Mathematical Modeling of Blood Flow through an Inclined Axially Non-Symmetric Stenosed Catheterized Artery with Body Acceleration

Lukendra Kakati¹, Nazibuddin Ahmed², Karabi Dutta Choudhury³

¹Research Scholar, Department of Mathematics, Assam University, Silchar- 788011, Assam, India.
²Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India.
³Department of Mathematics, Assam University, Silchar- 788011, Assam, India.

Correspondent Author: Lukendra Kakati

Abstract:
In this paper, a mathematical model has been developed to study the pulsatile flow of blood through an axially non-symmetric but radially symmetric stenosed inclined catheterized artery with periodic body acceleration and slip at wall. In our study, blood is assumed to be a Newtonian fluid, since in general stenosis is developed in large arteries. Here, perturbation as well as analytical methods are used to finding the solutions of non-linear partial differential equations of various flow variables. In this work, various interesting results are obtained regarding velocity field, volumetric flow rate, wall shear stress and effective viscosity of blood during catheterization in inclined arteries. The variation of flow variables with different parameters are obtained graphically and discussed elaborately. Here, it has been observed that velocity increases with increase of inclination, time, body acceleration to a certain limit and then decreases. Also, wall shear stress increases with increase of inclination but decrease with increase of body acceleration and slip velocity. Volumetric flow rate increases with mild increase of body acceleration and Effective viscosity also increase with increase of stenosis shape parameter.

Keywords: Newtonian Fluid, Pulsatile flow, Inclined, Stenosis, Body acceleration, Slip velocity.

Mathematical Subject Classification: 76Z05, 74G10.

INTRODUCTION
The one of the most leading cause of death in the world is heart diseases. The most common types of disease are ischemia, atherosclerosis and angina pectoris (Mekheimer and El Kot [1]). Ischemia is the temporary deficiency of oxygen in a part of body due to constriction or obstruction of blood in the blood vessels. Atherosclerosis is a type of arteriosclerosis, which comes from Greek word athero (meaning paste) and sclerosis (hardness). This is because of deposit of fatty substances, cholesterol, cellular waste products, calcium, fibrin etc. This abnormal growth in the lumen of an artery is called stenosis (atherosclerosis). Such sevır growth on the artery wall results in serious circulatory disorders as proposed by Young [2], Biswas [3], Bali and Awasthi [4], Biswas and Chakraborty [5] etc. These circulatory disorders may be included as narrowing in artery leading to the reduction and impediment to blood flow in the constricted artery regions, the blockage of artery in making the flow irregular, causing abnormality in blood flow.

With the development of medical technology, there has been considerable increase in use of catheter of various sizes for coronary Angiography, balloon Angioplasty, Renal Angiography etc. In medical science, a catheter is a thin tube made from medical grade materials inserted in the body to treat diseases or perform surgical procedure. The catheter has immense importance and standard tool for diagnosis and treatment of cardiovascular diseases.

In general, blood exhibits Non-Newtonian character at low shear rates [6], but in larger arteries with diameters about and above 1mm, blood behaves like Newtonian fluids [7]. In general, stenosis normally generated and developed in large diameter arteries in the range of 500 μm to 2000 μm, so we may consider the behaviour of blood as Newtonian fluid.

A good number of researchers viz. Haldar [8], Srivastava [9], Srivastava [10], Chakraborty and Mandal [11], Mandal [12], EI-Shahed [13], Jung et al.[14], Liu et al.[15], Sankar and Lee [16, 17], Srivastava and Srivastava[18] etc. have studied the flow of blood through stenosed arteries.

In recent years, considerable amount of researchers have studied the flow of blood through catheterized artery. The researchers have analysed the flow of blood in arteries by considering the catheter and the artery as rigid co-axial cylinders, assuming blood as either a Newtonian or a Non-Newtonian fluid.

