On The Numerical Solution for Pollution in a River and Its Remediation by Two-Dimensional Unsteady Aeration

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Abstract
The aim of this paper is to present the unsteady numerical solution for the problem of the pollution in a river and its remediation. The effect of aeration on the degradation of pollutants is investigated. The governing coupled pair of nonlinear unsteady-state equations for the pollutant concentration $P$ and the dissolved oxygen concentration $X$ are solved numerically by using explicit finite difference method. Effect of the time $t$ and the parameters controlling the flow on $P$ and $X$ have been studied. 2D-Graphics illustrating the calculated variables are drawn to predict their behavior and the results are discussed.

Keywords: Partial differential equations; Advection-dispersion equations; Finite difference method; Numerical solutions; Pollution in a river; Unsteady problems.

1. INTRODUCTION
When assessing the quality of water in a river, there are many factors to be considered: the level of dissolved oxygen, the presence of nitrates, chlorides, phosphates, the level of suspended solids, environmental hormones, chemical oxygen demand, such as heavy metals, and the presence of bacteria. Pollutants from agricultural operations can be considered a significant contributor to the impairment of the surface and the groundwater quality. Advection-dispersion equations are applicable in many disciplines like water pollution, groundwater hydrology, chemical engineering biosciences, environmental sciences and petroleum engineering. Analytical solutions for one-dimensional advection-dispersion equation of the pollutant concentration, pulse type input, semi-infinite media, variable coefficients and porous media are studied in many previous studies [1-9]. Numerical solutions have to be obtained for realistic engineering problems as in [10-13]. Analytical and numerical for two-dimensional advection-dispersion equations with different special cases are investigated in [14-19]. Massabo et al. [20] restudied some analytical solutions for two-dimensional convection-dispersion equation in cylindrical geometry. Alexander and Svetislav [21], studied the two-dimensional advection-dispersion equations with variable coefficients. The resulting equations are solved numerically by using explicit finite difference method. Mathematical water quality models date back to the well-known model in 1925 [22], where they described the balance of dissolved oxygen in rivers. Chapra [23] stated the standard equations of the water pollution by using advection-dispersion equations for the pollutant and dissolved oxygen concentrations. Advection diffusion equations for stream pollutant, temperature and dissolved oxygen are considered in [24-30]. Pimpunchat et al. [31] presented a simple mathematical model for river pollution and investigated the effect of aeration on the degradation of pollutant; their model consists of a pair of coupled reaction-diffusion-advection equations for the pollutant concentration and dissolved oxygen concentration, respectively. Hussain et al. [32] studied a mathematical model providing the ability to predict the contaminant concentration levels of a river; they presented a simple mathematical model for river pollution. They considered the steady state case in oneshotient dimensional and zero dispersion. Ibrahim et al. [33] study remediation of pollution in a river by unsteady aeration with arbitrary initial and boundary conditions.

The primary objective of the present study is to investigate the alleviation of pollution by aeration within a flowing river contaminated by distributed sources and the associated depletion of dissolved oxygen. The particular river whose water quality was the motivation for the study is the Nile River in Egypt. It is assumed that the pollutants are largely biological wastes that undergo various biochemical and biodegradation processes using dissolved oxygen. Our study is to develop a general solutions for the two-dimensional coupled partial differential equations representing the diffusion of both the pollutant concentration and dissolved oxygen concentration by using finite difference method.

2. FORMULATION OF THE PROBLEM
For our model, the advection-diffusion-equation may be a good approximation model of river pollution. We assumed that the river has a uniform crosssectional area. Consider the unsteady flow in the river as being two-dimensional characterized by double distances $x^*(m)$ and $y^*(m)$ measured from the origin $x^* = 0$ and $y^* = 0$. The flow is described by the coupled equations for the pollutant concentration $P^*(x^*, y^*)$, $x^*$, and $y^*$.
The last term in equation (1), \( D_{px}^* \frac{\partial^2 (A^* P^*)}{\partial x'^2} + D_{py}^* \frac{\partial^2 (A^* P^*)}{\partial y'^2} - \frac{\partial(u^* A^* P^*)}{\partial x^*} - \frac{\partial(v^* A^* P^*)}{\partial y^*} - k_1^* \frac{X^*}{X^* + k} A^* P^* + q^* \quad (0 \leq x^* \leq L^*, 0 \leq y^* \leq W^*, t^* \geq 0), \tag{1} \]

\[
\frac{\partial (A^* X^*)}{\partial t^*} = D_{xx}^* \frac{\partial^2 (A^* X^*)}{\partial x'^2} + D_{xy}^* \frac{\partial^2 (A^* X^*)}{\partial y'^2} - \frac{\partial(u^* A^* X^*)}{\partial x^*} - \frac{\partial(v^* A^* X^*)}{\partial y^*} - k_2^* \frac{X^*}{X^* + k} A^* P^* + \alpha^* (S^* - X^*), \quad (0 \leq x^* \leq L^*, 0 \leq y^* \leq W^*, t^* \geq 0). \tag{2} \]

