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Some Products On Max weighted finite state automaton

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Abstract: This paper introduces various kinds of products such as shuffle product, general direct product, cascade product, wreath product, and cartesian composition on max weighted finite automaton (mwfa). Some construction on mwfa are defined using inverse image, and direct image related to a morphism $h : \Gamma^* \to \Sigma^*$ (where Γ and Σ are two alphabets).

Keywords: Max Weighted finite state automaton, Weighted regular languages, Recognizable, product, morphism.

INTRODUCTION

Mathematical models in classical computation, automata have been an important area in theoretical computer science [3]. It started from a seminal paper of Kleene [5], and within a few years developed into a rich mathematical research topic. The theory of computation deals with the computational logic with respect to simple machines, termed as automata. It is the study of abstract computer devices or intangible machines. Warren Mc Culloch and Walter Pitts, two neurophysiologists, were the first to present a description of finite automata in 1943. Finite automata played a crucial role in the theory of programming languages, compiler constructions, switching circuit designing, computer controller, neuron net, text editor and lexical analyzer [1].

Weighted automata are classical finite automata in which the transition carry weights. The starting point of weighted automaton is to determine the number of ways a word can be accepted or the amount of resources used for this.

The principle of weighted automata is to consider non deterministic automata, that takes value in semiring. Max weighted finite state automaton belong to the wider family of weighted automaton, as introduced by schutzenberger [8].

Section 1 is a review of some basic concepts. which includes definitions and concepts, which are essential to establish the results proved in the subsequent section. The concepts of shuffle product, general direct product, Cascade product, wreath product play a prominent role in the study of automaton [4].

In section 2, we examine these ideas for Max weighted finite automaton (mwfa). In section 3 we study product of (mwfa) M_1 and M_2, written $M_1 \odot M_2$ and called cartesian composition of M_1 and M_2 as in [2]. we show that M_1 , M_2 and $M_1 \odot M_2$ share many similar structural properties.

In section 4 deals with the construction on mwfa, related to a morphism $h : \Gamma^* \to \Sigma^*$, where Σ and Γ are two alphabets. It is shown that if h is a morphism and $h^{-1}(\lambda) = \lambda$, then image of a recognizable subset of Γ^* is a recognizable subset of Σ^* and h is fine, then the inverse image of recognizable subset of Σ^* is recognizable.

PRELIMINARIES

This section introduces the notions of (mwfa) and a set recognized by a (mwfa).

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Definition 1. A Max weighted finite state automaton is a six tuple (mwfa) $M = (Q, \Sigma, W, \mu, i, f)$, where

(i) Q is a finite non-empty set of states.
(ii) Σ is a finite non-empty set of input symbols.
(iii) W is a weighting space. i.e., weighting space $W = ([0, \infty), \cdot, \max)$, where $\cdot$ is usual multiplication.
(iv) μ is a weighting function such as $\mu : Q \times \Sigma \times Q \to [0, \infty)$ is called a state transition function. The value of $\mu(p, a, q)$ represents the weighted transition from state p to state q when the input symbol is a.
(v) i is an initial distribution function, where $i : Q \to [0, \infty)$.
(vi) f is a final distribution function, where $f : Q \to [0, \infty)$.

Definition 2. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa , the extended weighting transition function for M is the weighted subset
$\mu^* : Q \times \Sigma^* \times Q \to [0, \infty)$ has defined as follows: $\forall p, q \in$Q, $a \in \Sigma$, $x \in \Sigma^*$,

$$\mu^*(p, \lambda, q) = \begin{cases} 1, \text{if } p = q \\ 0, \text{if } p \neq q \end{cases}$$

$$\mu^*(p, xa, q) = \max_{r \in Q}\{\mu^*(p, x, r) \cdot \mu(r, a, q)\}$$

Definition 3. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa. Let $x \in \Sigma^*$. Then x is said to be recognized by M if $L(x) > 0$, where

$$\begin{aligned} L(x) &= \max_{p,q \in Q}\{i(p) \cdot \mu^*(p, x, q) \cdot f(q)\} \\ &= \max_{p,q \in Q}\big\{i(p) \cdot \{\max_{r \in Q} \mu^*(p, y, r) \cdot \mu(r, a, q)\} \cdot f(q)\big\}, \\ &\quad x = ya. \end{aligned}$$

Theorem 4. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa. Then $\mu^*(p, xy, q) = \max_{r \in Q}\{\mu^*(p, x, r) \cdot \mu^*(r, y, q)\}$ $\forall\ p, q \in Q$ and $\forall\ x, y \in \Sigma^*$.

PRODUCTS OF mwfa

In this section, the concepts of shuffle product, general direct product, cascade product and wreath product of mwfa are given and also proved that the product of any two recognizable sets is recognizable.

Shuffle Product

Definition 5. Given subsets $L_1 \subset \Sigma^*, L_2 \subset \Gamma^*$ with $\Sigma \cap \Gamma = \phi$, the shuffle product $(L_1 ⧢ L_2) \subset (\Sigma \cup \Gamma)^*$ consists of all words of the form $s_1 g_1 s_2 g_2 \cdots s_n g_n \in (\Sigma \cup \Gamma)^*$ with $s_1 s_2 \cdots s_n \in L_1$, $g_1 g_2 \cdots g_n \in L_2$.

Theorem 6. Let $\pi : (\Sigma \cup \Gamma)^* \to \Sigma^* \times \Gamma^*$ be the morphism given by $\pi(a) = (a, \lambda)$, $\forall a \in \Sigma$, $\pi(a') = (\lambda, a')$, $\forall a' \in \Gamma$ then
$A ⧢ B = \pi^{-1}(A \times B)$, $A \subset \Sigma^*$, $B \subset \Gamma^*$.

Proof. Given $\pi(a) = (a, \lambda), \forall a \in \Sigma, \pi(a') = (\lambda, a'), \forall\, a' \in \Gamma$. Let $x \in \pi(A ⧢ B)$

$$x = \pi(a_1 a_1' a_2 a_2' \cdots a_n a_n'),$$

where $a_1 a_2 a_3 \cdots a_n \in A$, $a_1' a_2' a_3' \cdots a_n' \in B$.

$$\begin{aligned} &= \pi(a_1)\pi(a_1')\pi(a_2)\pi(a_2') \cdots \pi(a_n)\pi(a_n') \\ &= (a_1, \lambda)(\lambda, a_1')(a_2, \lambda)(\lambda, a_2') \cdots (a_n, \pi)(\lambda, a_n') \\ &= (a_1 a_1 a_2 a_3 \cdots a_n,\ a_1' a_1' a_2' a_3' \cdots a_n') \in A \times B \end{aligned}$$

Therefore $\pi(A ⧢ B) \subseteq A \times B$. (1)

Let $x \in A \times B$.

$$\begin{aligned} x &= (a, a'),\ \text{where } a \in A,\ a' \in B. \\ &= (a, \lambda)(\lambda, a') \\ &= \pi(a)\pi(a') \\ &= \pi(aa'), aa' \in A ⧢ B. \end{aligned}$$

$\therefore \pi(aa') \in \pi(A ⧢ B) \Rightarrow x \in \pi(A ⧢ B)$.

Therefore $A \times B \subseteq \pi(A ⧢ B)$. (2)

From (1) and (2), $A ⧢ B = \pi^{-1}(A \times B)$. ■

Definition 7. Let $M_1 = (Q_1, \Sigma, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Gamma, W, \mu_2, i_2, f_2)$ be two mwfa's ; $Q_1 \cap Q_2 = \phi$, the Shuffle product of M_1 and M_2 is the mwfa $M = M_1 ⧢ M_2$ by setting $Q = Q_1 \times Q_2, i = i_1 \times i_2, f = f_1 \times f_2$,
i.e., $M = (Q, (\Sigma \cup \Gamma), W, \mu, i, f)$, where

(i) $\mu : Q \times (\Sigma \cup \Gamma) \times Q \to [0, \infty)$ is defined as follows: $\forall p, q \in Q_1$, $\forall p', q' \in Q_2$, $a \in \Sigma \cup \Gamma$.

$$\mu((p, p'), a, (q, q')) = \begin{cases} \mu_1(p, a, q) & \text{, if } p' = q',\ a \in \Sigma \\ \mu_2(p', a, q') & \text{, if } p = q,\ \ a \in \Gamma \\ 0 & \text{, otherwise} \end{cases}$$

(ii) $i : Q \to [0, \infty)$ is defined by

$$i(p, p') = \begin{cases} i_1(p) \cdot i_2(p') & , \text{ if } p \in Q_1,\ p' \in Q_2 \\ 0 & , \text{ otherwise} \end{cases}$$

(iii) $f : Q \to [0, \infty)$ is defined by

$$f(p, p') = \begin{cases} f_1(p) \cdot f_2(p') & , \text{ if } p \in Q_1,\ p' \in Q_2 \\ 0 & , \text{ otherwise} \end{cases}$$

Lemma 8. if $x = a_1 b_1 a_2 b_2 \cdots a_n b_n \in (\Sigma \cup \Gamma)^*$, $a_i \in \Sigma, b_i \in \Gamma$, $i = 1, 2 \ldots, n$ then $\mu^*((p_i, p'_r), x, (q_k, q'_t)) = \mu_1^*(p_i, a_1 a_2 \cdots a_n, q_k) \cdot \mu_2^*(p'_r, b_1 b_2 \cdots b_n, q'_t)$.

