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Improved Likelihood Ratio Tests in Power Series Generalized Nonlinear Models

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Abstract: *The power series generalized nonlinear models*, recently proposed in the literature, is a class of discrete nonlinear regression models with the aim of generalizing several widely known counting regression models such as the generalized Poisson and generalized negative binomial, among others. In this paper, improved versions for likelihood ratio statistic for hypothesis testing in this class are presented. Performance of bootstrap-based improved versions of the test statistics are addressed. Based on Bartlett correction theory, it is possible to ensure, for the improved statistics, an asymptotic χ^2 distribution up to order $O(n^{-1})$. We proposed a bootstrap-based numeric estimation of such correction factor. Monte Carlo simulations show that the proposed improvements display reliable finite-sample behaviour, outperforming the original tests. The usefulness of the improved tests is also shown by means of a real data set.

Keywords: Bartlett correction, bootstrap, bootstrap-Bartlett correction, likelihood ratio statistic.

INTRODUCTION

Regression models are oftentimes used to model the behaviour of a variable of interest conditioned on a set of explanatory variables. In some scenarios, a nonlinear relation might arise between them. Such situation is modelled using a nonlinear link function involving the mean of the response variable and the nonlinear regression structure of the explanatory variables. The count data occur in several different areas where widely known models, such as Poisson and negative binomial, are likely to be used, see [1]. With the aim of generalize regression models for counting data into a single conceptual object, [2] proposed the power series generalized nonlinear model (PSGNLM) as a global alternative in order to deal with discrete data.

Inference over the parameter vector in the PSGNLM may be carried out using the likelihood ratio (LR) test, proposed in [3]. These tests are based on approximations of critical values of the asymptotic distribution (χ^2) of the test statistic under the null hypothesis. Thus, those are approximated tests and size distortions are likely to arise in small samples. This happens because when the number of observations is not large the exact null distribution of the test statistic is oftentimes poorly approximated by its asymptotic counterpart. It is a common practice to modify the test statistic in order to obtain more reliable results, see [4]. Bartlett procedures, introduced in [5], are a natural alternative to perform the transformation of the test statistic.

Bartlett corrections are not easy to obtain once they deliver asymptotic refinements based on the general result given by [6]. In [7] matrix notation for such refinements were introduced. One alternative to avoid the difficult derivation of Bartlett correction is calculating them numerically using bootstrap resampling, see [8]. Such procedure is called *bootstrap Bartlett correction* and was pioneered in [9]. As examples of such procedures, in [10] was derived the Bartlett correction for partial likelihood ratio test in Cox regression models and [11] where Bartlett and Bartlett-bootstrap factor were obtained for beta regression model.

Our main goal in this paper is to derive Bootstrap-Bartlett improved version of the LR test statistic in PSGNLM. After extensive computational simulations, we obtain the such improved version by assuming that the dispersion parameter is known. We also consider the bootstrap correction, i.e., numerical calculation of the respective percentile, respectively. We perform extensive Monte Carlo simulations where the behaviour of the improved test statistics are tested in finite samples allowing us to conclude the usefulness of the proposed improved test statistics and their outstanding performance in relation to the classical test statistic.

The paper unfolds as follows. Section 2 presents the PSGNLM introduced in [2]. Section 3 shows the theory highlights on the Bartlett correction factor. Section 4 is devoted to introduce the bootstrap based correction procedure for the Bartlett correction and percentile estimation. Section 5 displays all the results of the Monte Carlo simulations made in several scenarios to assess the performance of the proposed improved test statistics. An application to real data (not simulated) is shown in Section 6. Finally, concluding remarks are offered in Section 7.

POWER SERIES GENERALIZED NON-LINEAR REGRESSION MODEL

Consider the discrete random variables $Y_1, \cdots, Y_n$ which are independent and each Y_i having a family of distributions with mean $\mu_i > 0$ and dispersion parameter $\phi > 0$ defined by the probability mass function

$$\pi(y; \mu_i, \phi) = \frac{a(y, \phi) g(\mu_i, \phi)^y}{f(\mu_i, \phi)}, \qquad (1)$$

where $y \in A_\epsilon = \{\epsilon, \epsilon + 1, \cdots\}$ (ϵ positive integer), $a(y, \phi)$ is positive and the analytical functions $g(\mu, \phi)$ and $f(\mu, \phi)$ of the parameters μ e ϕ are positive, finite and twice differentiable. Some distributions belonging to this class of models sucha as Poisson, Negative Binomial (NB), Generalized Poisson (GPO), Generalized negative binomial (GNB) and Binomial Delta (BD), are displayed in Table 1. We consider ϕ to be known. Is this class of models, we have

$$E(Y_i) = \mu_i = \frac{f_i' g_i}{g_i' f_i} \quad \text{and} \quad V(Y_i) = V(\mu_i, \phi) = \frac{g_i}{g_i'}$$