Where \( A^* \) is the cross-section area (\( m^2 \)), \( D_{px}^* \) is the dispersion coefficient of pollutant in the \( x \)-direction (\( m^2 \text{ day}^{-1} \)), \( D_{py}^* \) is the dispersion coefficient of pollutant in the \( y \)-direction (\( m^2 \text{ day}^{-1} \)), \( u^* \) is the water velocity in the \( x \)-direction (\( m \text{ day}^{-1} \)), \( v^* \) is the water velocity in the \( y \)-direction (\( m \text{ day}^{-1} \)), \( k_1^* \) is the degradation rate coefficient for pollutant (\( \text{day}^{-1} \)), \( k_2^* \) is the half-saturated oxygen demand concentration for pollutant decay (\( \text{kg m}^{-3} \)), \( q^* \) is the added pollutant rate along the river (\( \text{kg m}^{-1} \text{ day}^{-1} \)), \( L^* \) is the length of river (m), \( W^* \) is the width of river (m), \( D_{xx}^* \) is the dispersion coefficient of dissolved oxygen in the \( x \)-direction (\( m^2 \text{ day}^{-1} \)), \( D_{xy}^* \) is the dispersion coefficient of dissolved oxygen in the \( y \)-direction (\( m^2 \text{ day}^{-1} \)), \( k_3^* \) is the de-aeration rate coefficient for dissolved oxygen (\( \text{day}^{-1} \)), \( \alpha^* \) is the mass transfer of oxygen from air to water (\( \text{m}^2 \text{ day}^{-1} \)) and \( S^* \) is the saturated oxygen concentration (\( \text{kg m}^{-3} \)). The last term in equation (1), represents the addition of pollutant at a rate \( q^* \), while the third term, in the right hand side, represents its removal by aeration. The rate of depletion of pollutant concentration \( P^* \), due to the biochemical reaction with dissolved oxygen concentration \( X^* \), has been described using a “Michaelis_Menten” term \( k_1^* \frac{X^* A^*}{X^* + k} P^* \). To simplify the equations, we introduce the initial and boundary conditions associated with equations (1) and (2) as:

\[
P^*(x^*, y^*, 0) = 0, \quad \text{and} \quad X^*(x^*, y^*, 0) = 0, \quad 0 < x^* \leq L^*, 0 < y^* \leq W^* \tag{3}.
\]

\[
P^*(0, y^*, t^*) = P^0^*(x^*,0^*,0^*) = P^0^* \quad \text{and} \quad X^*(0, y^*, t^*) = X^0^*(x^*,0^*,0^*) = X^0^*, \quad t^* > 0, \tag{4}
\]

\[
\frac{\partial P^*}{\partial x^*}(L^*, W^*, t^*) = 0, \quad \text{and} \quad \frac{\partial X^*}{\partial x^*}(L^*, W^*, t^*) = 0, \quad t^* \geq 0, \tag{5}
\]

where \( P^0^* \) and \( X^0^* \) are the pollutant concentration and dissolved oxygen concentration at the origin, respectively. \( P^0^* \) and \( X^0^* \) are taken constants. Let \( \text{scale of pollutant concentration} = 0.2P^0^* \) and \( \text{scale of dissolved oxygen concentration} = 1.9X^0^* \), where \( P^0^* \) and \( X^0^* \) are the scales of the pollutant concentration and the dissolved oxygen concentration, respectively[20]. The numerical values of \( P^0^* \) and \( X^0^* \) are chosen, so that they must not have the same numerical value at the origin \( x^* = 0 \) and \( y^* = 0 \) [10]. The following non-dimensional variables are used in the governing equations, initial and boundary conditions.

\[
\frac{x}{L^*} = \frac{x^*}{L}, \quad \frac{y}{L^*} = \frac{y^*}{L}, \quad \frac{t}{t^*} = k_2^* t^*, \quad P = \frac{P^*}{P^0^*} \quad \text{and} \quad X = \frac{X^*}{X^0^*}, \quad W = \frac{W^*}{L} \tag{6}
\]