Proof. Let $x = a_1 b_1 a_2 b_2 \cdots a_n b_n \in (\Sigma \cup \Gamma)^*$. We prove the result by induction on n. For $n = 1$, we have $x = a_1 b_1$, $a_1 \in \Sigma$, $b_1 \in \Gamma$, then

$$\begin{aligned} \mu^*((p_i, p'_r), a_1 b_1, (q_k, q'_t)) &= \max_{(p_j, q'_s) \in Q} \left\{ \mu((p_i, p'_r), a_1, (p_j, q'_s)) \cdot \mu((p_j, q'_s), b_1, (q_k, q'_t)) \right\} \\ &= \mu((p_i, p'_r), a_1, (q_k, p'_r)) \cdot \mu((q_k, p'_r), b_1, (q_k, q'_t)) \\ &\qquad [\text{ if } p'_r \neq q'_s,\ \mu((p_i, p'_r), a_1, (p_j, q'_s)) = 0, \\ &\qquad \text{ if } p_j \neq q_k,\ \mu((p_j, q'_s), b_1, (q_k, q'_t)) = 0\,] \\ &= \mu_1(p_i, a_1, q_k) \cdot \mu_2(p'_r, b_1, q'_t) \end{aligned}$$

Thus the result is true for $n = 1$. Assume that the result is true for $n = k - 1$, $k > 2$. Now, for $n = k$

$$\begin{aligned} &\mu^*((p_i, p'_r), a_1 b_1 a_2 b_2 \cdots a_k b_k, (q_k, q'_t)) \\ &\quad = \max_{(p_j, q'_s) \in Q} \left\{ \mu^*((p_i, p'_r), a_1 b_1 \cdots a_{k-1} b_{k-1}, (p_j, q'_s)) \cdot \mu((p_j, q'_s), a_k b_k, (q_k, q'_t)) \right\} \\ &\quad = \max_{(p_j, q'_s) \in Q} \left\{ \mu_1^*(p_i, a_1 a_2 \cdots a_{k-1}, p_j) \cdot \mu_2^*(p'_r, b_1 b_2 \cdots b_{k-1}, q'_s) \cdot \right. \\ &\qquad\qquad \left. \mu_1(p_j, a_k, q_k) \cdot \mu_2(q'_s, b_k, q'_t) \right\} \\ &\quad = \max_{p_j \in Q_1, q'_s \in Q_2} \left\{ \mu_1^*(p_i, a_1 a_2 \cdots a_{k-1}, p_j) \cdot \mu_1(p_j, a_k, q_k) \cdot \right. \\ &\qquad\qquad \left. \mu_2^*(p'_r, b_1 b_2 \cdots b_{k-1}, q'_s) \cdot \mu_2(q'_s, b_k, q'_t) \right\} \\ &\quad = \max_{p_j \in Q_1} \left\{ \mu_1^*(p_i, a_1 a_2 \cdots a_{k-1}, p_j) \cdot \mu_1(p_j, a_k, q_k) \right\} \cdot \\ &\qquad\qquad \max_{q'_s \in Q_2} \left\{ \mu_2^*(p'_r, b_1 b_2 \cdots b_{k-1}, q'_s) \cdot \mu_2(q'_s, b_k, q'_t) \right\} \\ &\quad = \mu_1^*(p_i, a_1 a_2 \cdots a_k, q_k) \cdot \mu_2^*(p'_r, b_1 b_2 \cdot b_k, q'_t) \end{aligned}$$

Hence the result for $n = k$. ■

Theorem 9. Let $M_1 = (Q_1, \Sigma, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Gamma, W, \mu_2, i_2, f_2)$ be two mwfa's, with L_1 and L_2 as the weighted regular language respectively. Then L is the weighted regular language accepted by mwfa $M = M_1 \sqcup\!\sqcup M_2$ such that $L(x) = L_1(x_1) \cdot L_2(x_2)$.

Proof. Let $x = a_1b_1a_2b_2\cdots a_nb_n \in (\Sigma \cup \Gamma)^*$, where $x_1 = a_1a_2\cdots a_n \in L_1 \subseteq \Sigma^*$, $x_2 = b_1b_2\cdots b_n \in L_2 \subseteq \Gamma^*$. L be the weighted regular language accepted by M. and $L(x) > 0$.

$$\begin{aligned}
L(x) &= \max_{(p_i,p'_r),(q_k,q'_t)\in Q} \left\{ i(p_i,p'_r)\cdot \mu^*((p_i,p'_r),x,(q_k,q'_t))\cdot f(q_k,q'_t)\right\}\\
&= \max_{(p_i,p'_r),(q_k,q'_t)\in Q} \left\{ i(p_i,p'_r)\cdot \mu^*((p_i,p'_r),a_1b_1a_2b_2\cdots a_nb_n,(q_k,q'_t))\cdot f(q_k,q'_t)\right\}\\
&= \max_{(p_i,p'_r),(q_k,q'_t)\in Q} \left\{ i_1(p_i)\cdot i_2(p'_r)\cdot \mu_1^*(p_i,a_1\cdots a_n,q_k)\cdot \mu_2^*(p'_r,b_1\cdots b_n,q'_t)\cdot f_1(q_k)\cdot f_2(q'_t)\right\}\\
&= \max_{(p_i,p'_r),(q_k,q'_t)\in Q} \left\{ i_1(p_i)\cdot \mu_1^*(p_i,a_1\cdots a_n,q_k)\cdot f_1(q_k)\cdot \right.\\
&\qquad \left. i_2(p'_r)\cdot \mu_2^*(p'_r,b_1\cdots b_n,q'_t)\cdot f_2(q'_t)\right\}\\
&= \max_{(p_i,q_k)\in Q_1} \left\{ i_1(p_i)\cdot \mu_1^*(p_i,a_1a_2\cdots a_n,q_k)\cdot f_1(q_k)\right\}\cdot\\
&\qquad \max_{(p'_r,q'_t)\in Q_2} \left\{ i_2(p'_r)\cdot \mu_2^*(p'_r,b_1b_2\cdots b_n,q'_t)\cdot f_2(q'_t)\right\}
\end{aligned}$$

Therefore $L(x) = L_1(x_1)\cdot L_2(x_2)$.

■

General direct product of M_1 and M_2

Definition 10. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. Let Σ' be a finite set and g be a function from Σ' into $\Sigma_1 \times \Sigma_2$. Let π_i be the projection map of $\Sigma_1 \times \Sigma_2$ onto Σ_i, $i = 1, 2$.
Let $M_1 * M_2 = (Q_1 \times Q_2, \Sigma', W, \mu_g, i_g, f_g)$ be the general direct product of M_1 and M_2, where

(i) $\mu_g : (Q_1 \times Q_2) \times \Sigma' \times (Q_1 \times Q_2) \to [0,\infty)$ is defined by
$\forall (p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$ and $\forall a' \in \Sigma'$,

$$\begin{aligned}
\mu_g((p_1,p_2),a',(q_1,q_2)) &= \mu_1\times\mu_2\big((p_1,p_2),(\pi_1(g(a')),\pi_2(g(a'))),(q_1,q_2)\big)\\
&= \mu_1(p_1,\pi_1(g(a')),q_1)\cdot \mu_2(p_2,\pi_2(g(a')),q_2)
\end{aligned}$$

(ii) $i_g : (Q_1 \times Q_2) \to [0,\infty)$ is defined by
$i_g(p_1,p_2) = (i_1 \times i_2)(p_1,p_2) = i_1(p_1)\cdot i_2(p_2)$.

(iii) $f_g : (Q_1 \times Q_2) \to [0,\infty)$ is defined by
$f_g(q_1,q_2) = (f_1 \times f_2)(q_1,q_2) = f_1(q_1)\cdot f_2(q_2)$.

Definition 11. If $\Sigma' = \Sigma_1 \times \Sigma_2$ and g is the identity map, then $M_1 * M_2$ is called the direct full product of M_1 and M_2 and we write $M_1 \times M_2$
for $M_1 * M_2$. Then, $\mu_1 \times \mu_2((p_1,p_2),(a_1,a_2),(q_1,q_2)) = \mu_1(p_1,a_1,q_1)\cdot\mu_2(p_2,a_2,q_2)$
$\forall\ (p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2, \forall\ (a_1,a_2) \in \Sigma_1 \times \Sigma_2$.

Definition 12. If $\Sigma_1 = \Sigma_2$, $\Sigma' = \{(a_1,a_2) | a_i \in \Sigma_i, i = 1,2, a_1 = a_2\}$, g is the identity map, then $M_1 * M_2$ is called restricted direct product of M_1 and M_2 and we write $M_1 \cdot M_2$ for $M_1 * M_2$. We could also let $\Sigma' = \Sigma_1 = \Sigma_2$ and $g : \Sigma' \to \{(a_1,a_2) | a_i \in \Sigma_i, i = 1,2, a_1 = a_2\}$ where $g(a) = (a,a)$ to obtain the restricted direct product.

Lemma 13. *Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. Consider the general direct product $M_1 * M_2$. Then*
$\forall p_1, q_1 \in Q_1$, $\forall p_2, q_2 \in Q_2$,

1. $\mu_g^*\big((p_1,p_2),\lambda,(q_1,q_2)\big) = \mu_1^*(p_1,\lambda,q_1)\cdot\mu_2^*(p_2,\lambda,q_2)$

2. $\mu_g^*\big((p_1,p_2), a_1'a_2'\cdots a_n', (q_1,q_2)\big) = \mu_1^*\big(p_1, \pi_1(g(a_1'))\pi_1(g(a_2'))\cdots\pi_1(g(a_n')), q_1\big)\cdot \mu_2^*\big(p_2, \pi_2(g(a_1'))\pi_2(g(a_2'))\cdots\pi_2(g(a_n')), q_2\big)$

where $a_i' \in \Sigma'$, $i = 1, 2, ...n$.

Proof. (1) is obvious
Now, (2)

$$
\begin{aligned}
&\mu_g^*\big((p_1,p_2), a_1'a_2'\cdots a_n', (q_1,q_2)\big)\\
&= \max_{(r_1^{(i)}, r_2^{(i)})\in Q_1\times Q_2, i=1,2,\dots,n-1} \Big\{\mu_g\Big((p_1,p_2), a_1', (r_1^{(1)}, r_2^{(1)})\Big)\cdot\mu_g\Big((r_1^{(1)}, r_2^{(1)}), a_2', (r_1^{(2)}, r_2^{(2)})\Big)\cdots\\
&\qquad\cdot\mu_g\Big((r_1^{(n-1)}, r_2^{(n-1)}), a_n', (q_1,q_2)\Big)\Big\}\\
&= \max_{(r_1^{(i)}, r_2^{(i)})\in Q_1\times Q_2, i=1,2,\dots,n-1} \Big\{(\mu_1\times\mu_2)\Big((p_1,p_2), (\pi_1(g(a_1')), \pi_2(g(a_1'))), (r_1^{(1)}, r_2^{(1)})\Big)\\
&\qquad\cdot(\mu_1\times\mu_2)\Big((r_1^{(1)}, r_2^{(1)}), (\pi_1(g(a_2')), \pi_2(g(a_2')), (r_1^{(2)}, r_2^{(2)})\Big)\cdots\\
&\qquad\cdot(\mu_1\times\mu_2)\Big((r_1^{(n-1)}, r_2^{(n-1)}), (\pi_1(g(a_n'), \pi_2(g(a_n')), (q_1,q_2)\Big)\Big\}\\
&= \max_{(r_1^{(i)}, r_2^{(i)})\in Q_1\times Q_2, i=1,2,\dots,n-1} \big\{\mu_1(p_1, \pi_1(g(a_1')), r_1^{(1)})\cdot\mu_2(p_2, \pi_2(g(a_1')), r_2^{(1)})\cdot\\
&\qquad\mu_1(r_1^{(1)}, \pi_1(g(a_2')), r_1^{(2)})\cdot\mu_2(r_2^{(1)}, \pi_2(g(a_2')), r_2^{(2)})\cdots\\
&\qquad\mu_1(r_1^{(n-1)}, \pi_1(g(a_n')), q_1)\cdot\mu_2(r_2^{(n-1)}, \pi_2(g(a_n')), q_2)\big\}\\
&= \max_{(r_1^{(i)}\in Q_1, r_2^{(i)}\in Q_2), i=1,2,\dots,n-1} \big\{\mu_1(p_1, \pi_1(g(a_1')), r_1^{(1)})\cdot\mu_1(r_1^{(1)}, \pi_1(g(a_2')), r_1^{(2)})\cdots\\
&\qquad\mu_1(r_1^{(n-1)}, \pi_1(g(a_n')), q_1)\cdot\mu_2(p_2, \pi_2(g(a_1')), r_2^{(1)})\cdot\\
&\qquad\mu_2(r_2^{(1)}, \pi_2(g(a_2')), r_2^{(2)})\cdots\mu_2(r_2^{(n-1)}, \pi_2(g(a_n')), q_2)\big\}\\
&= \max_{r_1^{(i)}\in Q_1, i=1,2,\dots,n-1} \big\{\mu_1(p_1, \pi_1(g(a_1')), r_1^{(1)})\cdot\mu_1(r_1^{(1)}, \pi_1(g(a_2')), r_1^{(2)})\cdots\\
&\qquad\mu_1(r_1^{(n-1)}, \pi_1(g(a_n')), q_1)\big\}\cdot\max_{r_2^{(i)}\in Q_2, i=1,2,\dots,n-1}\big\{\mu_2(p_2, \pi_2(g(a_1')), r_2^{(1)})\\
&\qquad\cdot\mu_2(r_2^{(1)}, \pi_2(g(a_2')), r_2^{(2)})\cdots\mu_2(r_2^{(n-1)}, \pi_2(g(a_n')), q_2)\big\}\\
&= \mu_1^*\big(p_1, \pi_1(g(a_1'))\pi_1(g(a_2'))\cdots\pi_1(g(a_n')), q_1\big)\cdot\mu_2^*\big(p_2, \pi_2(g(a_1'))\pi_2(g(a_2'))\cdots\pi_2(g(a_n')), q_2\big)
\end{aligned}
$$