Back [19], investigated the influence and size of the catheter on measurement of mean pressure drop and resistance to flow of blood through coronary vessels. Mc Donald [20], investigated the theoretical corrections of pressure gradient for pulsatile flow of blood through catheterized artery. Kerahalios [21] and Jayaraman and Tiwari [22] studied the effect of catheterization on various flow characteristics in a curved artery. Back and Denton [23], investigated and estimated the wall shear stress and its clinical importance in coronary Angioplasty. Daripa and Dash[24], studied the
pulsatile flow of blood in an eccentric catheterized artery, considering blood as Newtonian fluid. Sankar and Hemalata [25], studied the steady flow of Herschel–Bulkley fluid through catheterized artery. Sankar [26], has investigated the flow of blood through catheterized artery by considering blood as two-layer fluid with the core layer as a Casson fluid and the peripheral layer as Newtonian fluid. Biswas and Chakraborty [27] have studied the pulsatile flow of blood through a catheterized artery in presence of an axially non-symmetric mild stenosis with velocity slip at stenotic wall. Tanwar et al.[28], have studied the effect of body acceleration on pulsatile blood flow through a catheterized artery with an axially non-symmetric mild stenosis, assuming blood as Newtonian fluid.

On the basis of above motivations, in this paper, an attempt has been made to study some characteristics of flow of blood through an inclined axially non-symmetric stenosed catheterized artery with body acceleration and slip at wall. In our study, blood will be considered as Newtonian fluid.

Figure 1: Geometry of an axially non-symmetric stenosed artery with an inserted catheter inclined to the vertical.

**Mathematical Formulation:**

Here, we consider an axially symmetric, laminar, pulsatile and fully developed flow of blood through a catheterized inclined artery with mild stenosis as shown in Fig.1. Further it is to be assumed that the stenosis developed in the arterial wall in an axially non-symmetric but radially symmetric manner along the axial distance z and the artery is inclined to the vertical and a slip is imposed at stenosis region of the fluid. Here, the blood is assumed to be Newtonian fluid.

The geometry of the stenosis [29] is given by

\[
R(z) = \begin{cases} 
R_0 - A \left( L_0^{n-1} (z - d) - (z - d)^n \right) ; & \text{if } d \leq z \leq d + L_0 \\
R_0 ; & \text{Otherwise}
\end{cases}
\]  

(1)

Where \( R(z) \) is the radius of the stenosed region, \( L_0 \) is the length of the stenosis , \( R_0 \) is the radius of the normal artery, \( d \) indicates the location and \( n \geq 2 \) is a parameter, called shaped parameter which determines the shape of the stenosis (the symmetric stenosis occurred when \( n = 2 \)).

The parameter \( A \) is given by \( A = \frac{\delta n^{n+1}}{L_0(n-1)} \), where \( \delta \) denotes the maximum height of the stenosis located at \( z = d + L_0 \left/ \frac{n}{n+1} \right. \) such that \( \delta / R_0 << 1 \).

It has been observed that the radial velocity is negligibly small and can be neglected for low Reynolds number flow in a tube with mild stenosis [16, 27].

The equation of motion governing the fluid flow are given by

\[
\rho \frac{\partial \vec{u}}{\partial t} = -\nabla \rho - \frac{1}{\rho} \frac{\partial}{\partial r} (r \vec{u}) + G(T) + \rho g \cos \beta
\]  

(2)

\[
\frac{\partial \rho}{\partial r} = 0
\]  

(3)

Where \( \vec{u} \) is the fluid velocity in the axial direction , \( \rho \) is the density and \( \rho \) is the pressure ,\( G(T) \) is the body acceleration , \( \beta \) is the angle of inclination of the artery with the vertical and \( g \) is the acceleration due to gravity.

The constitutive equation of Newtonian fluid is given by

\[
\tau = -\mu \frac{\partial \vec{u}}{\partial r}
\]  

(4)

Where \( \mu \) is the co-efficient of viscosity and \( \tau \) is the shear stress.

The boundary conditions are

\[
\vec{u} = \vec{u}_s \text{ at } r = R(z)
\]  

(5)

\[
\vec{u} = 0 \text{ at } r = R_i
\]  

(6)
Where \( \bar{u} \) is the slip velocity at the stenotic wall and \( R_0 (<< R) \) is the radius of the catheter.