where $f_i = f(\mu_i, \phi)$ $g_i = g(\mu_i, \phi)$ and the symbol "$'$" denotes the differentiation in relation to μ. The mean of Y_i is related with the systematic component through a link function of the form

$$h(\mu_i) = \eta_i = \eta(x_i; \boldsymbol{\beta}), \qquad i = 1, \cdots, n, \qquad (2)$$

where $h(\,\cdot\,)$ is a known link function twice differentiable, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_p)^\top$ is a vector of size p ($p < n$) of unknown parameters to be estimated, $x_i = (x_{i1}, \cdots, x_{ik})^\top$ represents the value of the k explanatory variables and $\eta(\cdot, \cdot)$ is a possibly nonlinear function continuous and differentiable with respect to the components of $\boldsymbol{\beta}$. Let $\eta = (\eta_1, \cdots, \eta_n)^\top$ and from now on, we use the notation $h_i = h(\mu_i)$. The PSGNLM is defined by equations (1) and (2).

For a given PSGNLM, we are interested in the estimation of the parameter vector $\boldsymbol{\beta}$. Denote the sample observations by $\boldsymbol{y} = (y_1, \cdots, y_n)^\top$ and the corresponding total log-likelihood function is given by

$$\ell(\boldsymbol{\beta}; \boldsymbol{y}) = \sum_{i=1}^{n} [\,\log\{a(y_i, \phi)\} + y_i \log\{g(\mu_i, \phi)\} - \log\{f(\mu_i, \phi)\}]. \qquad (3)$$

The score vector function is expressed as

$$U_{\boldsymbol{\beta}} = \widetilde{X}^\top (T\,\boldsymbol{y} - Q), \qquad (4)$$

where $T = diag\{t_1, \ldots, t_n\}$ is a $n \times n$ diagonal matrix whose ith element is $t_i = \frac{g_i'}{g_i h_i'}$, $Q = (q_1, \ldots, q_n)^\top$ is a $n \times 1$ vector whose ith element is $q_i = \frac{f_i'}{f_i h_i'}$ and $\widetilde{X} = \widetilde{X}(\boldsymbol{\beta}) = \partial \eta / \partial \boldsymbol{\beta}^\top$ is a full rank matrix. Functions q_i and t_i for some PSNLGM are displayed in Table 2 The elements of $\widetilde{X}$ are, in general, functions of the parameters vector $\boldsymbol{\beta}$.

The total information matrix of $\boldsymbol{\beta}$ is given by

$$K_{\boldsymbol{\beta}} = \widetilde{X}^\top W \widetilde{X}, \qquad (5)$$

where W is a $n \times n$ diagonal matrix of weights defined as

$$w_i = \left(q_i' - \frac{f_i' g_i}{f_i g_i'} t_i' \right) \frac{1}{h_i'}.$$

The maximum likelihood estimator (MLE) of $\boldsymbol{\beta}$, say $\widehat{\boldsymbol{\beta}}$, must satisfy the nonlinear equation system $U_{\widehat{\boldsymbol{\beta}}} = 0$ derived in (4). Under mild regularity conditions, see [12], $\widehat{\boldsymbol{\beta}}$ is, indeed, the solution of such system.

BARTLETT CORRECTION

We assume $\ell(\boldsymbol{\beta}; \boldsymbol{y})$ to be regular with respect the components of $\boldsymbol{\beta}$ up to fourth order. Consider the partition $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^\top, \boldsymbol{\beta}_2^\top)^\top$, where $\boldsymbol{\beta}_1 = (\beta_1, \ldots, \beta_q)^\top$ is the vector of parameters of interest and $\boldsymbol{\beta}_2 = (\beta_{q+1}, \ldots, \beta_p)^\top$ the

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Table 1. Functions $f(\mu_i, \phi)$, $g(\mu_i, \phi)$ and $a(y_i, \phi)$ for some distributions.