■

Theorem 14. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's with L_1 and L_2 as a weighted regular languages respectively. Then L is the weighted regular language accepted by mwfa $M_1 * M_2$ such that $L(x) = L_1(x_1)\cdot L_2(x_2)$, $\forall x \in L$.

Proof. Let $M = M_1 * M_2$ and L be the weighted regular language accepted by M.
Let $x = a'_1 a'_2 \cdots a'_n \in \Sigma'^*$ and $L(x) > 0$.

$$\begin{aligned} L(x) &= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)} \left\{ i_g(p_1,p_2)\cdot \mu_g^*\big((p_1,p_2), a'_1a'_2\cdots a'_n, (q_1,q_2)\big)\cdot f_g(q_1,q_2)\right\} \\ &= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)} \Big\{ i_1(p_1)\cdot i_2(p_2)\cdot \mu_1^*\big(p_1, \pi_1(g(a'_1))\pi_1(g(a'_2))\cdots\pi_1(g(a'_n)), q_1\big)\cdot \\ &\qquad \mu_2^*\big(p_2, \pi_2(g(a'_1))\pi_2(g(a'_2))\cdots\pi_2(g(a'_n)), q_2\big)\cdot f_1(q_1)\cdot f_2(q_2)\Big\} \\ &= \max_{p_1,q_1\in Q_1} \left\{ i_1(p_1)\cdot \mu_1^*\big(p_1, \pi_1(g(a'_1))\pi_1(g(a'_2))\cdots\pi_1(g(a'_n)), q_1\big)\cdot f_1(q_1)\right\} \\ &\qquad \max_{p_2,q_2\in Q_2} \left\{ i_2(p_2)\cdot \mu_2^*\big(p_2, \pi_2(g(a'_1))\pi_2(g(a'_2))\cdots\pi_2(g(a'_n)), q_2\big)\cdot f_2(q_2)\right\} \\ &\qquad\qquad \text{since } \pi_1(g(a'_1))\pi_1(g(a'_2))\cdots\pi_1(g(a'_n)) = x_1 \in \Sigma_1^* \text{ and} \\ &\qquad\qquad \pi_2(g(a'_1))\pi_2(g(a'_2))\cdots\pi_2(g(a'_n)) = x_2 \in \Sigma_2^* \end{aligned}$$

Therefore $L(x) = L_1(x_1)\cdot L_2(x_2)$. ∎

Cascade product of M_1 and M_2

Definition 15. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. Let ζ be the function of $(Q_2 \times \Sigma_2)$ into Σ_1. Let $M_1\ \zeta\ M_2 = (Q_1 \times Q_2, \Sigma_2, W, \mu^\zeta, i^\zeta, f^\zeta)$ be the cascade product of M_1 and M_2, where

(i) $\mu^\zeta : (Q_1 \times Q_2) \times \Sigma_2 \times (Q_1 \times Q_2) \to [0,\infty)$ is defined by
$\forall (p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$, $\forall b \in \Sigma_2$,
$\mu^\zeta((p_1,p_2), b, (q_1,q_2)) = \mu_1\big(p_1, \zeta(p_2,b), q_1\big)\cdot \mu_2(p_2,b,q_2)$

(ii) $i^\zeta : (Q_1 \times Q_2) \to [0,\infty)$ is defined by $i^\zeta(p_1,p_2) = i_1(p_1)\cdot i_2(p_2)$.

(iii) $f^\zeta : (Q_1 \times Q_2) \to [0,\infty)$ is defined by $f^\zeta(q_1,q_2) = f_1(q_1)\cdot f_2(q_2)$.

Proposition 16. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. Let $M = M_1 \zeta M_2$ for some ζ. Then $\forall\ (p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$ and $y = y_1y_2\cdots y_n \in \Sigma_2^*$,

$$\begin{aligned} &\mu^{\zeta *}\big((p_1,p_2), y_1y_2\cdots y_n, (q_1,q_2)\big) \\ &\qquad = \mu_1^*(p_1, \zeta(p_2,y_1)\zeta(p_2^{(1)},y_2)\cdots\zeta(p_2^{(n-1)},y_n), q_1)\cdot \mu_2^*(p_2, y_1y_2\cdots y_n, q_2) \\ &\qquad\qquad \text{for some } p_2^{(i)} \in Q_2,\ i = 1,2,\ldots,n-1. \end{aligned}$$

Proof.

$$\begin{aligned} &\mu^{\zeta *}\big((p_1,p_2), y, (q_1,q_2)\big) \\ &\quad = \mu^{\zeta *}\big((p_1,p_2), y_1y_2\cdots y_n, (q_1,q_2)\big) \\ &\quad = \max_{(p_1^{(1)},p_2^{(1)}),\ldots,(p_1^{(n-1)},p_2^{(n-1)})\in Q_1\times Q_2} \Big\{ \mu^\zeta((p_1,p_2), y_1, (p_1^{(1)},p_2^{(1)}))\cdot \mu^\zeta((p_1^{(1)},p_2^{(1)}), y_2, (p_1^{(2)},p_2^{(2)}))\cdots \\ &\qquad \mu^\zeta((p_1^{(n-1)},p_2^{(n-1)}), y_n, (q_1,q_2))\Big\} \\ &\quad = \max_{(p_1^{(1)},p_2^{(1)}),\ldots,(p_1^{(n-1)},p_2^{(n-1)})\in Q_1\times Q_2} \Big\{ \mu_1(p_1, \zeta(p_2,y_1), p_1^{(1)})\cdot \mu_2(p_2,y_1,p_2^{(1)})\cdot \mu_1(p_1^{(1)}, \zeta(p_2^{(1)},y_2), p_1^{(2)})\cdot \\ &\qquad\qquad \mu_2(p_2^{(1)}, y_2, p_2^{(2)})\cdots \mu_1(p_1^{(n-1)}, \zeta(p_2^{(n-1)},y_n), q_1)\cdot \mu_2(p_2^{(n-1)}, y_n, q_2)\Big\} \end{aligned}$$

$$= \max_{(p_1^{(1)},p_2^{(1)}),\ldots,(p_1^{(n-1)},p_2^{(n-1)})\in Q_1\times Q_2} \big\{\mu_1(p_1,\zeta(p_2,y_1),p_1^{(1)})\cdot\mu_1(p_1^{(1)},\zeta(p_2^{(1)},y_2),p_1^{(2)})\cdots \mu_1(p_1^{(n-1)},\zeta(p_2^{(n-1)},y_n),q_1)\cdot\mu_2(p_2,y_1,p_2^{(1)})\cdot\mu_2(p_2^{(1)},y_2,p_2^{(2)})\cdots\mu_2(p_2^{(n-1)},y_n,q_2)\big\}$$

$$= \max_{p_1^{(1)},\ldots,p_1^{(n-1)}\in Q_1}\big\{\mu_1(p_1,\zeta(p_2,y_1),p_1^{(1)})\cdot\mu_1(p_1^{(1)},\zeta(p_2^{(1)},y_2),p_1^{(2)})\cdots\mu_1(p_1^{(n-1)},\zeta(p_2^{(n-1)},y_n),q_1)\big\} \cdot \max_{p_2^{(1)},\ldots,p_2^{(n-1)}\in Q_2}\big\{\mu_2(p_2,y_1,p_2^{(1)})\cdot\mu_2(p_2^{(1)},y_2,p_2^{(2)})\cdots\mu_2(p_2^{(n-1)},y_n,q_2)\big\}$$

$$= \mu_1^*(p_1,\zeta(p_2,y_1)\zeta(p_2^{(1)},y_2)\cdots\zeta(p_2^{(n-1)},y_n),q_1)\cdot\mu_2^*(p_2,y_1y_2\cdots y_n,q_2)$$

■

Theorem 17. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's with L_1 and L_2 as a weighted regular languages respectively. Then L is the weighted regular language accepted by mwfa $M = M_1\zeta M_2$ such that $L(y) = L_1(x)\cdot L_2(y)$, $\forall y \in L$.

