase system with server vacations. Madan (2000) has studied an M/G/1 with two stage heterogeneous service and Bernoulli schedule with single vacation policy. Madan and Baklizi (2002) analyzed a single server with Poisson arrivals, two stages of heterogeneous service with different general service time distributions. Choi and Kim (2003) have studied analysis of two phase queueing systems with vacations and Bernoulli feedback. Artalejo and Choudhury (2004) have studied the steady state analysis of an M/G/1 queue with repeated attempts and two phase service.

Choudhury (2005) has analyzed an M/G/1 queue with an additional phase of second service where the concept of D - policy is introduced. Atencia and Moreno (2006) considered a discrete time Geo/G/1 retrial queue in which all the arriving customers demand a first essential service where as only some of them ask for second optional service. Zadeh and Shankar(2008) has considered an M/G/1 queue with two phases of heterogeneous services, Bernoulli feedback and Bernoulli vacation. Thangaraj and Vanitha (2010) have studied a M/G/1 queue with two stages of heterogeneous services subject to compulsory server vacations and random breakdowns. Bhagat and Jain (2013) investigated M/G/1 retrial queue with unreliable server and general retrial times.Lakshmi and Ramanath (2014) considered a Poisson arrival queueing system with single server and two essential phases of heterogemeous service.

Madan (2015) has studied M/G/1 queue with third optional service subject to deterministic server vacations.

Pavai Madheswari and Suganthi (2016) have snalyzed M/G/1 retrial queue with second optional service and starting failure under modified Bernoulli vacation. Varalakshmi et al. (2017) have studied steady state behaviour of M/G/1 retrial queueing system with two phases of services and immediate feedbacks under working vaction policy where the regular busy server is affected due to the arrival of negative customers. Madan (2018) has studied a single server queue provinding one of the two types of first essential service followed by one of the two types of optional additional service. On completion of services chosen by the customers the server has the option to take the deterministic vacation of fixed duration d with probability α or else with probability $1-\alpha$, the server may continue staying in the system. Many several other authors have contributed to the theory of two phase queueing systems due to its enormous applications.

In the present paper, we consider an M/G/1 queue with feedback subject to deterministic server vacation times of fixed length $d(>0)$ using supplementary variable technique. Each arriving customer has to undergo two stages of heteroneous services provided by a single server and the services times of two stages are assumed to follow general distribution. After completion of second stage of service, if the customer is dissatisfied with its service for certain reason or if it received unseccessful service, the customer may immediately join the tail of the original queue with probability p, $(0<p<1)$. Otherwise the customer may depart forever from the system with probability $q=1-p$. We assume that whenever the server takes a vacation, it is of constant duration $d(>0)$.

The rest of the paper is organised as follows. The mathematical description of our model is in section 2 and the equations governing the model are given in section 3. The time dependent solution have been obtained in section 4 using supplemenraty variable technique and the corresponding steady state results have been derived explicitly in section 5.

2. ASSUMPTIONS UNDERLYING THE MODEL

1. Customers arrive at the system one by one in according to a Poisson stream with arrival rate $(\lambda > 0)$.
2. Each customer undergoes two stages of heterogeneous service provided by a single server on a fitst come first served basis. The service time of two stages follow different general (arbitrary) distributions with distribution function $B_j(v)$ and the density function $b_j(v)$, $j=1,2$.
3. After completion of second stage of service, if the customer is dissatisfied with its service for certain reason or if it received unseccessful service, the customer may immediately join the tail of the original queue with probability p $(0<p<1)$. Otherwise the customer may depart forever from the system with probability $q=1-p$.
4. Let $\mu_i(x)\mathrm{d}x$ be the conditional probability of completion of the i^{th} stage of service during the interval $(x, x+\mathrm{d}x)$ given that elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}, i=1,2, \tag{2.1}$$

and therefore

$$b_i(v) = \mu_i(v) e^{-\int_0^v \mu_i(x)\mathrm{d}x}, i=1,2. \tag{2.2}$$

5. As soon as the service of second stage of a customer is complete, then with probability θ the server decides to take a vacation and with probability $1-\theta$, he continues to be available for the next service.
6. We assume that whenever the server takes a vacation, it is of constant duration $d(>0)$.
7. Various stochastic processes involved in the system are independent of each other.