Distribution	$f(\mu_i, \phi)$	$g(\mu_i, \phi)$	$a(y_i, \phi)$
1. Poisson	e^{μ_i}	μ_i	$\frac{1}{y_i!}$
2. Binomial	$\left(1 + \frac{\mu_i}{m - \mu_i}\right)^m$	$\frac{\mu_i}{m - \mu_i}$	$\binom{m}{y_i}$
3. Generalized Poisson	$e^{\mu_i (1+\mu_i \phi)^{-1}}$	$\frac{1}{(1+\phi\mu_i)^2 h'}$	$\frac{(1+\phi y_i)^{y_i - 1}}{y_i!}$
4. Consul	$\frac{\mu_i - 1}{\mu_i(\phi - 1) + 1}$	$\phi^{-\phi}(1 - \frac{1}{\mu_i})(\phi - 1 + \frac{1}{\mu_i})^{\phi - 1}$	$\frac{\tau(\phi y_i + 1)}{y_i! \tau(\phi y_i - y_i + 2)}$
5. Generalized Negative Binomial	$\left(\frac{\phi - 1 + \nu/\mu_i}{\phi + \nu/\mu_i}\right)^{-\nu}$	$\frac{1}{\phi + \nu/\mu_i}\left(\frac{\phi - 1 + \nu/\mu_i}{\phi + \nu/\mu_i}\right)^{\phi - 1}$	$\frac{\nu \tau(\phi y_i + \nu + 1)}{(\phi y_i + \nu) y_i! \tau(\phi y_i - y_i + \nu + 1)}$
6. Delta Binomial	$\left\{\frac{\mu_i - m}{\mu_i(\phi - 1) + m}\right\}^m$	$\frac{1}{\phi^\phi}\left(1 - \frac{m}{\mu_i}\right)\left(\phi - 1 + \frac{m}{\mu_i}\right)^{\phi - 1}$	$\frac{m\tau(\phi y + 1)}{y(y - m)! \tau(\phi y - y + m + 1)}$

Table 2. Functions $t(\mu_i, \phi)$ and $q(\mu_i, \phi)$ for some distributions.

Distribution	$t(\mu_i, \phi)$	$q(\mu_i, \phi)$
1. Poisson	$\frac{1}{\mu_i h'}$	$\frac{1}{h'}$
2. Binomial	$\frac{m}{\mu_i(m - \mu)h'}$	$\frac{m}{(m - \mu_i)h'}$
3. Generalized Poisson	$\frac{1}{\mu_i(1+\phi\mu_i)^2 h'}$	$\frac{1}{(1+\phi\mu_i)^2 h'}$
4. Consul	$\frac{\phi}{\mu_i(\mu_i - 1)\{\mu_i(\phi - 1) + 1\}h'}$	$\frac{\phi}{(\mu_i - 1)\{\mu_i(\phi - 1) + 1\}h'}$
5. Generalized Negative Binomial	$\frac{\nu^2}{\mu_i(\nu + \mu_i\phi)\{\nu + \mu_i(\phi - 1)\}h'}$	$\frac{\nu^2}{(\nu + \mu_i\phi)\{\nu + \mu_i(\phi - 1)\}h'}$
6. Delta Binomial	$\frac{m^2\phi}{\mu_i\{\mu_i(\phi - 1) + m\}(\mu_i - m)h'}$	$\frac{m^2\phi}{\{\mu_i(\phi - 1) + m\}(\mu_i - m)h'}$

vector of nuisance parameters. Such decomposition leads us to the partition $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)$, where $\widetilde{X}_1 = \partial\eta/\partial\boldsymbol{\beta}_1$ and $\widetilde{X}_2 = \partial\eta/\partial\boldsymbol{\beta}_2$. In many situations, we are interested in testing the hypothesis $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^{(0)}$ against $H_1 : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^{(0)}$, where $\boldsymbol{\beta}_1^{(0)}$ is a specified vector of dimension q $(q \leq p)$.

In order to perform such hypothesis testing procedure, the LR statistic is given by

$$LR = 2\{\ell(\widehat{\boldsymbol{\beta}}; \boldsymbol{y}) - \ell(\widetilde{\boldsymbol{\beta}}; \boldsymbol{y})\} \quad (6)$$

where $\widehat{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}_1^\top, \widehat{\boldsymbol{\beta}}_2^\top)^\top$ is the unrestricted MLE of $\boldsymbol{\beta}$ and $\widetilde{\boldsymbol{\beta}} = (\boldsymbol{\beta}_1^{(0)\top}, \widetilde{\boldsymbol{\beta}}_2^\top)^\top$ is the restricted MLE of $\boldsymbol{\beta}$.

Under H_0, the LR statistic has a χ^2_q reference distribution. So, one rejects the null hypothesis H_0, with significance level α, if $LR > \chi^2_{(\alpha;q)}$ with error up to order n^{-1} in large samples sizes, where $\chi^2_{(\alpha;q)}$ is the $(1 - \alpha)$ percentile of the distribution χ^2_q. It is noteworthy that, in small samples size, approximation of the LR statistics by a χ^2 distribution might not be accurate, thus, leading to distorted results. For the purpose of correcting the distortion, likely to arise in such small sample size scenarios, a correction factor for the LR statistic was proposed in [5], leading to a modified version, say LR^*, whose mean is closer to the expected value of the χ^2_q distribution. Under mild regularity conditions, see [13], a Taylor series expansion of $\ell(\widehat{\beta})$ is obtained in [6] under H_0, involving derivatives up to fourth order of the log likelihood function. It was proven that

$$2E\left[\ell(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2) - \ell(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right] = p + \epsilon_p + O(n^{-2}), \quad (7)$$

where p is the number of parameters of $\boldsymbol{\beta}$ and the term ϵ_p of order n^{-1} can be expressed as