Proof. Let $M = M_1\zeta M_2$ and L be the weighted regular language accepted by M. Let $y = y_1y_2\cdots y_n \in \Sigma_2^*$ and $L(y) > 0$. Now,

$$\begin{aligned}
L(y) &= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)}\big\{i^\zeta(p_1,p_2)\cdot\mu^{\zeta*}\big((p_1,p_2),y,(q_1,q_2)\big)\cdot f^\zeta(q_1,q_2)\big\}\\
&= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)}\big\{i^\zeta(p_1,p_2)\cdot\mu^{\zeta*}\big((p_1,p_2),y_1y_2\cdots y_n,(q_1,q_2)\big)\cdot f^\zeta(q_1,q_2)\big\}\\
&= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)}\big\{i_1(p_1)\cdot i_2(p_2)\cdot\mu_1^*(p_1,\zeta(p_2,y_1)\zeta(p_2^{(1)},y_2)\cdots\zeta(p_2^{(n-1)},y_n),q_1)\cdot\\
&\qquad \mu_2^*(p_2,y_1y_2\cdots y_n,q_2)\cdot f_1(q_1)\cdot f_2(q_2)\big\}\\
&= \max_{p_1,q_1\in Q_1}\big\{i_1(p_1)\cdot\mu_1^*(p_1,\zeta(p_2,y_1)\zeta(p_2^{(1)},y_2)\cdots\zeta(p_2^{(n-1)},y_n),q_1)\cdot f_1(q_1)\big\}\\
&\qquad \max_{p_2,q_2\in Q_2}\big\{i_2(p_2)\cdot\mu_2^*(p_2,y_1y_2\cdots y_n,q_2)\cdot f_2(q_2)\big\}\\
&\qquad \text{since } \zeta(p_2,y_1)\zeta(p_2^{(1)},y_2)\cdots\zeta(p_2^{(n-1)},y_n) = x\in\Sigma_1^*
\end{aligned}$$

Therefore $L(y) = L_1(x)\cdot L_2(y)$. ■

Example 18. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ be a mwfa, where $Q_1 = \{q_1, q_2\}$, $\Sigma_1 = \{a, b\}$, $\mu_1 : Q_1 \times \Sigma_1 \times Q_1 \to [0,\infty)$ is defined as follows:

$$\mu_1(q_1,a,q_1) = 3 \qquad \mu_1(q_2,b,q_2) = 4$$
$$\mu_1(q_1,a,q_2) = 2 \qquad \mu_1(q_1,b,q_2) = 2.5$$

$i_1 : Q_1 \to [0,\infty)$ is defined by $i_1(q_1) = 4$.
$f_1 : Q_1 \to [0,\infty)$ is defined by $f_1(q_1) = 5$.

The transition diagram is shown below:

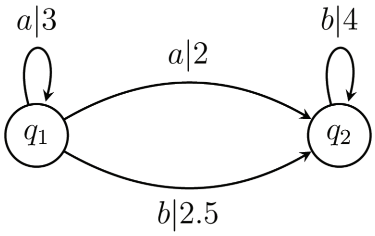

FIGURE 1.

The language accepted by M_1 is a weighted subset $L_1 : \Sigma_1^* \to [0, \infty)$ such that

$$L_1(x) = \begin{cases} w_1, & w_1 \geq 40 \quad \text{if } x \in a^*ab^* \\ w_2, & w_2 \geq 50 \quad \text{if } x \in a^*bb^* \\ 0\,, & \text{otherwise} \end{cases}$$

Let $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be a mwfa, where $Q_2 = \{q_1^{'}, q_2^{'}\}$, $\Sigma_2 = \{a', b'\}$, $\mu_2 : Q_2 \times \Sigma_2 \times Q_2 \to [0, \infty)$ is defined as follows:

$$\mu_2(q_1', b', q_1') = 3 \qquad \mu_2(q_1', b', q_2') = 5 \qquad \mu_2(q_1', a', q_2') = 4$$
$$\mu_2(q_2', a', q_2') = 2 \qquad \mu_2(q_2', b', q_2') = 6$$

$i_2 : Q_2 \to [0, \infty)$ is defined by $i_2(q_1') = 6$.
$f_2 : Q_2 \to [0, \infty)$ is defined by $f_2(q_2') = 7$.

The transition diagram is shown below:

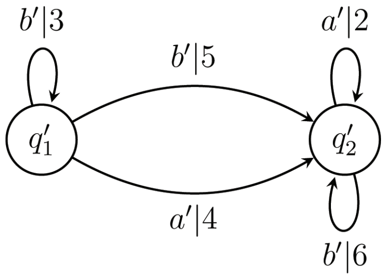

FIGURE 2.

The language accepted by M_2 is a weighted subset $L_2 : \Sigma_2^* \to [0, \infty)$ such that

$$L_2(x) = \begin{cases} w_1, & w_1 \geq 210 \quad \text{if } x \in b^{'*}b'\{a', b'\}^* \\ w_2, & w_2 \geq 168 \quad \text{if } x \in b^{'*}a'\{a', b'\}^* \\ 0\,, & \text{otherwise} \end{cases}$$

Let $\zeta : (Q_2 \times \Sigma_2) \to \Sigma_1$ is defined by
$\zeta(q_1', a') = a,\ \zeta(q_2', b') = b,\ \zeta(q_1', b') = b,\ \zeta(q_2', a') = a.$
Let $M_1\ \zeta\ M_2 = (Q_1 \times Q_2, \Sigma_2, W, \mu^\zeta, i^\zeta, f^\zeta)$ be the cascade product of M_1 and M_2, where
$Q_1 \times Q_2 = \{(q_1, q_1'), (q_1, q_2'), (q_2, q_1'), (q_2, q_2')\}$,
$\mu^\zeta : (Q_1 \times Q_2) \times \Sigma_2 \times (Q_1 \times Q_2) \to [0, \infty)$ is such that
$\mu^\zeta((q_1, q_1'), a', (q_2, q_2')) = \mu_1(q_1, \zeta(q_1', a'), q_2).\mu_2(q_1', a', q_2') = 2 \cdot 4 = 8$

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$\mu^\zeta((q_1, q_1'), a', (q_1, q_2')) = \mu_1(q_1, \zeta(q_1', a'), q_1).\mu_2(q_1', a', q_2') = 3 \cdot 4 = 12$
$\mu^\zeta((q_1, q_1'), b', (q_2, q_1')) = \mu_1(q_1, \zeta(q_1', b'), q_2).\mu_2(q_1', b', q_1') = 2.5 \cdot 3 = 7.5$
$\mu^\zeta((q_1, q_1'), b', (q_2, q_2')) = \mu_1(q_1, \zeta(q_1', b'), q_2).\mu_2(q_1', b', q_2') = 2.5 \cdot 5 = 12.5$
$\mu^\zeta((q_1, q_2'), a', (q_1, q_2')) = \mu_1(q_1, \zeta(q_2', a'), q_1).\mu_2(q_2', a', q_2') = 3 \cdot 2 = 6$
$\mu^\zeta((q_2, q_1'), b', (q_2, q_1')) = \mu_1(q_2, \zeta(q_1', b'), q_2).\mu_2(q_1', b', q_1') = 4 \cdot 3 = 12$
$\mu^\zeta((q_1, q_2'), a', (q_2, q_2')) = \mu_1(q_1, \zeta(q_2', a'), q_2).\mu_2(q_2', a', q_2') = 2 \cdot 2 = 4$
$\mu^\zeta((q_2, q_1'), b', (q_2, q_2')) = \mu_1(q_2, \zeta(q_1', b'), q_2).\mu_2(q_1', b', q_2') = 4 \cdot 5 = 20$
$\mu^\zeta((q_2, q_2'), b', (q_2, q_2')) = \mu_1(q_2, \zeta(q_2', b'), q_2).\mu_2(q_2', b', q_2') = 4 \cdot 6 = 24$
$\mu^\zeta((q_1, q_2'), b', (q_2, q_2')) = \mu_1(q_1, \zeta(q_2', b'), q_2).\mu_2(q_2', b', q_2') = 2.5 \cdot 6 = 15$
and μ^ζ is 0 elsewhere.
$i^\zeta : (Q_1 \times Q_2) \to [0, \infty)$ is defined by $i^\zeta(q_1, q_1') = i_1(q_1) \cdot i_2(q_1') = 24$.
$f^\zeta : (Q_1 \times Q_2) \to [0, \infty)$ is defined by $f^\zeta(q_2, q_2') = f_1(q_2) \cdot f_2(q_2') = 35$.

The transition diagram is shown below:

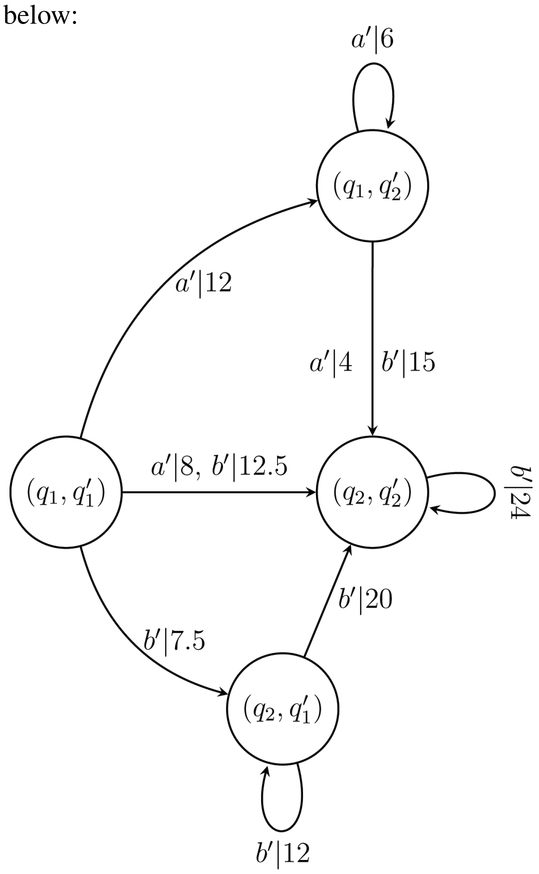

FIGURE 3.

The language accepted by M is a weighted subset $L : \Sigma_2^* \to [0, \infty)$ such that

$$L(y) = \begin{cases} w_1, & w_1 \geq 1,51,200 \qquad \text{if } x \in a'a'^*b'b'^* \\ w_2, & w_2 \geq 40,320 \qquad \text{if } x \in a'a'^*a'b'^* \\ w_3, & w_3 \geq 10,500 \qquad \text{if } x \in b'b'^* \\ w_4, & w_4 \geq 6,720 \qquad \text{if } x \in a'b'^* \\ w_5, & w_5 \geq 1,26,000 \qquad \text{if } x \in b'b'^*b'b'^* \\ 0, & \text{otherwise} \end{cases}$$

From the construction of M it is clear that, $L(y) = L_1(x) \cdot L_2(y)$, $\forall y \in L$.

Wreath product of M_1 and M_2

Definition 19. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. Let h be a function from Q_2 into Σ_1.
Let $M_1 \varpi M_2 = (Q_1 \times Q_2, \Sigma_1^{Q_2} \times \Sigma_2, W, \mu^{\varpi}, i^{\varpi}, f^{\varpi)}$ be the wreath product of M_1 and M_2, where

(i) $\Sigma_1^{Q_2} = \{h_i \mid h_i : Q_2 \to \Sigma_1,\ i = 1, 2 \ldots n\}$.
(ii) $\mu^{\varpi} : (Q_1 \times Q_2) \times (\Sigma_1^{Q_2} \times \Sigma_2) \times (Q_1 \times Q_2) \to [0, \infty)$ is defined by
$\forall\, ((p_1, p_2), (h, b), (q_1, q_2)) \in (Q_1 \times Q_2) \times (\Sigma_1^{Q_2} \times \Sigma_2) \times (Q_1 \times Q_2)$,
$\mu^{\varpi}\big((p_1, p_2), (h, b), (q_1, q_2)\big) = \mu_1(p_1, h(p_2), q_1) \cdot \mu_2(p_2, b, q_2)$
(iii) $i^{\varpi} : (Q_1 \times Q_2) \to [0, \infty)$ is defined by $i^{\varpi}(p_1, p_2) = i_1(p_1) \cdot i_2(p_2)$.
(iv) $f^{\varpi} : (Q_1 \times Q_2) \to [0, \infty)$ is defined by $f^{\varpi}(q_1, q_2) = f_1(q_1) \cdot f_2(q_2)$.