Since the pressure gradient is a function of \( z \) and \( T \), we take

\[
- \frac{\partial P}{\partial z}(z, T) = \bar{q}(z) f(T)
\]

(7)

where \( \bar{q}(z) = - \frac{\partial P}{\partial z}(z, 0) \),

\( f(T) = 1 + a \sin \bar{\omega}_p T \) and \( G(T) = a_0 + \cos(\bar{\omega}_b t + \phi) \)

Where \( a \) is the amplitude of blood flow and \( \bar{\omega}_p \) is the angular frequency of blood flow, \( a_0 \) is the amplitude of body acceleration, \( \bar{\omega}_b \) is the angular frequency of body acceleration, \( \phi \) is phase angle of body acceleration, \( \bar{p} \) is the pressure and \( \bar{T} \) is time.

Let us introduce the non-dimensional variables

\[
z = \frac{z}{R_0}, \quad R(z) = \frac{R(z)}{R_0}, \quad R_1 = \frac{R_1}{R_0},
\]

\[
r = \frac{r}{R_0}, \quad t = \frac{t}{\bar{T}}, \quad L_0 = \frac{L_0}{R_0}, \quad d = \frac{d}{R_0},
\]

\[
\delta = \frac{\delta}{R_0}, \quad A = \frac{A}{R_0^{n-1}}, \quad u = \frac{u}{\bar{u}_0 R_0^{\alpha}},
\]

\[
\omega = \frac{\bar{\omega}_p}{\bar{\omega}_0}, \quad B = \frac{a_0}{\bar{\omega}_0}, \quad \bar{u}_s = \frac{\bar{u}_s}{\bar{u}_0 R_0^{\alpha}}, \quad \alpha^2 = \frac{\bar{R}_0^2 \bar{\omega}_p \bar{p}}{\bar{u}_0}, \quad 
\]

\(
\tau = \frac{\bar{\tau}}{\bar{\tau}_0 R_0^{\alpha}}, \quad \mu_e = \frac{\bar{\mu}_e}{8\bar{\pi}}, \quad \bar{A}_s = \frac{\bar{A}_s \bar{R}_0}{\bar{q}_0},
\)

(8)

Where \( \alpha \) is the pulsatile Reynolds number for Newtonian fluid and \( \bar{q}_0 \) is the constant pressure gradient in a uniform tube without catheter in the negative direction.

The non-dimensional form of geometry of the stenosis is given by

\[
R(z) = \begin{cases} 
1 - A[L_0^{-1}(z-d) - (z-d)^\alpha] & ; \quad d \leq z \leq d + L_0 \\
1 & ; \quad \text{Otherwise}
\end{cases}
\]

(9)

Now let us introduce non-dimensional substitutions from (8) in the expression of (2) and (4), we get,

\[
\alpha^2 \frac{\partial u}{\partial t} = 4\{1 + a \sin t + B \cos(\omega t + \phi) + A_s \cos \beta\} - \frac{2}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) \tag{12}
\]

and

\[
\tau = - \frac{1}{2} \frac{\partial u}{\partial r} \tag{11}
\]

Now, using the expression of (11) in the expression (10), we get

\[
\alpha^2 \frac{\partial u}{\partial t} = 4F(t) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \tag{12}
\]

where \( F(t) = 1 + a \sin t + B \cos(\omega t + \phi) + A_s \cos \beta \)

The expressions for boundary conditions in (5) and (6) in dimensionless form reduces to

\[
u = u_s \quad at \quad r = R(z) \tag{13}
\]

\[
u = 0 \quad at \quad r = R_1 \tag{14}
\]

The non-dimensional volumetric flow rate is given by

\[
Q = 4 \int_{R_1}^{R(z)} r u(r, z, t) dr \tag{15}
\]

where \( Q(t) = \pi(R_0)^4 \frac{q_0}{8\pi} \)

\[
\bar{Q}(T) = 2\pi \int_{R_1}^{R(z)} r \bar{u}(r, z, T) , \text{ is the volumetric flow rate.}
\]

The effective viscosity \( \mu_e \) is defined as

\[
\bar{\mu}_e = \left( \frac{\partial \bar{p}}{\partial \bar{z}} \right) \left( \bar{R}(z) \right)^4 \tag{16}
\]

This can be expressed in non-dimensional form as

\[
\mu_e = \frac{(R(z))^4 (1 + a \sin t)}{Q(t)} \tag{17}
\]

Where \( Q(t) \) is defined as in the expression (15).