3. DEFINITIONS, NOTATIONS AND THE TIME - DEPENDANT EQUATIONS GOVERNING THE SYSTEM

The system has then the following set of differential - difference equations

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + (\lambda+\mu_1(x))P_n^{(1)}(x,t) = \lambda P_{n-1}^{(1)}(x,t), n=1,2,\ldots \tag{3.1}$$

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + (\lambda+\mu(x))P_0^{(1)}(x,t) = 0, \tag{3.2}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + (\lambda+\mu_2(x))P_n^{(2)}(x,t) = \lambda P_{n-1}^{(2)}(x,t), n=1,2,\ldots,$$

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$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_0^{(2)}(x,t)+(\lambda+\mu_2(x))P_0^{(2)}(x,t)=0, \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0(t) = \theta q\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)\mathrm{d}x + V_0(t)[\,-K_0-K_1-K_2\cdots\,], \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}V_n(t) = \theta q\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)\mathrm{d}x + \theta p\int_0^\infty P_{n-1}^{(2)}(x,t)\mu_2(x)\mathrm{d}x + V_n(t)[\,-K_0-K_1-K_2\ldots\,],\ n=1,2,\ldots, \tag{3.5}$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t)+V_0(t)K_0+(1-\theta)q\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)\mathrm{d}x. \tag{3.6}$$

Equations (3.2) - (3.7) are to be solved subject to the following boundary conditions

$$P_0^{(1)}(0,t) = Q(t)\lambda+V_0(t)K_1+V_1(t)K_0+(1-\theta)p\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)\mathrm{d}x, +(1-\theta)q\int_0^\infty P_1^{(2)}(x,t)\mu_2(x)\mathrm{d}x\,, \tag{3.7}$$

$$P_n^{(1)}(0,t) = V_0(t)K_{n+1}+V_1(t)K_n+\cdots+V_n(t)K_1+V_{n+1}(t)K_0+ +(1-\theta)p\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)\mathrm{d}x, +(1-\theta)q\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)\mathrm{d}x\,,\quad n=1,2,\ldots, \tag{3.8}$$

$$P_n^{(2)}(0,t) = \int_0^\infty P_n^{(1)}(x,t)\mu_1(x)\mathrm{d}x,\ n=0,1,\ldots, \tag{3.9}$$

We assume that initially there is no customer in the system and the server is idle so that the initial conditions are

$$Q(0)=1,\ P_n^{(j)}(0)=0,\ V_0(0)=0=V_n(0), n\geq 0, j=1,2,\ldots,. \tag{3.10}$$

4. GENERATING FUNCTIONS OF THE QUEUE LENGTH: THE TIME-DEPENDENT SOLUTION

We define the probability generating functions,

$$\left.\begin{aligned} P^{(j)}(x,z,t) &= \sum_{n=0}^\infty z^n P^{(j)}(x,t),\\ P^{(j)}(z,t) &= \sum_{n=0}^\infty z^n P^{(j)}(t),\ j=1,2,\ldots,\\ V(z,t) &= \sum_{n=0}^\infty z^n V_n(t). \end{aligned}\right\} \tag{4.1}$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t, \ \Re(s) > 0. \tag{4.2}$$

Taking the Laplace transforms of equations (3.1) to (3.10) and using (3.11), we obtain

$$\frac{\partial}{\partial x}\overline{P}_n^{(1)}(x,s) + (s+\lambda+\mu(x))\overline{P}_n^{(1)}(x,s) = \lambda\overline{P}_{n-1}^{(1)}(x,s), \quad n = 1,2,\ldots \tag{4.3}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(1)}(x,s) + (s+\lambda+\mu(x))\overline{P}_0^{(1)}(x,s) = 0, \tag{4.4}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(2)}(x,s) + (s+\lambda+\mu_2(x))\overline{P}_n^{(2)}(x,s) = \lambda\overline{P}_{n-1}^{(2)}(x,s), \quad n = 1,2,\ldots, \tag{4.5}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(2)}(x,s) + (s+\lambda+\mu_2(x))\overline{P}_0^{(2)}(x,s) = 0, \tag{4.6}$$

$$s\overline{V}_0(s) = \alpha\overline{Q}(s) + \overline{V}_0(s)[-K_0 - K_1 - K_2 \ldots] + \theta q\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x, \tag{4.7}$$