Proposition 20. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. consider the wreath product $M_1 \varpi M_2$. Then
$\forall\, (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $\forall\, (h_1, b_1) \cdots (h_n, b_n) \in (\Sigma_1^{Q_2} \times \Sigma_2)^*$,

$$\begin{aligned} &\mu^{\varpi *}\big((p_1, p_2), (h_1, b_1) \cdots (h_n, b_n), (q_1, q_2)\big) \\ &\quad = \mu_1^*(p_1, h_1(p_2) h_2(p_2^{(1)}) \cdots h_n(p_2^{(n-1)}), q_1) \cdot \mu_2^*(p_2, b_1 b_2 \cdots b_n, q_2) \\ &\qquad\qquad \text{for some } p_2^{(i)} \in Q_2,\ i = 1, 2, \ldots, n-1. \end{aligned}$$

Proof.

$$\begin{aligned} &\mu^{\varpi *}\big((p_1, p_2), (h_1, b_1) \cdots (h_n, b_n), (q_1, q_2)\big) \\ &= \max_{p_1^{(1)}, \ldots, p_1^{(n-1)} \in Q_1,\, p_2^{(1)}, \ldots, p_2^{(n-1)} \in Q_2} \big\{\mu_1(p_1, h_1(p_2), p_1^{(1)}) \cdot \mu_2(p_2, b_1, p_2^{(1)}) \\ &\quad \cdot \mu_1(p_1^{(1)}, h_2(p_2^{(1)}), p_1^{(2)}) \cdot \mu_2(p_2^{(1)}, b_2, p_2^{(2)}) \cdots \mu_1(p_1^{(n-1)}, h_n(p_2^{(n-1)}), q_1) \cdot \mu_2(p_2^{(n-1)}, b_n, q_2)\big\} \\ &= \max_{p_1^{(1)}, \ldots, p_1^{(n-1)} \in Q_1,\, p_2^{(1)}, \ldots, p_2^{(n-1)} \in Q_2} \big\{\mu_1(p_1, h_1(p_2), p_1^{(1)}) \cdot \mu_1(p_1^{(1)}, h_2(p_2^{(1)}), p_1^{(2)}) \\ &\quad \cdots \mu_1(p_1^{(n-1)}, h_n(p_2^{(n-1)}), q_1) \cdot \mu_2(p_2, b_1, p_2^{(1)}) \cdot \mu_2(p_2^{(1)}, b_2, p_2^{(2)}) \cdots \mu_2(p_2^{(n-1)}, b_n, q_2)\big\} \\ &= \max_{p_1^{(1)}, \ldots, p_1^{(n-1)} \in Q_1} \big\{\mu_1(p_1, h_1(p_2), p_1^{(1)}) \cdot \mu_1(p_1^{(1)}, h_2(p_2^{(1)}), p_1^{(2)}) \cdots \mu_1(p_1^{(n-1)}, h_n(p_2^{(n-1)}), q_1)\big\} \cdot \\ &\quad \max_{p_2^{(1)}, \ldots, p_2^{(n-1)} \in Q_2} \big\{\mu_2(p_2, b_1, p_2^{(1)}) \cdot \mu_2(p_2^{(1)}, b_2, p_2^{(2)}) \cdots \mu_2(p_2^{(n-1)}, b_n, q_2)\big\} \\ &= \mu_1^*(p_1, h_1(p_2) h_2(p_2^{(1)}) \cdots h_n(p_2^{(n-1)}), q_1) \cdot \mu_2^*(p_2, b_1 b_2 \cdots b_n, q_2) \end{aligned}$$

■

Theorem 21. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's with L_1 and L_2 as a weighted regular languages respectively. Then L is the weighted regular language accepted by mwfa $M_1 \varpi M_2$ such that $L(x) = L_1(x_1) \cdot L_2(x_1)$, $\forall x \in L$.

Proof. Let $M = M_1 \varpi M_2$ and L be the weighted regular language accepted by M. Let $x = (h_1, b_1)(h_2, b_2)\cdots(h_n, b_n) \in (\Sigma_1^{Q_2} \times \Sigma_2)^*$ and $L(x) > 0$. Now,

$$\begin{aligned}
L(x) &= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)} \big\{ i^{\varpi}(p_1,p_2)\cdot \mu^{\varpi *}\big((p_1,p_2),(h_1,b_1)(h_2,b_2)\cdots(h_n,b_n),(q_1,q_2)\big)\cdot \\
&\qquad\qquad f^{\varpi}(q_1,q_2)\big\} \\
&= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)} \big\{ i_1(p_1)\cdot i_2(p_2)\cdot \mu_1^*(p_1, h_1(p_2)h_2(p_2^{(1)})\cdots h_n(p_2^{(n-1)}), q_1)\cdot \\
&\qquad\qquad \mu_2^*(p_2, b_1b_2\cdots b_n, q_2)\cdot f_1(q_1)\cdot f_2(q_2)\big\} \\
&= \max_{p_1,q_1\in Q_1} \big\{ i_1(p_1)\cdot \mu_1^*(p_1, h_1(p_2)h_2(p_2^{(1)})\cdots h_n(p_2^{(n-1)}), q_1)\cdot f_1(q_1)\big\} \\
&\qquad\qquad \max_{p_2,q_2\in Q_2} \big\{ i_2(p_2)\cdot \mu_2^*(p_2, b_1b_2\cdots b_n, q_2)\cdot f_2(q_2)\big\} \\
&\qquad\qquad \text{since } h_1(p_2)h_2(p_2^{(1)})\cdots h_n(p_2^{(n-1)}) = x_1 \in \Sigma_1^* \text{ and} \\
&\qquad\qquad b_1b_2\cdots b_n = x_2 \in \Sigma_2^*
\end{aligned}$$

Therefore $L(x) = L_1(x_1)\cdot L_2(x_1),\ \forall x \in L$.

■

CARTESIAN COMPOSITION OF M_1 AND M_2

This section presents a study on Cartesian Composition of two max weighted finite automatons.

Definition 22. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. and $\Sigma_1 \cap \Sigma_2 = \phi$. Let $M_1 \odot M_2 = (Q_1 \times Q_2, \Sigma_1 \cup \Sigma_2, W, \mu_1 \odot \mu_2, i_1 \odot i_2, f_1 \odot f_2)$, where

(i) $(\mu_1 \odot \mu_2) : (Q_1 \times Q_2) \times (\Sigma_1 \cup \Sigma_2) \times (Q_1 \times Q_2) \to [0, \infty)$ is defined by

$$(\mu_1 \odot \mu_2)\big((p_1,p_2), a, (q_1,q_2)\big) = \begin{cases} \mu_1(p_1, a, q_1), & \text{if } a \in \Sigma_1 \text{ and } p_2 = q_2 \\ \mu_2(p_2, a, q_2), & \text{if } a \in \Sigma_2 \text{ and } p_1 = q_1 \\ 0 & \text{, otherwise} \end{cases}$$

$$\forall (p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2,\ a \in \Sigma_1 \cup \Sigma_2.$$

(ii) $i_1 \odot i_2 : (Q_1 \times Q_2) \to [0, \infty)$ is defined by $i_1 \odot i_2(p_1, p_2) = i_1(p_1)\cdot i_2(p_2)$.

(iii) $f_1 \odot f_2 : (Q_1 \times Q_2) \to [0, \infty)$ is defined by $f_1 \odot f_2(q_1, q_2) = f_1(q_1)\cdot f_2(q_2)$.

Then $M_1 \odot M_2$ is a mwfa, called the Cartesian composition of M_1 and M_2.

Theorem 23. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's, with $\Sigma_1 \cap \Sigma_2 = \phi$. Let $M_1 \odot M_2 = (Q_1 \times Q_2, \Sigma_1 \cup \Sigma_2, \mu_1 \odot \mu_2, i_1 \odot i_2, f_1 \odot f_2)$ be the Cartesian composition of M_1 and M_2. Then $\forall x \in \Sigma_1^* \cup \Sigma_2^*,\ x \neq \lambda$,

$$(\mu_1 \odot \mu_2)^*\big((p_1,p_2), x, (q_1,q_2)\big) = \begin{cases} \mu_1^*(p_1, x, q_1), & \text{if } x \in \Sigma_1^* \text{ and } p_2 = q_2 \\ \mu_2^*(p_2, x, q_2), & \text{if } x \in \Sigma_2^* \text{ and } p_1 = q_1 \\ 0 & \text{, otherwise} \end{cases}$$

Proof. Let $x \in \Sigma_1^* \cup \Sigma_2^*,\ x \neq \lambda$. i.e.,$x \in \Sigma_1^*$ (or) $x \in \Sigma_2^*$. [since $\Sigma_1 \cap \Sigma_2 = \phi$]
We prove this result by induction on $|x| = n$. Suppose that $x \in \Sigma_1^*$. For $n = 1$, then $x = a \in \Sigma_1$. Now,
$(\mu_1 \odot \mu_2)\big((p_1,p_2), a, (q_1,q_2)\big) = \mu_1(p_1, a, q_1)$ if $p_2 = q_2$.