$$\epsilon_p = \sum (\lambda_{rstu} - \lambda_{rstuvw}), \quad (8)$$

where $\sum$ varies over the elements of $\boldsymbol{\beta}$ and the quantities λ_{rstu} and λ_{rstuvw} are functions of the joint cumulants of derivatives of the log-likelihood function and derivatives of such cumulants, computed as

$$\lambda_{rstu} = \kappa^{rs}\kappa^{tu}\left(\kappa_{rstu}/4 - \kappa_{rst}^{(u)} + \kappa_{rt}^{(su)}\right) \quad \text{and}$$

$$\lambda_{rstuvw} = \kappa^{rs}\kappa^{tu}\kappa^{vw}\Big\{\kappa_{rtv}\left(\kappa_{suw}/6 - \kappa_{sw}^{(u)}\right) + \kappa_{rtu}\left(\kappa_{svw}/4 - \kappa_{sw}^{(v)}\right) + \kappa_{rt}^{(v)}\kappa_{sw}^{(u)} + \kappa_{rt}^{(u)}\kappa_{sw}^{(v)}\Big\}.$$

Terms of the form $\kappa^{rs} = -\kappa^{r,s}$ are referred as the (r, s)th element of the inverse of information matrix defined in (5). Definitions of the other terms involved, besides the quantities λ_{rstu} and λ_{rstuvw}, are given in Appendix 1.

According to [6], we can express

$$2E\left[\ell(\boldsymbol{\beta}_1^{(0)}, \widetilde{\boldsymbol{\beta}}_2) - \ell(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right] = p - q + \epsilon_{p-q} + O(n^{-2}), \tag{9}$$

where ϵ_{p-q}, of order n^{-1}, is defined analogously to ϵ_p given in (8) extending $\sum$ just over the parameters of the $p - q$ vector $\boldsymbol{\beta_2}$ of nuisance parameters.

The LR statistic defined in (6) can be re-written as

$$LR = 2\left[\left\{\ell(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2) - \ell(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right\} - \left\{\ell(\boldsymbol{\beta}_1^{(0)}, \widetilde{\boldsymbol{\beta}}_2) - \ell(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right\}\right],$$

and then, its expected value can be computed as

$$\begin{aligned} E(LR) &= 2E\left[\left\{\ell(\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2) - \ell(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right\} - \left\{\ell(\boldsymbol{\beta}_1^{(0)}, \widetilde{\boldsymbol{\beta}}_2) - \ell(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right\}\right] \\ &= q + \epsilon_p - \epsilon_{p-q} + O(n^{-2}) \\ &= q\left(1 + \frac{\epsilon_p - \epsilon_{p-q}}{q}\right) + O(n^{-2}). \end{aligned}$$

Thus, the approximation of the distribution of the LR statistic by the χ_q^2 distribution might be corrected by substituting LR by the improved statistic $LR^* = LR/(1 + d)$ or $LR_1^* = LR(1 - d)$, where the Bartlett correction factors $1/(1 + d)$ and $(1 - d)$ are determined from

$$d = \frac{\epsilon_p - \epsilon_{p-q}}{q}. \tag{10}$$

Asymptotically and under the null hypothesis, the modified test statistics LR^* and LR_1^* have a χ_q^2 distribution up to order n^{-1} under H_0. It is noteworthy that, in the case of a simple null hypothesis $H_0 : \beta = \beta^{(0)}$, the quantity d given in (10) determines the Bartlett correction, reduces to $d = \epsilon_p/p$. The effect of such quantity is the one to be estimated via the following bootstrap procedure.

BOOTSTRAP-BARTLETT CORRECTION

In statistical inference problems, either in terms of analytical solutions or setting particular theoretical frameworks, are likely to arise. In these scenarios, the computational methods play an important role allowing the researcher to obtain approximate numerical solutions for such problems. The bootstrap method, introduced in [8], is one of the most suitable procedures to be used. Our main goal is to assess the performance of the bootstrap improved versions of the LR statistics. This will be carried out proposing an estimative for the required percentiles, rather than use the real corresponding percentile of the reference χ^2 distribution.

Denote as $\boldsymbol{y} = (y_1, \ldots, y_n)^\top$ the random sample. The bootstrap correction scheme may be explained using the following steps:

1. Build up B random pseudo-samples $(\boldsymbol{y}^{*1}, \ldots, \boldsymbol{y}^{*B})$, under the null hypothesis from the original sample $\boldsymbol{y}$.
2. Compute the value of the LR statistic for each pseudo-sample, i.e., for each pseudo-sample $\boldsymbol{y}^{*b}$, compute the corresponding test statistic, say LR^{*b}, for $b = 1, \ldots, B$ a
$$LR^{*b} = 2\left\{\ell\left(\widehat{\boldsymbol{\beta}}^{*b}; \boldsymbol{y}^{*b}\right) - \ell\left(\widetilde{\boldsymbol{\beta}}^{*b}; \boldsymbol{y}^{*b}\right)\right\}, \tag{11}$$
where $\widehat{\boldsymbol{\beta}}^{*b}$ and $\widetilde{\boldsymbol{\beta}}^{*b}$ are the MLEs of $\boldsymbol{\beta}$, under H_0 and H_1, respectively, obtained from the maximization of $\ell(\boldsymbol{\beta}; \boldsymbol{y}^{*b})$.
The percentile $1 - \alpha$ of LR^{*b} is estimated as the value $\widehat{q}_{(1-\alpha)}$ such that
$$\frac{1}{B}\sum_{b=1}^{B} I\left(\left\{LR^{*b} \le \widehat{q}_{(1-\alpha)}\right\}\right) = 1 - \alpha,$$
where $I(\,\cdot\,)$ is the indicator function. This is, $I(\{LR^{*b} \le \widehat{q}_{(1-\alpha)}\}) = 1$ if $LR^{*b} \le \widehat{q}_{(1-\alpha)}$ and zero, otherwise.
3. The value $\widehat{q}_{(1-\alpha)}$ is computed as follows: Order from smaller to bigger the B bootstrap test statistics LR^{*b}, $b = 1, \ldots, B$ previously obtained. The replication $B \times (1 - \alpha)$ is the estimated percentile, z_α, assuming $B \times (1-\alpha)$ integer. If not, use the following procedure: Let $k = \lfloor (B + 1) \times \alpha \rfloor$ be the biggest integer $\le (B + 1) \times \alpha$. Then, the quantity $\widehat{q}_{(1-\alpha)}$ is given by the $(B + 1 - k)$th element ordered of LR^{*b}.

The bootstrap LR test is based on rejecting H_0 if $LR > \widehat{q}_{(1-\alpha)}$. From now on, we denote by LR_{boot} the statistic obtained from this procedure, i.e., LR_{boot} is the same LR

statistic but is compared with the estimated percentile rather than the reference quantile of the χ^2 distribution. Further details of bootstrap based tests are available in [14].

A different approach might be as follows. In [9], a numerical alternative for calculating the Bartlett correction was introduced based on bootstrap re-sampling. The cornerstone idea is to estimate the test statistic expected value. Let B be the number of bootstrap re-samples and $(\boldsymbol{y}^{*1}, \ldots, \boldsymbol{y}^{*B})$ be the artificial re-sample generated under H_0. For each pseudo-sample y^{*b}, $b = 1, \ldots, B$ the LR statistic is computed as is equation (11). The bootstrap Bartlett corrected statistic is computed by

$$LR^*_{boot} = q\frac{LR}{\overline{LR^*}},$$

where $\overline{LR^*} = B^{-1}\sum_{b=1}^{B} LR^{*b}$.

The latter approach is expected to be more computationally efficient than the former, i.e., we expect that the bootstrap Bartlett correction (numerical computation of the Bartlett correction) yields to accurate inferences with only 200 replications since it estimates the mean of the distribution. The former approach (estimation of the respective critical quantile) is expected to perform well with 1,000 bootstrap replications, due to the fact that this procedures involves tail estimations.

NUMERICAL RESULTS

This section presents the Monte Carlo simulation results for under some scenarios with the aim to study the performance of the LR test, and two improved versions, namely, the bootstrap test (LR_{boot}), the Bartlett-bootstrap test (LR^*_{boot}). We set the number of Monte Carlo replications in 10,000, in 600 the bootstrap replications for LR_{boot} and in 300 for LR^*_{boot}. All simulations are carried out using `OxConsole`, see [15]. The non-linear maximization of the relevant log-likelihoods are carried out using the quasi-Newton algorithm known as the *BFGS*, see [16, Chap. 8] and [17, Chap 12].

To assess the performance of the five tests, we set the hypothesis $H_0 : (\beta_5, \beta_6) = (0, 0)$ against $H_1 : (\beta_5, \beta_6) \neq (0, 0)$. In order to study the size and power of the proposed tests, we set the sample sizes $n = 20, 30, 40, 50$ and the significance levels $\alpha = 1\%, 5\%, 10\%$. The Consul, generalized Poisson (GP) and generalized negative binomial (GNB) were the used distributions for the response variable, details of these distributions are displayed in table 1. We also used the logarithm as the link function, i.e., $h(\mu_i) = \log(\mu_i)$ and studied the impact when the number of nuisance regression parameters increases. We consider the non-linear models listed in Table 3.

The true values of the response variable is generated based on $\beta_5 = \beta_6 = 0$ and $\beta_i = 0.05$ for $i \in \{1, 2, 3, 4, 7\}$, while the covariates are generated as random variables from the following distributions: $LN(0, 1)$ (Log-normal), $F(2, 5)$, Cauchy, χ^2_3, Beta$(2, 3)$, $N(0, 2)$, $N(0, 1)$, respectively. Also, it is assumed that ϕ is fixed and known. For the GNB model, we set $\phi = 1$ and $\nu = 3$ while for the GP and Consul models, we set $\phi = 0.2$ and $\phi = 1$, respectively.