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Substituting equation (6), into equations (1) - (5), these equations are reduced to:

\[
\begin{align*}
\frac{\partial P}{\partial t} + D_P \nabla^2 P + D_v \frac{\partial^2 P}{\partial x^2} + D_v \frac{\partial^2 P}{\partial y^2} - v \frac{\partial P}{\partial x} - \frac{X}{X + k} P &= q \quad (0 \leq x \leq 1, 0 \leq y \leq W, t \geq 0) \\
D_P \nabla^2 X + D_v \frac{\partial^2 X}{\partial x^2} + D_v \frac{\partial^2 X}{\partial y^2} - \frac{k_x}{k_y} \frac{X}{X + k} P &= \alpha (S - X) \quad (0 \leq x \leq 1, 0 \leq y \leq W, t \geq 0),
\end{align*}
\]

where the dimensionless parameters controlling the flow are:

\[
\begin{align*}
P &= u v, & q &= q^* \\
X &= X^* \\
D_P &= \frac{D_P^*}{k_x k_y}, & D_v &= \frac{D_v^*}{k_x k_y}, & u &= u^*, & v &= v^*, & k_x &= k_x^*, & k_y &= k_y^*, \alpha &= \alpha^*, & S &= S^*, & P &= P^* \text{ for } D_{P_x} = 1 \text{ and } 5.
\end{align*}
\]

where \(P_x\) and \(X_x\) are the dimensionless pollutant and the dimensionless dissolved oxygen concentrations at the origin, respectively.

3. NUMERICAL RESULTS

Equations (7) and (8) have been solved by using finite difference method. We take in our model the domain of dimensionless longitudinal direction in the region \(0 \leq x \leq 1, 0 \leq y \leq 0.1\), and the dimensionless time \(0 \leq t \leq 1\). In the numerical calculations, the step lengths \(\Delta x = 0.1, \Delta y = 0.01\) and the step of time \(\Delta t = 0.0001\) have been used to achieve the stability of the finite difference scheme [21]. \(P_x\) and \(X_x\) are taken equal to 0.2 and 1.9, respectively [27]. The variations of \(P\) and \(X\) along the river for several values of the time \(t\) and the parameters \(D_{P_x}, k, u, v, q\) and \(D_{X_x}\) are given in figures (1-12), taking \(D_{X_y} = 0.1, D_{P_y} = 0.1, k_3 = 0.5, a = 2\) and \(S = 2\).

Figures (1) and (2), show the variation of \(P(x,y,t)\) with \(t\) for \(t = 0.01\) and 5. From figures (1) and (2), it is clear that, for any fixed point \((x, y)\), the values of \(P\) increase as \(t\) increases. Numerical studies show that, at \(t < 0.05\), as \(x\) and \(y\) increase \(P\) decreases and at \(t > 0.05\), as \(x\) and \(y\) increase \(P\) increases. This is due to the fact that at any point \((x, y)\) the accumulation of the pollutant increases especially for small values of \(u\) and \(v\). It is obvious that, for any cross-section \(x = constant\), \(P\) decreases with increasing \(y\). Also for any cross-section \(y = constant\), \(P\) decreases with increasing \(x\). The decrease in \(C\) in the range \(0 \leq x \leq 1\) is very large compared to the decrease in \(P\) in the range \(0 \leq y \leq 0.1\). This is due to the fact that, the flow velocity is in the \(x\) - direction. In general, as \(t\) increases, the value of \(P\) increases at any point \((x, y)\). Those results agree with that obtained by Yadav et al. [5] and Yadav and Jain [15].

Figures (1) and (3), show the variation of \(P(x,y,t)\) with \(D_{P_x}\) (for \(D_{P_x} = 1\) and 5). From figures (1) and (3), we can conclude that, for any fixed point \((x, y)\), the values of \(P\) increase as \(D_{P_x}\) increases and for any cross section \(y = constant\), \(P\) increase with \(D_{P_x}\) increases.

Figures (1) and (4), show the variation of \(P(x,y,t)\) with \(k\) (for \(k = 1\) and 5). From figures (1) and (4), we can see that, for any fixed point \((x, y)\), the values of \(P\) increase as \(k\) increases. For any cross section \(y = constant\), the values of \(P\) increase with \(k\) increases.