Thus, the result is true for $n = 1$. Suppose the result is true $\forall y \in \Sigma_1^*$, such that $|y| = n - 1,\ n > 1$. Let $x = ya$ where $a \in \Sigma_1$ and $y \in \Sigma_1^*$. Now,

$$\begin{aligned}
&(\mu_1 \odot \mu_2)^*\big((p_1, p_2), x, (q_1, q_2)\big)\\
&\quad = (\mu_1 \odot \mu_2)^*\big((p_1, p_2), ya, (q_1, q_2)\big)\\
&\quad = \max_{(r_1,r_2)\in Q_1\times Q_2} \big\{(\mu_1 \odot \mu_2))^*\big((p_1, p_2), y, (r_1, r_2)\big) \cdot (\mu_1 \odot \mu_2)\big((r_1, r_2), a, (q_1, q_2)\big)\big\}\\
&\quad = \max_{r_1\in Q_1} \big\{(\mu_1 \odot \mu_2))^*\big((p_1, p_2), y, (r_1, p_2)\big) \cdot (\mu_1 \odot \mu_2)\big((r_1, p_2), a, (q_1, q_2)\big)\big\}\\
&\qquad\qquad\qquad\qquad [\text{Since } (\mu_1 \odot \mu_2))^*\big((p_1, p_2), y, (r_1, r_2)\big) = 0 \text{ if } r_2 \neq p_2]\\
&\quad = \max_{r_1\in Q_1} \big\{\mu_1^*(p_1, y, r_1) \cdot (\mu_1 \odot \mu_2)\big((r_1, p_2), a, (q_1, q_2)\big)\big\}\\
&\quad = \begin{cases} \max\limits_{r_1\in Q_1} \{\mu_1^*(p_1, y, r_1) \cdot \mu_1(r_1, a, q_1)\}, & \text{if } p_2 = q_2\\ 0, & \text{otherwise}\end{cases}\\
&\quad = \begin{cases} \mu_1^*(p_1, ya, q_1), & \text{if } p_2 = q_2\\ 0, & \text{otherwise}\end{cases}\\
&\quad = \begin{cases} \mu_1^*(p_1, x, q_1), & \text{if } p_2 = q_2\\ 0, & \text{otherwise}\end{cases}
\end{aligned}$$

The proof is similar, if $x \in \Sigma_2^*$. ■

Theorem 24. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. and let $\Sigma_1 \cap \Sigma_2 = \phi$. Then $\forall x \in \Sigma_1^*,\ \forall y \in \Sigma_2^*$,

$$\begin{aligned}
(\mu_1 \odot \mu_2)^*\big((p_1, p_2), xy, (q_1, q_2)\big) &= \mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, y, q_2)\\
&= (\mu_1 \odot \mu_2)^*\big((p_1, p_2), yx, (q_1, q_2)\big)\\
&\forall (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2.
\end{aligned}$$

Proof. Let $x \in \Sigma_1^*,\ y \in \Sigma_2^*,\ (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$.
Case (1): If $x = \lambda = y$, then $xy = \lambda$. Suppose $(p_1, p_2) = (q_1, q_2)$. Then $p_1 = q_1$ and $p_2 = q_2$. Hence $(\mu_1 \odot \mu_2)^*\big((p_1, p_2), xy, (q_1, q_2)\big) = 1 = 1 \cdot 1 = \mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, y, q_2)$. If $(p_1, p_2) \neq (q_1, q_2)$, then either $p_1 \neq q_1$ or $p_2 \neq q_2$. Thus $\mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, y, q_2) = 0$. Hence $(\mu_1 \odot \mu_2)^*\big((p_1, p_2), xy, (q_1, q_2)\big) = 0 = \mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, y, q_2)$. Therefore, $(\mu_1 \odot \mu_2)^*\big((p_1, p_2), xy, (q_1, q_2)\big) = \mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, y, q_2)$.
Case (2): If $x = \lambda$ and $y \neq \lambda$ or $x \neq \lambda$ and $y = \lambda$, then the result follows by theorem 23.
Case (3): Suppose $x \neq \lambda$ and $y \neq \lambda$. Now,

$$\begin{aligned}
&(\mu_1 \odot \mu_2)^*\big((p_1, p_2), xy, (q_1, q_2)\big)\\
&\quad = \max_{(r_1,r_2)\in Q_1\times Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1, p_2), x, (r_1, r_2)\big) \cdot (\mu_1 \odot \mu_2)^*\big((r_1, r_2), y, (q_1, q_2)\big)\big\}\\
&\quad = \max_{r_1\in Q_1, r_2\in Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1, p_2), x, (r_1, r_2)\big) \cdot (\mu_1 \odot \mu_2)^*\big((r_1, r_2), y, (q_1, q_2)\big)\big\}\\
&\quad = \max_{r_1\in Q_1} \big\{\max_{r_2\in Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1, p_2), x, (r_1, r_2)\big) \cdot (\mu_1 \odot \mu_2)^*\big((r_1, r_2), y, (q_1, q_2)\big)\big\}\big\}\\
&\quad = \max_{r_1\in Q_1} \big\{(\mu_1 \odot \mu_2)^*\big((p_1, p_2), x, (r_1, p_2)\big) \cdot (\mu_1 \odot \mu_2)^*\big((r_1, p_2), y, (q_1, q_2)\big)\big\}\\
&\qquad\qquad\qquad\qquad [\text{Since } (\mu_1 \odot \mu_2))^*\big((p_1, p_2), x, (r_1, r_2)\big) = 0 \text{ if } r_2 \neq p_2]\\
&\quad = \mu_1 \odot \mu_2)^*\big((p_1, p_2), x, (q_1, p_2)\big) \cdot (\mu_1 \odot \mu_2)^*\big((q_1, p_2), y, (q_1, q_2)\big)\\
&\qquad\qquad\qquad\qquad [\text{Since } (\mu_1 \odot \mu_2))^*\big((r_1, p_2), y, (q_1, q_2)\big) = 0 \text{ if } r_1 \neq q_1]\\
&\quad = \mu_1^*(p_1, x, q_1) \cdot \mu_2^*(p_2, y, q_2).
\end{aligned}$$

Similarly, $(\mu_1 \odot \mu_2)^*\big((p_1,p_2), yx, (q_1,q_2)\big) = \mu_1^*(p_1,x,q_1)\cdot\mu_2^*(p_2,y,q_2)$. Therefore, $(\mu_1 \odot \mu_2)^*\big((p_1,p_2), xy, (q_1,q_2)\big) = (\mu_1 \odot \mu_2)^*\big((p_1,p_2), yx, (q_1,q_2)$. Hence the theorem. ■

Theorem 25. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. and let $\Sigma_1 \cap \Sigma_2 = \phi$. Then $\forall w \in (\Sigma_1^* \cup \Sigma_2^*)$, $\exists\, u \in \Sigma_1^*,\ v \in \Sigma_2^*$ such that $(\mu_1 \odot \mu_2)^*\big((p_1,p_2), w, (q_1,q_2)\big) = (\mu_1 \odot \mu_2)^*\big((p_1,p_2), uv, (q_1,q_2)\big)$ $\forall (p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$.

Proof. Let $w \in (\Sigma_1 \cup \Sigma_2)^*$ and $(p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$. If $w = \lambda$, then we can choose $u = \lambda = v$. In this case the result is trivially true. Suppose $w \neq \lambda$. If $w \in \Sigma_1^*$ or $w \in \Sigma_2^*$, then the result is trivially true.
Case (1): If $w = xy,\ x \in \Sigma_1^+,\ y \in \Sigma_2^+$, then the result follows by theorem 24.
Case (2): Suppose $w = x_1 y x_2,\ x_1, x_2 \in \Sigma_1^*$ and $y \in \Sigma_2^*$, x_i and y are non-empty strings, $i = 1, 2$. Let $u = x_1x_2 \in \Sigma_1^*$ and $v = y \in \Sigma_2^*$. Thus,

$$\begin{aligned}
&(\mu_1 \odot \mu_2)^*\big((p_1,p_2), x_1 y x_2, (q_1,q_2)\big)\\
&\quad= \max_{(r_1,r_2)\in Q_1\times Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1,p_2), x_1, (r_1,r_2)\big)\cdot\\
&\qquad\qquad (\mu_1 \odot \mu_2)^*\big((r_1,r_2), y x_2, (q_1,q_2)\big)\big\}\\
&\quad[\text{ Now by theorem 24.}\\
&\quad(\mu_1 \odot \mu_2)^*\big((r_1,r_2), x_2 y, (q_1,q_2)\big) = (\mu_1 \odot \mu_2)^*\big((r_1,r_2), y x_2, (q_1,q_2)\big)\\
&\qquad\qquad \forall (r_1,r_2),(q_1,q_2) \in Q_1 \times Q_2.]\\
&\quad= \max_{(r_1,r_2)\in Q_1\times Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1,p_2), x_1, (r_1,r_2)\big)\cdot\\
&\qquad\qquad (\mu_1 \odot \mu_2)^*\big((r_1,r_2), x_2 y, (q_1,q_2)\big)\big\}\\
&\quad= (\mu_1 \odot \mu_2)^*\big((p_1,p_2), x_1 x_2 y, (q_1,q_2)\big)
\end{aligned}$$

Case (3): Suppose $w = y_1 x y_2,\ y_1, y_2 \in \Sigma_2^*$ and $x \in \Sigma_1^*$, y_i and x are non-empty strings, $i = 1, 2$.
Let $v = y_1 y_2 \in \Sigma_2^*$ and $u = x$. The proof of this case is similar to Case(2).
Case (4): Suppose $w = x_1 y_1 x_2 y_2,\ x_1, x_2 \in \Sigma_1^*$ and $y_1, y_2 \in \Sigma_2^*$, x_i and y_i are non-empty strings, $i = 1, 2$. Let $u = x_1 x_2 \in \Sigma_1^*$ and $v = y_1 y_2 \in \Sigma_2^*$. Then

$$\begin{aligned}
&(\mu_1 \odot \mu_2)^*\big((p_1,p_2), x_1 y_1 x_2 y_2, (q_1,q_2)\big)\\
&\quad= \max_{(r_1,r_2)\in Q_1\times Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1,p_2), x_1, (r_1,r_2)\big)\cdot\\
&\qquad\qquad (\mu_1 \odot \mu_2)^*\big((r_1,r_2), y_1 x_2 y_2, (q_1,q_2)\big)\big\}\\
&\quad= \max_{(r_1,r_2)\in Q_1\times Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1,p_2), x_1, (r_1,r_2)\big)\cdot\\
&\qquad\qquad (\mu_1 \odot \mu_2)^*\big((r_1,r_2), x_2 y_1 y_2, (q_1,q_2)\big)\big\}[\text{by Case(3)}]\\
&\quad= (\mu_1 \odot \mu_2)^*\big((p_1,p_2), x_1 x_2 y_1 y_2, (q_1,q_2)\big)\\
&\quad= (\mu_1 \odot \mu_2)^*\big((p_1,p_2), uv, (q_1,q_2)\big).
\end{aligned}$$

Case (5): Suppose $w = y_1 x_1 y_2 x_2,\ x_1, x_2 \in \Sigma_1^*,\ y_1, y_2 \in \Sigma_2^*$. Let $u = x_1 x_2 \in \Sigma_1^*$ and $v = y_1 y_2 \in \Sigma_2^*$. The proof of this case is similar to Case (4).
Case (6): Let $w = (\Sigma_1 \cup \Sigma_2)^*$.
Then $w = x_1 y_1 x_2 y_2 \cdots x_n y_n$ or $w = y_1 x_1 y_2 x_2 \cdots y_n x_n$, $x_i \in \Sigma_1^*$, $y_i \in \Sigma_2^*$, x_i and y_i are non-empty strings, $i = 1, 2, \cdots, n$. To be specific, let $w = x_1 y_1 x_2 y_2 \cdots x_n y_n$. Let $u = x_1 x_2 \cdots x_n \in \Sigma_1^*$, and $v = y_1 y_2 \cdots y_1 \in \Sigma_2^*$. To prove: $(\mu_1 \odot \mu_2)^*\big((p_1,p_2), w, (q_1,q_2)\big) = (\mu_1 \odot \mu_2)^*\big((p_1,p_2), uv, (q_1,q_2)\big)$
We prove this result by induction on n. If $n = 0, 1$ or 2, then the result is true by the previous cases. Suppose the result for all $z = x_1 y_1 x_2 y_2 \cdots x_{n-1} y_{n-1} \in (\Sigma_1 \cup \Sigma_2)^*$, $n \geq 2$. Let $u_1 = x_1 x_2 \cdots x_{n-1}$, $v_1 = y_1 y_2 \cdots y_{n-1}$, $u = u_1 x_n$ and