For each sample size and each nominal level, we compute the rejection rates of the tests based on the LR and LR^*_{boot} statistics, as the estimation, using Monte Carlo simulation, of the following probabilities: $P(LR \geq \chi^2_{(\alpha,q)})$, $P(LR^*_{\text{boot}} \geq \chi^2_{(\alpha,q)})$, respectively, where $\chi^2_{(\alpha,q)}$ is the $(1-\alpha)$ percentile of the χ^2_q distribution. For the test based on the LR_{boot}, the rejection rate is obtained from the estimation of the probability $P(LR \geq \widehat{q}_{(1-\alpha)})$, where $\widehat{q}_{(1-\alpha)}$ was presented in Section 4.

Rejection rates for the current tests for several sample sizes are given in Table 4. For each of the three general cases (varying the distribution of the response variable), it is noteworthy to note the largely liberal behaviour of the LR test, i.e., size distortions are likely to arise specially in small sample sizes scenarios. Nevertheless, the corrected test LR^*_{boot} display a considerably more reliable behaviour, through various sample sizes, than the original test. We report that such tests outperform the original test in terms of proximity to the real rejection rate for each case and decreasing the size distortions. For instance, in the GNB model with $n = 20$ and $\alpha = 10\%$, the rejection rate of LR is 14.7, which is a liberal rate, and the corrected test LR^*_{boot} yields the rate and 9.9, respectively.

In Table 5, simulation results for several nuisance parameter vector size (denoted as p), with $n = 30$, are presented. In this case, we report the changes of the LR test depending on the value of p, for example, going from 1.0 to 2.3, from 10.5% to 14.3% and from 5.1% to 8.1%, respectively, in the Consul model. It is noteworthy that the LR test is better suited when p is very small. Although the corrected test, outperform the original test displaying better rates for all cases (varying p), this is, reducing the original distortion size. For instance, in the Consul model with $\alpha = 10\%$ and $p = 8$, the rejection rate of LR is 14.3, which is a liberal rate, and the corrected test LR^*_{boot} yields the rate 9.3.

Table 3. Proposed models to assess the perform of the proposed improved test statistics

1. $\eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \beta_6 x_{i6} + \exp(\beta_7 x_{i7})$	$p = 8, p - q = 6$
2. $\eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_5 x_{i5} + \beta_6 x_{i6} + \exp(\beta_7 x_{i7})$	$p = 7, p - q = 5$
3. $\eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_5 x_{i5} + \beta_6 x_{i6} + \exp(\beta_7 x_{i7})$	$p = 6, p - q = 4$
4. $\eta_i = \beta_0 + \beta_1 x_{i1} + \beta_5 x_{i5} + \beta_6 x_{i6} + \exp(\beta_7 x_{i7})$	$p = 5, p - q = 3$
5. $\eta_i = \beta_0 + \beta_5 x_{i5} + \beta_6 x_{i6} + \exp(\beta_7 x_{i7})$	$p = 4, p - q = 2$
6. $\eta_i = \beta_5 x_{i5} + \beta_6 x_{i6} + \exp(\beta_7 x_{i7})$	$p = 3, p - q = 1$

Table 4. Rejection rates of $H_0 : \beta_5 = \beta_6 = 0$ of the LR, LR_{boot} and LR^*_{boot} statistics when $p = 8$ and various values of n.

n	$\alpha(\%)$	GNB model			Consul model			GPO model		
		LR	LR_{boot}	LR^*_{boot}	LR	LR_{boot}	LR^*_{boot}	LR	LR_{boot}	LR^*_{boot}
20	1	2,2	1,5	1,1	2,5	0,8	0,9	1,8	0,9	1,0
	5	8,2	8,1	5,3	8,6	4,7	4,8	7,3	4,0	4,9
	10	14,7	15,1	9,9	15,5	7,9	9,8	12,9	7,8	10,2
30	1	1,5	1,2	1,0	1,6	0,8	0,9	1,4	1,0	0,9
	5	6,4	5,9	5,0	7,1	5,8	4,5	6,3	4,7	5,0
	10	12,1	10,5	10,2	13,3	10,6	9,3	11,9	10,2	10,1
40	1	1,4	1,3	1,1	1,7	1,1	1,1	1,4	0,9	0,9
	5	6,1	5,9	5,0	6,8	4,8	4,9	6,1	5,0	5,0
	10	11,7	8,8	9,9	12,7	9,6	9,9	11,9	8,6	10,1
50	1	1,1	0,6	1,1	1,3	0,7	1,1	1,0	1,0	1,0
	5	5,5	4,0	5,1	5,9	4,9	5,2	5,5	4,5	4,9
	10	11,0	8,5	10,1	11,7	8,6	9,8	10,8	8,1	9,8

Power sizes for the current tests are also considered and given in Table 6. This is carried out by considering the variation on $\beta_5 = \beta_6 = \beta^{(0)}$ from 0.05 to 0.4 and then assessing the estimated rejection rates. In general, there is no loss of power in a significant level for any of the considered scenarios. In other words, all tests are equivalent in terms of power size. Thus, we have no loss of power in a significant level, replacing the original LR test by some of its corrected versions.