Figures (1) and (5), show the variation of \(P(x,y,t)\) with \(u\) for \(u = 0.1\) and 3. From figures (1) and (5), we can see that, for any fixed point \((x, y)\), the value of \(P\) increase as \(u\) increases. For any cross section \(y = constant\), the values of \(P\) increase with \(u\) increases.

Figures (1) and (6), show the variation of \(P(x,y,t)\) with \(v\) for \(v = 0.1\) and 3. From figures (1) and (6), it is obvious that, for any fixed point \((x, y)\), the values of \(X\) increase as \(v\) increases. As \(x\) and \(y\) increase, the values of \(P\) increase with \(v\) increases. This is due to the fact that, the velocity represents the flux of...
the polluted water per unit area at any cross section. \( P \) increases as \( u \) and \( v \) increase. This is due to the fact that the velocity represents the flux of the polluted water per unit area. Hence, as expected, \( P \) increases as \( u \) and \( v \) increase.

Figures (1) and (7), show the variation of \( P(x,y,t) \) with \( q \) for \( q = 1 \) and \( 4 \). From figures (1) and (7), we can note that: for any fixed point \((x, y)\), the values of \( P \) increase as \( q \) increases. (ii) As \( x \) and \( y \) increase, the values of \( P \) decrease with \( q \) increases.

Figures (8) and (9), show the variation of \( X(x,y,t) \) with \( t \) for \( t = 0.01 \) and \( 0.5 \). From figures (8) and (9), we recognize that: for any fixed point \((x, y)\), the values of \( X \) increase as \( t \) increases. As \( x \) and \( y \) increase, the values of \( X \) increase with \( t \) increases.

Figures (8) and (10), show the variation of \( X(x,y,t) \) with \( D_{Xx} \) (for \( D_{Xx} = 1 \) and \( 5 \)). From figures (8) and (10), we can perceive that: for any fixed point \((x, y)\), the values of \( X \) increase as \( D_{Px} \) increases. As \( x \) and \( y \) increase, the values of \( X \) decrease with \( D_{Xx} \) increases.

Figures (8) and (11), show the variation of \( X(x,y,t) \) with \( v = 0.1 \) and \( 3 \). From figures (8) and (11), we can notice that: for any fixed point \((x, y)\), the values of \( X \) increases as \( v \) increases. As \( x \) and \( y \) increase, the values of \( P \) increase with \( v \) increases.
Fig. 6. Pollutant concentration $P$ versus space $y$ and time $t$ for $v = 3$.

Fig. 7. Pollutant concentration $P$ versus space $y$ and time $t$ for $q = 5$.

Fig. 8. Pollutant concentration $P$ versus space $y$ and time $t$ for $D_{x_{x}} = 1$, $D_{y_{y}} = 0.1$, $u = 1$, $v = 0.1$, $k = 1$, $k_s = 0.5$, $q = 1$, $D_{x_{x}} = 1$, $D_{x_{y}} = 0.1$, $\alpha = 2$, $S = 2$ and $t = 0.01$.

Fig. 9. Dissolved oxygen concentration $X$ versus space $y$ and time $t$ for $t = 0.5$.

Fig. 10. Dissolved oxygen concentration $X$ versus space $y$ and time $t$ for $D_{x_{x}} = 5$.

Fig. 11. Dissolved oxygen concentration $X$ versus space $y$ and time $t$ for $v = 3$.
4. CONCLUSION

We have presented a mathematical model for river pollution and investigated the effect of aeration on the degradation of pollution for all the parameters controlling the flow. We consider the flow two-dimensional along the river. The governing coupled pair of nonlinear unsteady-state equations for river pollutant and dissolved oxygen concentrations are non-dimensionalized by using appropriate transformations. The resulting equations are solved numerically by using explicit finite difference method and the results are plotted. It is found that: (I) For any fixed point \((x, y)\), the values of \(X\) and \(P\) increase as \(t\) increases. (ii) For any cross section \(y = \text{constant}\), the values of \(P\) increase as the dimensionless of the dispersion coefficient of pollutant in the \(y\)-direction \(D_{py}\) increases. (iii) For any fixed point \((x, y)\), the values of \(X\) increase as dimensionless of the dispersion coefficient of dissolved oxygen in the \(x\)-direction \(D_{px}\) increases. (iv) For any fixed point \((x, y)\), the values of \(P\) increase with the dimensionless of water velocity in the \(x\)-direction \(u\), the dimensionless of water velocity in the \(y\)-direction \(v\) and the dimensionless of addition of pollutant at a rate \(q\) increase.

REFERENCES