$v = v_1 y_n$. Now,

$$
\begin{aligned}
&(\mu_1 \odot \mu_2)^*\big((p_1, p_2), w, (q_1, q_2)\big) \\
&= (\mu_1 \odot \mu_2)^*\big((p_1, p_2), x_1 y_1 x_2 y_2 \cdots x_n y_n, (q_1, q_2)\big) \\
&= \max_{(r_1,r_2)\in Q_1\times Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1, p_2), x_1 y_1 x_2 y_2 \cdots x_{n-1} y_{n-1}, (r_1, r_2)\big) \\
&\qquad\qquad \cdot (\mu_1 \odot \mu_2)^*\big((r_1, r_2), x_n y_n, (q_1, q_2)\big)\big\} \\
&= \max_{(r_1,r_2)\in Q_1\times Q_2} \big\{(\mu_1 \odot \mu_2)^*\big((p_1, p_2), u_1 v_1, (r_1, r_2)\big) \cdot \\
&\qquad\qquad (\mu_1 \odot \mu_2)^*\big((r_1, r_2), x_n y_n, (q_1, q_2)\big)\big\} \\
&= (\mu_1 \odot \mu_2)^*\big((p_1, p_2), u_1 v_1 x_n y_n, (q_1, q_2)\big) \\
&= (\mu_1 \odot \mu_2)^*\big((p_1, p_2), uv, (q_1, q_2)\big)
\end{aligned}
$$

$(\mu_1 \odot \mu_2)^*\big((p_1, p_2), w, (q_1, q_2)\big) = (\mu_1 \odot \mu_2)^*\big((p_1, p_2), uv, (q_1, q_2)\big)$. ■

Theorem 26. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's with L_1 and L_2 as a weighted regular languages respectively. Then L is the weighted regular language accepted by mwfa $M_1 \odot M_2$ such that $L(w) = L_1(u) \cdot L_2(v)$, $\forall w \in L$.

Proof. Let $M = M_1 \odot M_2$ and L be the weighted regular language accepted by M. Let Let $w \in (\Sigma_1 \cup \Sigma_2)^*$ and $L(w) > 0$.

$$
\begin{aligned}
L(w) &= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)} \big\{i_1 \odot i_2(p_1, p_2) \cdot (\mu_1 \odot \mu_2)^*\big((p_1, p_2), w, (q_1, q_2)\big) \cdot f_1 \odot f_2(q_1, q_2)\big\} \\
&\qquad \Big[\text{ Since } \forall w \in (\Sigma_1^* \cup \Sigma_2^*),\ \exists\, u \in \Sigma_1^*,\ v \in \Sigma_2^* \text{ such that} \\
&\qquad (\mu_1 \odot \mu_2)^*\big((p_1, p_2), w, (q_1, q_2)\big) = (\mu_1 \odot \mu_2)^*\big((p_1, p_2), uv, (q_1, q_2)\big)\Big] \\
&= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)} \big\{i_1 \odot i_2(p_1, p_2) \cdot (\mu_1 \odot \mu_2)^*\big((p_1, p_2), uv, (q_1, q_2)\big) \cdot f_1 \odot f_2(q_1, q_2)\big\} \\
&= \max_{(p_1,p_2),(q_1,q_2)\in(Q_1\times Q_2)} \big\{i_1(p_1) \cdot i_2(p_2) \cdot \mu_1^*(p_1, u, q_1) \cdot \mu_2^*(p_2, v, q_2) \cdot f_1(q_1) \cdot f_2(q_2)\big\} \\
&= \max_{p_1,q_1\in Q_1} \big\{i_1(p_1) \cdot \mu_1^*(p_1, u, q_1) \cdot f_1(q_1)\big\} \cdot \\
&\qquad \max_{p_2,q_2\in Q_2} \big\{i_2(p_2) \cdot \mu_2^*(p_2, v, q_2) \cdot f_2(q_2)\big\}
\end{aligned}
$$

Therefore $L(w) = L_1(u) \cdot L_2(v)$. ■

Definition 27. Let $M = (Q, \Sigma, W, \mu, i, f)$ be a mwfa. M is said to be retrievable if $\forall p \in Q$, $\forall u \in \Sigma^*$ if $\exists\, q \in Q$ such that $\mu^*(p, u, q) > 0$, then $\exists\, v \in \Sigma^*$ such that $\mu^*(q, v, p) > 0$.

Theorem 28. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's. and let $\Sigma_1 \cap \Sigma_2 = \phi$. Then the Cartesian composition $M_1 \odot M_2$ is retrievable if and only if M_1 and M_2 are retrievable.

Proof. Suppose that M_1 and M_2 are retrievable. Let $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$ and $w \in (\Sigma_1 \cup \Sigma_2)^*$ be such that $(\mu_1 \odot \mu_2)^*\big((p_1, p_2), w, (q_1, q_2)\big) > 0$. Let $w = uv$ be the standard form of w, $u \in \Sigma_1^*$, $v \in \Sigma_2^*$. Then

$$
\begin{aligned}
(\mu_1 \odot \mu_2)^*\big((p_1, p_2), w, (q_1, q_2)\big) &= (\mu_1 \odot \mu_2)^*\big((p_1, p_2), uv, (q_1, q_2)\big) \\
&= \mu_1^*(p_1, u, q_1) \cdot \mu_2^*(p_2, v, q_2)
\end{aligned}
$$

Thus $\mu_1^*(p_1, u, q_1) > 0$, and $\mu_2^*(p_2, v, q_2) > 0$. Since M_1 and M_2 are retrievable, $\exists u' \in \Sigma_1^*$, $v' \in \Sigma_2^*$ such that $\mu_1^*(q_1, u', p_1) > 0$, and $\mu_2^*(q_2, v', p_2) > 0$. Thus $(\mu_1 \odot \mu_2)^*\big((q_1, q_2), u'v', (p_1, p_2)\big) = \mu_1^*(q_1, u', p_1) \cdot \mu_2^*(q_2, v', p_2) > 0$. Hence M_1 and M_2 are retrievable. Conversely, suppose that $M_1 \odot M_2$ is retrievable. Let $p_1, q_1 \in Q_1$ and $y \in \Sigma_1^*$ be such that $\mu_1^*(p_1, y, q_1) > 0$. Then $\forall q_2 \in Q_2, (\mu_1 \odot \mu_2)^*\big((p_1, q_2), y, (q_1, q_2)\big) = \mu_1^*(p_1, y, q_1) > 0$. Thus $\exists\, w \in (\Sigma_1 \cup \Sigma_2)^*$

such that $(\mu_1 \odot \mu_2)^*\big((q_1, q_2), w, (p_1, q_2)\big) > 0$. Let $w = uv$ be the standard form of w, $u \in \Sigma_1^*$, $v \in \Sigma_2^*$. Then $(\mu_1 \odot \mu_2)^*\big((q_1, q_2), w, (p_1, q_2)\big) > 0$
$\mu_1^*(q_1, u, p_1) \cdot \mu_2^*(q_2, v, p_2) > 0$. Thus $\mu_1^*(q_1, u, p_1) > 0$. Hence M_1 is retrievable. Similarly M_2 is retrievable. ■

Definition 29. Let $M = (Q, \Sigma, W, \mu, i, f)$ be an mwfa. Let $p, q \in Q$. Then p and q are said to be connected if either $q = p$ or there exists $q_0, q_1, \ldots, q_k \in Q, p = q_0, q = q_k$ and there exists $a_1, a_2, a_3, \ldots, a_k \in \Sigma$ such that $\forall i = 1, 2, \cdots, k$, either $\mu(q_{i-1}, a_i, q_i) > 0$ or $\mu(q_i, a_i, q_{i-1}) > 0$.

Theorem 30. Let $M_1 = (Q_1, \Sigma_1, W, \mu_1, i_1, f_1)$ and $M_2 = (Q_2, \Sigma_2, W, \mu_2, i_2, f_2)$ be two mwfa's and let $\Sigma_1 \cap \Sigma_2 = \phi$. Then the Cartesian composition $M_1 \odot M_2$ is connected if and only if M_1 and M_2 are connected.

Proof. Suppose that M_1 and M_2 are connected. Let $(p, p')(q, q') \in Q_1 \times Q_2$. Now $\exists\, p_0, p_1, \cdots, p_n \in Q_1$, $p = p_0$, $q = p_n$ and $\exists\, a_1, a_2, \cdots, a_n \in \Sigma_1$ such that $\forall i = 1, 2, \cdots, n$ either $\mu_1(p_{i-1}, a_i, p_i) > 0$ or $\mu_1(p_i, a_i, p_{i-1}) > 0$. and $\exists p_0', p_1', \cdots p_m' \in Q_2$, $p' = p_0'$, $q' = p_m'$ and $\exists b_1, b_2, \cdots, b_m \in \Sigma_2$ such that $\forall i = 1, 2, \cdots, m$ either $\mu_2(p_{i-1}, b_i, p_i) > 0$ or $\mu_2(p_i', b_i, p_{i-1}') > 0$. Consider the sequence of states,

$$\begin{aligned}(p, p') &= (p_0, p_0'), (p_1, p_0'), \cdots, (p_n, p_0'), (p_n, p_1'), \cdots, (p_n, p_m') \\ &= (q, q') \in Q_1 \times Q_2.\end{aligned}$$

and the sequence $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m \in \Sigma_1 \cup \Sigma_2$. Then $\forall i = 1, 2, \cdots, n$ either $(\mu_1 \odot \mu_2)\big((p_{i-1}, p_0'), a_i, (p_i, p_0')\big) > 0$ or $(\mu_1 \odot \mu_2)\big((p_i, p_0'), a_i, (p_{i-1}, p_0')\big) > 0$. $\forall j = 1, 2, \cdots, m$ either $(\mu_1 \odot \mu_2)\big((p_n, p_{j-1}'), b_j, (p_n, p_j')\big) > 0$ or $(\mu_1 \odot \mu_2)\big((p_n, p_j'), b_j, (p_n, p_{j-1}')\big) > 0$. Hence (p, p') and (q, q') are connected. Conversely, Suppose that $M_1 \odot M_2$ is connected. Let $p, q \in Q_1$ and let $r \in Q_2$. If $p = q$ then p and q are connected. Suppose $p \neq q$, then $\exists\, (p, r) = (p_0, q_0), (p_1, q_1), \cdots, (p_n, q_n) = (q, r) \in Q_1 \times Q_2$ and $a_1, a_2, \cdots, a_n \in \Sigma_1 \cup \Sigma_2$ such that $\forall i = 1, 2, \cdots, n$ either $(\mu_1 \odot \mu_2)\big((p_{i-1}, q_{i-1}), a_i, (p_i, q_i)\big) > 0$ or $(\mu_1 \odot \mu_2)\big((p_i, q_i), a_i, (p_{i-1}, q_{i-1})\big) > 0$. Clearly, if $p_{i-1} \neq p_i$ then $q_{i-1} = q_i$ and if $q_{i-1} \neq q_i$ then $p_{i-1} = p_i$ $\forall i = 1, 2, \cdots, n$. Let $\{p = p_{i_1}, p_{i_2} \cdots, p_{i_k} = q\}$ be the set of all distinct $p_i \in \{p_0, p_1, \cdots, p_n\}$ and let $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ be the corresponding $a_i's$. Then $a_{i_1}, a_{i_2}, \cdots, a_{i_k} \in \Sigma_1$ and $\forall j = 1, 2, \cdots, k$ either $\mu_1(p_{i_{j-1}}, a_{i_j}, p_{i_{j-1}}) > 0$ or $\mu_1(p_{i_j}, a_{i_j}, p_{i_{j-1}}) > 0$. Thus, p and q are connected and hence M_1 is connected. Similarly M_2 is connected. ■

IMAGES OF A MORPHISM

In this section, we give some construction on mwfa, related to a morphism $h : \Gamma^* \to \Sigma^*$, where Σ and Γ are two alphabets.