APPLICATION

In this section, our aim is to present the usefulness of the proposed statistics, previously defined, fitted to a real data set referring to the number of certain fish specie in one lake (dependent variable) and the natural logarithm of such lake area, given in km^2. Those data were initially studied in [18], lately by [19] and finally by [2]. The latter research discusses the flexibility of the GPSNLM in fitting these data employing the following predictors

$$\eta_i = \beta_0 + \beta_1 \log(x_i) \qquad (12)$$

and

$$\eta_i = \beta_0 + \beta_1 \log(x_i) + \beta_2 \{\log(x_i)\}^2 , \qquad (13)$$

$i = 1, \ldots, 70$, where $\eta_i = \log(\mu_i - m)$, and m denotes the minimum value of the support of the associated distribution. Note that the first model presented is linear and the second one is nonlinear. We perform the hypothesis test $H_0 : \beta_2 = 0$ against $H_1 : \beta_2 \neq 0$, i.e., if the model (12) is more adequate to represent the data set.

For doing this, we consider the models examined by [2]. They are: Poisson, NB, GPO, GNB and BD. The latter was the most adequate model in order to adjust the number of fish species, once it delivered the lower AIC value (Akaike information criteria). Such criterion were 610.9 and 614.1 using the non-linear predictors (12) and (13), respectively.

The values of the statistics LR and LR^*_{boot} given by the NB model were: 2.7447, and 2.2710, respectively, and the respective descriptive levels: 0.0976 and 0.13182. For the LR_{boot}, the estimated percentile was computed in 2.5627, leading to LR_{boot} to reject the null hypothesis. We note that when for the 10% nominal level, the involved tests lead to divergent results and only the LR and LR_{boot} indicate the rejection of H_0.

Table 5. Rejection rates of $H_0 : \beta_5 = \beta_6 = 0$ of the LR, LR_{boot} and LR^*_{boot} statistics when $n = 30$ and various values of p.

$\alpha(\%)$	p	GNB Model			Consul Model			GPO Model		
		LR	LR_{boot}	LR^*_{boot}	LR	LR_{boot}	LR^*_{boot}	LR	LR_{boot}	LR^*_{boot}
1	3	1,3	1,5	1,1	1,1	1,1	1,0	0,9	0,9	1,0
	4	0,9	0,8	1,0	1,0	0,9	1,0	0,9	0,9	1,0
	5	0,9	1,1	1,1	0,9	0,5	1,0	1,0	1,8	1,1
	6	1,1	1,1	0,9	1,1	0,7	1,0	1,0	1,2	1,1
	7	1,2	0,8	0,9	1,4	0,8	1,1	1,2	1,5	1,0
	8	1,5	1,2	1,0	1,6	0,8	0,9	1,4	1,0	0,9
5	3	5,8	5,8	5,2	5,8	5,9	5,1	5,3	5,5	5,0
	4	5,5	4,6	5,0	5,5	5,6	5,1	5,0	4,5	5,2
	5	5,7	5,7	5,3	5,8	4,7	5,1	5,3	5,6	5,1
	6	5,4	6,5	5,1	5,9	5,3	5,1	5,3	6,9	5,5
	7	6,5	4,8	5,3	7,0	5,7	5,0	6,1	7,2	4,9
	8	6,4	5,9	5,0	7,1	5,8	4,5	6,3	4,7	5,0
10	3	11,4	10,5	10,1	11,5	10,6	10,3	10,9	9,3	10,1
	4	11,0	9,3	10,2	11,2	9,7	10,2	10,7	9,3	10,1
	5	10,9	10,6	10,2	11,3	10,2	10,0	10,9	9,6	9,9
	6	10,9	13,5	10,1	11,6	10,4	9,8	10,6	10,2	10,4
	7	12,4	12,3	10,2	13,2	10,3	9,9	12,0	12,0	9,6
	8	12,1	10,5	10,2	13,3	10,6	9,3	11,9	10,2	10,1

Table 6. Power of the LR, LR_{boot} and LR^*_{boot} tests with $n = 30$, $p = 4$ and $\alpha = 10\%$