**Method of solutions:**

Consider the Womersley parameter to be very small, the velocity \( u \) can be expressed in the following form

\[
u(z, r, t) = u_s(z, r, t) + \alpha^2 u_s(z, r, t) + \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)
\]

Substituting the expression of \( u \) from (18) in the expression (12) and neglecting the higher powers of \( \alpha \) we get

\[
\frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = -4F(t) \tag{19}
\]

and

\[
\frac{\partial u_s}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_s}{\partial r} \right) \tag{20}
\]

Substituting the expression of \( u \) from (18) in the expressions of (13) and (14), we get

15292
Now, integrating equation (20) with respect to \( r \) twice and incorporating with the boundary conditions of (21), we obtain the following expressions for \( u_0 \) and \( u_1 \) as

\[
\begin{align*}
u_0 &= 1 - \left[ \log \left( \frac{r}{R} \right) \right] u_s + F(t) \left( R^2 - r^2 \right) \left( \log \left( \frac{R}{r} \right) \right) \\
u_1 &= F'(t) \left[ \log \left( \frac{r}{R} \right) \right] + \left( R^2 - r^2 \right) \left( \log \left( \frac{R}{r} \right) \right)
\end{align*}
\]

Hence, the expression for velocity field \( u \) is obtained from (18) by substituting the values of \( u_0 \) and \( u_1 \) from the expressions (22) and (23).

\[
i.e. \quad u = 1 - \left[ \log \left( \frac{r}{R} \right) \right] u_s + F(t) \left( R^2 - r^2 \right) \left( \log \left( \frac{R}{r} \right) \right)
\]

\[
\frac{\alpha^2 F'(t)}{4} = \frac{5R^4 - 2r^4 - 3R^2}{\log \left( \frac{R}{r} \right)} \left( r^2 \log \left( \frac{R}{r} \right) - r^2 + R \right)
\]

It may be noted that our expression for \( u_1 \) and that of the same kind of work done by Tanwar et al. (2016) for \( \beta = \pi/2 \) is differ perhaps due to differences in arrangement of terms in calculations.

The expression for shear stress \( \tau_w \) is obtained from the expressions of (11) and (18) as

\[
\tau_w = - \frac{1}{2} \left( \frac{\partial u_0}{\partial r} + \alpha^2 \frac{\partial u_1}{\partial r} \right)_{r=R(z)}
\]

Now, substituting the velocity expressions of (22) and (23) in the expression (25), we may get the expression for wall shear stress in the form

\[
\tau_w = \frac{u_s}{2R \log \left( \frac{R}{r} \right)} + F(t) \left[ \frac{R^4 - R^2}{2R \log \left( \frac{R}{r} \right)} \right]
\]

\[
- \alpha^2 \frac{F'(t)}{8} \left[ \frac{3}{2} R^2 + \frac{5}{4} \left( R^2 - R^2 \right)^2 \right]
\]

Now the volumetric flow rate is obtained from the expression (15) and (24) in the form

\[
Q = 2R^2 \left[ \frac{R^2 - R^2}{\log \left( \frac{R}{r} \right)} \right] u_s + F(t) \left[ \frac{R^4 - R^2}{\log \left( \frac{R}{r} \right)} \right]
\]

\[
- \alpha^2 \frac{F'(t)}{16} \left[ \frac{7}{3} \left( R^2 - R^2 \right)^2 + \frac{27}{4} \left( R^2 - R^2 \right)^2 \right]
\]

The effective viscosity \( \mu_e \) is obtained from the expression (17) and (27).

**RESULTS AND DISCUSSIONS**

This model has been developed to analyses the velocity profile, volumetric flow rate, wall shear stress and effective viscosity for pulsatile flow of blood through a catheterized artery, inclined to the vertical with periodic body acceleration and slip at stenosis wall.

The value 0.5 is taken for amplitude of blood flow \( \alpha \), pulsatile Reynolds’ number \( \alpha \), value of gravitational force parameter \( A_0 \) and angular frequency parameter \( \omega \). Further, the phase angle \( \phi \) is taken as \( \pi/12 \). The value of the shape parameter ‘ \( n \) ’ of the stenosis is taken from 2 to 5.
The variation of axial velocity \( u \) with radial distance \( r \) for different values of angle of inclination \( \beta \) with fixed values of \( R=0.8, \ R_i = 0.1, \ u_x = 0.05, t = \frac{\pi}{4}, \ B = 0.5 \) is presented in Fig.2. Here, it is found that the axial velocity increases with increase of \( \beta \) from \( \beta = \frac{\pi}{12} \) to \( \frac{\pi}{3} \).