$$s\overline{V}_n(s) = \theta q\int_0^\infty \overline{P}_n^{(2)}(x,s)\mu_2(x)\mathrm{d}x + \theta p\int_0^\infty \overline{P}_{n-1}^{(2)}(x,s)\mu_2(x)\mathrm{d}x + \overline{V}_n(s)[-K_0 - K_1 - K_2 \ldots], \ n = 1,2,\ldots, \tag{4.8}$$

$$(s+\lambda)\overline{Q}(s) = 1 + \overline{V}_0(s)K_0 + (1-\theta)q\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x, \tag{4.9}$$

$$\overline{P}_0^{(1)}(0,s) = \overline{Q}(s)\lambda + \overline{V}_0(s)K_1 + \overline{V}_1(s)K_0 + (1-\theta)p\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x + (1-\theta)q\int_0^\infty \overline{P}_1^{(2)}(x,s)\mu_2(x)\mathrm{d}x, \tag{4.10}$$

$$\overline{P}_n^{(1)}(0,s) = \overline{V}_0(s)K_{n+1} + \overline{V}_1(s)K_n + \cdots + \overline{V}_n(s)K_1 + \overline{V}_{n+1}(s)K_0 + +(1-\theta)p\int_0^\infty \overline{P}_n^{(2)}(x,s)\mu_2(x)\mathrm{d}x, +(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(2)}(x,s)\mu_2(x)\mathrm{d}x, \quad n = 1,2,\ldots, \tag{4.11}$$

$$\overline{P}_n^{(2)}(0,s) = \int_0^\infty \overline{P}_n^{(1)}(x,s)\mu_1(x)\mathrm{d}x, \ n = 0,1,\ldots, \tag{4.12}$$

Now multiplying equation (4.3) by z^n and summing over n from 1 to ∞, adding to equation (4.4) and using the generating functions defined in (4.1), we get

$$\frac{\partial}{\partial x}\overline{P}^{(1)}(x,z,s) + (s+\lambda-\lambda z+\mu_1(x))\overline{P}(x,z,s) = 0, \tag{4.13}$$

Performing similar operations on equations (4.5) to (4.8) we obtain

$$\frac{\partial}{\partial x}\overline{P}^{(2)}(x,z,s)+(s+\lambda-\lambda z+\mu_2(x))\overline{P}^{(2)}(x,z,s)=0, \tag{4.14}$$

$$(s+1)\overline{V}(z,s) = \theta(q+pz)\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x, \tag{4.15}$$

For the boundary conditions, we multiply both sides of equation (4.10) by z, multiply both sides of equation (4.11) by z^{n+1}, sum over n from 1 to ∞, add the two results and use equation (4.1) to get

$$\begin{aligned} z\overline{P}(0,z,s) &= \lambda z\overline{Q}(s)+\overline{V}(z,s)e^{-\lambda d[1-z]}+(1-\theta)(q+pz)\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x \\ &-(1-\theta)q\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x-\overline{V}_0(s)K_0. \end{aligned} \tag{4.16}$$

Performing similar operation on equation (4.12), we have

$$\overline{P}^{(2)}(0,z,s) = \int_0^\infty \overline{P}^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x. \tag{4.17}$$

Using equation (4.9), equation (4.16) become

$$\begin{aligned} z\overline{P}(0,z,s) &= \lambda z\overline{Q}(s)+\overline{V}(z,s)e^{-\lambda d[1-z]}+(1-\theta)(q+pz)\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x \\ &+1-(s+\lambda)\overline{Q}(s). \end{aligned} \tag{4.18}$$

Integrating equation (4.13) from 0 to x yields

$$\overline{P}^{(1)}(x,z,s)=\overline{P}^{(1)}(0,z,s)\,e^{-(s+\lambda-\lambda z)x-\int_0^x \mu_1(t)dt}, \tag{4.19}$$

where $\overline{P}^{(1)}(0,z,s)$ is given by equation (4.18). Again integrating equation (4.19) by parts with respect to x yields

$$\overline{P}^{(1)}(z,s)=\overline{P}^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{4.20}$$

where

$$\overline{B}_1(s+\lambda-\lambda z)=\int_0^\infty e^{-(s+\lambda-\lambda z)x}dB_1(x) \tag{4.21}$$

is the Laplace-Stieltjes transform of the essential service time $B_1(x)$. Now multiplying both sides of equation (4.19) by $\mu_1(x)$ and integrating over x, we obtain

$$\int_0^\infty \overline{P}^{(1)}(x,z,s)\mu_(x)\mathrm{d}x=\overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z). \tag{4.22}$$