Definition 31. Let $h : \Gamma^* \to \Sigma^*$ (where Γ and Σ are two alphabets) be a morphism if $h(uv) = h(u)h(v)$, for all $u, v \in \Gamma^*$ and h is fine morphism if $h(a) \in \Sigma \cup \{\lambda\}$ for all $a \in \Gamma$.

Theorem 32. Inverse image: Let $h : \Gamma^* \to \Sigma^*$ be a fine morphism. If $L_1 \subset \Sigma^*$ is recognized by a Σ- mwfa, then there exists a Γ-mwfa with weighted regular language L such that, $L(h^{-1}(x)) = L_1(x)$, for all $x \in L_1$.

Proof. Let $M_1 = (Q, \Sigma, W, \mu_1, i_1, f_1)$ be a $\Sigma-$ mwfa. L_1 be the weighted regular language accepted by M_1. Define a $\Gamma-$ mwfa M by $M = (Q, \Gamma, W, \mu, i, f)$, where

(i) $\mu : Q \times \Gamma \times Q \to [0, \infty)$ is defined as follows:

  (a) $\mu(p, a, q) = \mu_1(p, a', q)$, if $h(a) = a', a \in \Gamma, a' \in \Sigma$.
  (b) $\mu(q, a, q) = 1, \forall q \in Q$ if $h(a) = \lambda$.

(ii) $i : Q \to [0, \infty]$ is defined by

$$i(p) = \begin{cases} i_1(p) & , \text{ if } p \in Q \\ 0 & , \text{ otherwise} \end{cases}$$

(iii) $f : Q \to [0, \infty)$ is defined by

$$f(p) = \begin{cases} f_1(p) & , \text{ if } p \in Q \\ 0 & , \text{ otherwise} \end{cases}$$

Let L be the weighted regular language accepted by M. To prove: $L(h^{-1}(x)) = L_1(x)$, for all $x \in L_1$. Let $x = a'_1 a'_2 \cdots a'_n \in \Sigma^*$ and $L_1(x) > 0$. Since h is fine, assume that $h(a_i) = a'_i$, for $i = 1, 2, \ldots, n$. So,

$$\begin{aligned} h^{-1}(x) =& h^{-1}(a'_1 a'_2 \cdots a'_n) \\ =& h^{-1}(a'_1) h^{-1}(a'_2) \cdots h^{-1}(a'_n) \\ h^{-1}(x) =& a_1 a_2 \cdots a_n \end{aligned}$$

First we have to prove $\mu^*(p, h^{-1}(x), q) = \mu_1^*(p, x, q)$ $\forall x \in \Sigma^*$ and $\forall p, q \in Q$. We prove this result by induction on $|x| = n$. If $n = 1$, we have $x = a'_1$. Now $\mu(p, h^{-1}(a'_1), q) = \mu(p, a_1, q) = \mu_1(p, a'_1, q)$. Thus the result is true for $n = 1$. Assume that the result is true for all strings of length $\leq n - 1$. Consider

$$\begin{aligned} \mu^*\big(p, h^{-1}(a'_1 a'_2 \cdots a'_{n-1} a'_n), q\big) =& \mu^*\big(p, h^{-1}(a'_1 a'_2 \cdots a'_{n-1}) h^{-1}(a'_n), q\big) \\ =& \max_{r \in Q} \left\{ \mu^*(p, h^{-1}(a'_1 a'_2 \cdots a'_{n-1}), r) \cdot \mu(r, h^{-1}(a'_n), q) \right\} \\ =& \max_{r \in Q} \left\{ \mu_1^*(p, a'_1 a'_2 \cdots a'_{n-1}, r) \cdot \mu_1(r, a'_n, q) \right\} \\ =& \mu_1^*(p, a'_1 a'_2 \cdots a'_n, q) \\ \mu^*(p, h^{-1}(x), q) =& \mu_1^*(p, x, q) \end{aligned}$$

Now, for $x \in \Sigma^*$.

$$\begin{aligned} L(h^{-1}(x)) =& \max_{p,q \in Q} \{ i(p) \cdot \mu^*(p, h^{-1}(x), q) \cdot f(q) \} \\ =& \max_{p,q \in Q} \{ i_1(p) \cdot \mu_1^*(p, x, q) \cdot f_1(q) \} \\ =& L_1(x) \\ L(h^{-1}(x)) =& L_1(x) \ \forall x \in L_1. \end{aligned}$$

■

Definition 33. A function $h : \Gamma^* \to \Sigma^*$ be a morphism satisfying
$h(a) \neq \lambda$, $\forall a \in \Gamma$ (or equivalently $h^{-1}(\lambda) = \lambda$ or still equivalently that $|x| \leq |h(x)|$ for all $x \in \Gamma^*$).

Theorem 34. Direct image: Let $h : \Gamma^* \to \Sigma^*$ be a morphism such that $h(a) \neq \lambda, \forall a \in \Gamma$. If $L \subset \Gamma^*$ is recognized by a Γ- mwfa, then there exists a Σ-mwfa with weighted regular language L_1 such that, $L_1(h(x)) = L(x)$, for all $x \in L$.

Proof. Let $M = (Q, \Gamma, W, \mu, i, f)$ be a $\Gamma-$ mwfa.
Let L be the weighted regular language accepted by M. Define a $\Sigma-$ mwfa $M_1 = (Q_1, \Sigma, W, \mu_1, i_1, f_1)$ with $Q \subseteq Q_1$, where

(i) $\mu_1 : Q_1 \times \Sigma \times Q_1 \to [0, \infty)$ is defined as

(a) If $h(a) = a^{'}$, and $\mu(p, a, q) = m > 0$, then include
$\mu_1(p, a^{'}, q) = \mu(p, a, q)$.

(b) If $x = a_1 a_2 \cdots a_n \in \Gamma^*$, then

$$\begin{aligned} h(x) =& h(a_1 a_2 \cdots a_n) \\ =& h(a_1) h(a_2) \cdots h(a_n) \\ h(x) =& a'_1 a'_2 \cdots a'_n. \end{aligned}$$

then include $\mu_1^*(p, a'_1 a'_2 \cdots a'_n, q) = \mu_1(p, a'_1, p_1) \cdot \mu_1(p_1, a'_2, p_2) \cdots \mu_1(p_{n-1}, a'_n, q)$, where $p_1,\ p_2,\ p_3, , ,\ p_{n-1}$ are new states that are included to in Q_1.

(ii) $i_1 : Q_1 \to [0, \infty]$ is defined by

$$i_1(p) = \begin{cases} i(p) & , \quad \text{if } p \in Q \\ 0 & , \quad \text{otherwise} \end{cases}$$

(iii) $f_1 : Q_1 \to [0, \infty)$ is defined by

$$f_1(p) = \begin{cases} f(p) & , \quad \text{if } p \in Q \\ 0 & , \quad \text{otherwise} \end{cases}$$

L_1 be the weighted regular language accepted by M_1.
Claim: $L_1(h(x)) = L(x)$, $\forall x \in L$. Let $x = a_1 a_2 \cdots a_n \in \Gamma^*$, and $L(x) > 0$. $h(x) = h(a_1)h(a_2)\cdots h(a_n) = a'_1 a'_2 \cdots a'_n$, $h(a_i) = a'_i$, $i = 1, 2, \ldots, n$. First we have to prove $\mu_1^*(p, h(x), q) = \mu^*(p, x, q)$ $\forall x \in \Gamma^*$ and $\forall p, q \in Q$. We prove this result by induction on $|x| = n$.
If $n = 1$, we have $x = a_1$. Then
$\mu_1(p, h(a_1), q) = \mu_1(p, a'_1, q) = \mu(p, a_1, q)$.
Therefore, $\mu_1(p, h(x), q) = \mu(p, x, q)$. Thus the result is true for $n = 1$. Assume that the result is true for all strings of length $\leq n - 1$. Now

$$\begin{aligned} \mu_1^*\big(p, h(x), q\big) =& \mu_1^*\big(p, h(a_1 a_2 \cdots a_{n-1} a_n), q\big) \\ =& \max_{r \in Q} \big\{\mu_1^*(p, h(a_1 a_2 \cdots a_{n-1}), r) \cdot \mu_1(r, h(a_n), q)\big\} \\ =& \max_{r \in Q} \big\{\mu^*(p, a_1 a_2 \cdots a_{n-1}, r) \cdot \mu(r, a_n, q)\big\} \\ =& \mu^*(p, a_1 a_2 \cdots a_n, q) \\ \mu_1^*(p, h(x), q) =& \mu^*(p, x, q) \end{aligned}$$

Now, for $x \in \Sigma^*$,

$$\begin{aligned} L_1(h(x)) =& \max_{p,q \in Q_1} \{i_1(p) \cdot \mu_1^*(p, h(x), q) \cdot f_1(q)\} \\ =& \max_{p,q \in Q} \{i(p) \cdot \mu^*(p, x, q) \cdot f(q)\} \\ =& L(x) \\ L_1(h(x)) =& L(x) \ \forall x \in L. \end{aligned}$$

■

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CONCLUSION

In this paper, the structure of a mwfa is examined through the concepts of shuffle product, general direct product, cascade product, wreath product and some results are obtained. It is shown that the Cartesian composition of a mwfa preserve the properties of retrievability and connectedness.

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