$\beta^{(0)}$	GNB Model			Consul Model			GPO Model		
	LR	LR_{boot}	LR^*_{boot}	LR	LR_{boot}	LR^*_{boot}	LR	LR_{boot}	LR^*_{boot}
0,05	12,0	12,1	12,0	10,9	11,2	10,1	11,8	11,5	10,7
0,1	20,2	19,3	19,0	14,8	14,7	13,9	18,0	16,5	17,8
0,15	34,9	35,0	34,1	21,7	22,3	20,3	28,9	27,3	28,7
0,2	53,7	53,4	53,3	30,7	28,9	29,4	43,7	43,1	43,5
0,25	71,9	71,9	71,1	42,6	41,5	41,1	60,5	61,0	60,2
0,3	86,1	85,6	85,7	55,8	53,7	54,0	74,7	74,4	74,3
0,35	94,2	94,0	94,1	68,1	67,4	66,6	86,2	85,6	85,9
0,4	98,3	98,0	98,2	79,5	78,7	78,0	93,3	94,1	93,2

FINAL REMARKS

Power Series Generalized non Linear Models (PSGNLM), proposed in [2], constitute a very important generalization of regression models for counting data. The proposed model has a general structure for extending several important models previously established and widely used such as the generalized Poisson, generalized negative binomial, Consul, among others. Hypothesis testing for the PSGNLM is typically based on likelihood ratio test (LR) statistic using critical values obtained from the reference limiting null distribution.

In this paper, Bartlett correction for the LR statistic in PSGNLM is derived by means of bootstrap procedures. Such alternative lead to improved tests statistics whose rejection rates are closer to the nominal rates, i.e, reducing the distortion size, and whose means and variances are closer to those of the reference distribution χ^2_q. The numerical results support the higher performance of the corrected test LR^*_{boot} over the original test and even, over the single bootstrap-based quantile estimation statistic LR_{boot}. We strongly recommend practitioners to base inference on the proposed test LR^*_{boot} when research over counting regression models is going to be made.

REFERENCES

[1] A.C. Cameron and H.K. Trivedi. *Regression analysis of count data*. Number 53. Cambridge university press, 1998.

[2] G.M. Cordeiro, M.G. Andrade, and M. de Castro. Power series generalized nonlinear models. *Computational*

Statistics & Data Analysis, 53(4):1155–1166, 2009.

[3] S.S. Wilks. The large-sample distribution of the likelihood ratio for testing composite hypotheses. *The Annals of Mathematical Statistics*, 9(1):60–62, 1938.

[4] L. Pace, A. Salvan, and N. Sartori. Adjusting composite likelihood ratio statistics. *Statistica Sinica*, 21(1):129, 2011.

[5] M.S. Bartlett. Properties of sufficiency and statistical tests. *Proceedings of the Royal Society of London. Series A-Mathematical and Physical Sciences*, 160(901):268–282, 1937.

[6] D.N. Lawley. A general method for approximating to the distribution of likelihood ratio criteria. *Biometrika*, 43:295–303, 1956.

[7] G.M. Cordeiro, S.L.P. Ferrari, and G.A. Paula. Improved score tests for generalized linear models. *Journal of the Royal Statistical Society. Series B (Methodological)*, 55(3):661–674, 1993.

[8] B. Efron. Bootstrap methods: another look at the jackknife. *The annals of Statistics*, 7(1):1–26, 1979.

[9] D.M. Rocke. Bootstrap Bartlett adjustment in seemingly unrelated regression. *Journal of the American Statistical Association*, 84(406):598–601, 1989.

[10] M. Gu and Z. Zheng. On the bartlett adjustment for the partial likelihood ratio test in the cox regression model. *Statistica Sinica*, 3:543–555, 1993.

[11] F.M. Bayer and F. Cribari-Neto. Bartlett corrections in beta regression models. *Journal of Statistical Planning and Inference*, 143(3):531–547, 2013.

[12] D.R Cox and N. Reid. Parameter orthogonality and approximate conditional inference. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 1–39, 1987.

[13] T. Hayakawa. The likelihood ratio criterion and the asymptotic expansion of its distribution. *Annals of the Institute of Statistical Mathematics*, 29(1):359–378, 1977.

[14] B Efron and R J Tibshirani. *An introduction to the bootstrap*, volume 57. CRC press, 1994.

[15] J A Doornik. An object-oriented matrix language–ox. *Timberlake Consultants Press, London*, 107(440):256–259, 2006.

[16] J Nocedal and S J Wright. *Numerical Optimization*. Springer Series in Operations Research. Springer, 1999.

[17] D.G. Luenberger. *Introduction to linear and nonlinear programming*, volume 28. Addison-Wesley Reading, MA, 1973.

[18] C. D. Barbour and J.H. Brown. Fish species diversity in lakes. *American Naturalist*, pages 473–489, 1974.

[19] R.A Rigby, M.D. Stasinopoulos, and C. Akantziliotou. A framework for modelling overdispersed count data, including the poisson-shifted generalized inverse gaussian distribution. *Computational Statistics & Data Analysis*, 53(2):381–393, 2008.