Similarly, on integrating equation (4.14) from 0 to x, we get

$$\overline{P}^{(2)}(x,z,s)=\overline{P}^{(2)}(0,z,s)\,e^{-(s+\lambda-\lambda z)x-\int_0^x \mu_2(t)dt}, \tag{4.23}$$

where $\overline{P}^{(2)}(0,z,s)$ is given by equation (4.17). Again integrating equation (4.23) by parts with respect to x yields

$$\overline{P}^{(2)}(z,s)=\overline{P}^{(2)}(0,z,s)\left[\frac{1-\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{4.24}$$

where

$$\overline{B}_2(s+\lambda-\lambda z+\alpha)=\int_0^\infty e^{-(s+\lambda-\lambda z)x}dB_2(x) \tag{4.25}$$

is the Laplace-Stieltjes transform of n optional service times $B_2(x)$. We see that by virtue of equation (4.23), we have

$$\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x=\overline{P}^{(2)}(0,z,s)\overline{B}_2(s+\lambda-\lambda z). \tag{4.26}$$

By using equation (4.22), equation (4.17) reduces to

$$\overline{P}^{(2)}(0,z,s)=\overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z). \tag{4.27}$$

Using equation (4.27), equation (4.26) becomes

$$\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x=\overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z). \tag{4.28}$$

By using above equation (4.18) reduces to

$$\overline{P}^{(1)}(0,z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)}. \tag{4.29}$$

Substituting the value of $\overline{P}^{(1)}(0,z,s)$ into equation (4.20), we get

$$\overline{P}^{(1)}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)}\left[\frac{1-\overline{b}(s+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right]. \tag{4.30}$$

Now using equations (4.27) and (4.29), equation (4.24) become

$$\overline{P}^{(2)}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)}\overline{B}_1(s+\lambda-\lambda z)\left[\frac{1-\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]. \tag{4.31}$$

From equation (4.15)

$$(s+1)\overline{V}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]} + [1-s\overline{Q}(s)] + \lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)} \theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z).$$

which further reduces to

$$\overline{V}(z,s) = \left[\frac{[1-s\overline{Q}(s)] + \lambda\overline{Q}(s)[z-1]}{(s+1)\left[z-(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z) + \theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)[1-e^{-\lambda d[1-z]}]\right]}\right] \theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z). \tag{4.32}$$

Let $\overline{P}(z,s) = \overline{P}^{(1)}(z,s) + \overline{P}^{(2)}(z,s)$ denote the probability generating function of the number in the queue irrespective of the type of service being provided. Then adding equations (4.30) and (4.31) we have

$$\overline{P}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]} + [1-s\overline{Q}(s)] + \lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)} \left[\frac{1-\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)}{(s+\lambda-\lambda z)}\right]. \tag{4.33}$$

Thus substituting the value of $\overline{V}(z,s)$ from equation (4.32) into equation (4.33) we get

$$\overline{P}(z,s) = \frac{\overline{N}(z,s)}{\overline{D}(z,s)} \tag{4.34}$$

$$\begin{aligned}\overline{N}(z,s) = {} & \Big((s+1)\Big[z-(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z) \\ & +\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\left[1-e^{-\lambda d[1-z]}\right]\Big] \\ & \Big[1-s\overline{Q}(s)+\lambda\overline{Q}(s)[z-1]\Big] + \theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z) \\ & e^{-\lambda d[1-z]}\Big[1-s\overline{Q}(s)+\lambda\overline{Q}(s)[z-1]\Big]\Big)\Big[1-\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\Big]\end{aligned} \tag{4.35}$$

$$\begin{aligned}\overline{D}(z,s) = {} & (s+\lambda-\lambda z)\Big[(s+1)\Big[z-(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z) \\ & +\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\left[1-e^{-\lambda d[1-z]}\right]\Big] \\ & \Big[z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\Big]\Big]\end{aligned} \tag{4.36}$$

If we let $z = 1$ in equation (4.34), we can easily verify that

$$\overline{Q}(s) + \overline{V}(z,s) + \overline{P}(z,s) = \frac{1}{s}, \tag{4.37}$$

as it should be.

Thus $\overline{V}(z,s)$, $\overline{P}^{(1)}(z,s)$ and $\overline{P}^{(2)}(z,s)$ are completely determined from equations (4.32), (4.30) and (4.31) respectively.

5. STEADY STATE SOLUTION

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0}\overline{f}(s) = \lim_{t\to\infty} f(t). \tag{5.1}$$

In order to determine $\overline{P}^{(1)}(z,s)$, $\overline{P}^{(2)}(z,s)$, and $\overline{V}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (4.30), (4.31) and (4.32) by using initial conditions (4.29) and (4.27). For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + V(1) + Q = 1. \tag{5.2}$$

Thus multiplying both sides of equations (4.30), (4.31) and (4.32) by s, taking limit as $s \to 0$, applying property (5.1) and simplifying we have

$$P^{(1)}(z) = \left[\frac{V(z)e^{-\lambda d[1-z]} + \lambda Q(z-1)}{z-(q+pz)(1-\theta)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)}\right]\left[\frac{1-B_1(\lambda-\lambda z)}{(\lambda-\lambda z)}\right] \tag{5.3}$$

$$P^{(2)}(z) = \left[\frac{V(z)e^{-\lambda d[1-z]} + \lambda Q(z-1)}{z-(q+pz)(1-\theta)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)}\right]B_1(\lambda-\lambda z)\left[\frac{1-B_2(\lambda-\lambda z)}{(\lambda-\lambda z)}\right] \tag{5.4}$$

and

$$V(z) = \frac{\theta(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)Q(z-1)}{z-(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)+\theta(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)[1-e^{-\lambda d[1-z]}]}, \tag{5.5}$$

where

$$\begin{aligned} P(z) &= P^{(1)}(z) + P^{(2)}(z) \\ &= \left[\frac{V(z)e^{-\lambda d[1-z]} + \lambda Q(z-1)}{z-(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)}\right]\left[\frac{1-B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)}{(\lambda-\lambda z)}\right] \end{aligned} \tag{5.6}$$

Now using equation (5.5) in equation (5.6) we get,

$$P(z) = \frac{[B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)-1]Q}{z-(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)+\theta(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)[1-e^{-\lambda d[1-z]}]} \tag{5.7}$$

We see that for $z = 1$, the right hand side of both equations (5.5) and (5.6) are indeterminate of the form $\frac{0}{0}$. Therefore, applying L'Hopital's rule we obtain

$$V(1) = \frac{\theta\lambda Q\mu_1\mu_2}{q\mu_1\mu_2 - \lambda(\mu_1+\mu_2) - \theta\lambda\mu_1\mu_2 d} \tag{5.8}$$

$$P(1) = \frac{\lambda(\mu_1+\mu_2)Q}{q\mu_1\mu_2 - \lambda(\mu_1+\mu_2) - \theta\lambda\mu_1\mu_2 d} \tag{5.9}$$

wherein we have used the facts that $\overline{B}_1(0) = \overline{B}_2(0) = 1, -\overline{B}_1^{'}(0) = \frac{1}{\mu_1}$ and $-\overline{B}_2^{'}(0) = \frac{1}{\mu_2}$. Now to determine the only unknown constant Q, we see (5.8) and (5,9) in the normalizing condition $Q + W(1) + V(1) = 1$, and have

$$Q = \frac{q\mu_1\mu_2 - \lambda(\mu_1 + \mu_2) - \theta\lambda\mu_1\mu_2 d}{\mu_1\mu_2[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2} \quad (5.10)$$

$$= 1 - \frac{\lambda[\mu_1\mu_2 + \theta\mu_1\mu_2]}{\mu_1\mu_2[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2} \quad (5.11)$$

where $\lambda < \mu_1\mu_2[q - \theta\lambda d]$.

Equation (5.11) gives the stready state probability that there is no customer in the system and the server is idle.

Also from equation (5.11), we obtain ρ, the utilisation of the factor of the system as

$$\rho = 1 - Q = \frac{\lambda[\mu_1\mu_2 + \theta\mu_1\mu_2]}{\mu_1\mu_2[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2} < 1 \quad (5.12)$$

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