In the investigation of Fig.3, it is observed that axial velocity increases with increase of inclination from 0 to \( \pi/3 \), but axial velocity shows better result with catheter radius \( R_i = 0.15 \) than that of catheter radius \( R_i = 0.5 \). Hence we observed that, for inclined artery, finer the radius of catheter, will exhibits higher the axial velocity with increase of angle of inclination.

Fig.4, represents the variation of axial velocity \( u \) with the variation of radial distance \( r \) for different values of periodic time \( t \) and with fixed values of \( R=0.8 \), \( R_i = 0.1, \ u_x = 0.05, \ \beta = \pi/6, \ B = 0.5 \). From this figure, it is observed that the axial velocity increases with increase of time from \( t = 0 \) to \( t = \frac{\pi}{2} \) and then decrease continuously from \( t = \pi/2 \) to \( t = 4\pi/3 \).

Fig.5, represents the variation of axial velocity \( u \) with variation of radial distance \( r \) for different values of body acceleration parameter \( B \) and with fixed values of \( R = 0.8, \ R_i = 0.1, \ u_x = 0.05, \ \beta = \pi/6 \). From this figure, it is observed that the magnitude of axial velocity increases with increase of body acceleration parameter \( B \) from \( B = 0 \) to \( B = 1.5 \) and then decrease continuously with increase of \( B \) from \( B = 1.5 \) to \( B = 3.5 \).

Hence, we observed that the axial velocity increases with increase of body acceleration to a certain level and whenever the magnitude of body acceleration crosses to a certain limit, the axial velocity began to decline.
Fig5: Axial velocity $u$ versus radial distance $r$ for various values of Body acceleration parameter $B$.

Fig.6: Variation of volumetric flow rate $Q$ versus catheter radius $R_1$ for various values of Body acceleration parameter $B$.

Fig.7: Variation of Shear Stress $\tau_w$ versus catheter radius $R_1$ for various values of inclination $\beta$.

Fig.8: Variation of Shear stress $\tau_w$ versus catheter radius $R_1$ for various values of Body acceleration parameter $B$.

Fig.8, represents the variation of wall shear stress $\tau_w$ with the variation of catheter radius $R_1$ for various values of inclination $\beta = (\pi/12, \pi/6, \pi/4, \pi/3)$ and with fixed values of $R = 0.8$, $t = \pi/3$, $B = 0.5$, $u_s = 0.05$. From this figure, it is observed that the wall shear stress increases with increase of inclination from $\beta = \pi/12$ to $\pi/3$. Further, for narrow the catheter, the effect of inclination on wall shear stress observed to be very low.

Fig.8, shows the variation of wall shear stress $\tau_w$ with variation of catheter radius $R_1$ for various values of body acceleration parameter $B$. From this figure, we observed that wall shear stress decreases with increase of body acceleration parameter. Hence, from Fig.5, Fig.6 and Fig.8, we observed that a mild periodic body acceleration is advisable during catheterization for diagnosis or treatment of cardiovascular diseases in inclined artery to keep the cardiac condition of the patient stable.
Fig. 9: Shear Stress $\tau_w$ versus Catheter radius $R_i$ for various values of slip velocity $u_s$.

Fig. 9 represents the variation of wall shear stress $\tau_w$ with the variation of catheter radius $R_i$ for various values of slip velocity $u_s$ and with fixed values of $R = 0.8$, $t = \pi/2$, $\beta = \pi/4$, $B = 0.5$. From the figure, it is observed that the wall shear stress decreases with increase of slip velocity.

Fig. 10: Shear Stress $\tau_w$ versus time $t$ for various values of slip velocity $u_s$.

Fig. 10 depicts the variation of wall shear stress $\tau_w$ with the variation of time $t$ for various values of slip velocity $u_s$ and with fixed values of $R = 0.8$, $B = 0.5$, $\beta = \pi/4$, $R_i = 0.1$. Here also, as like in Fig. 9, wall shear stress decreases with increase of slip velocity.

Hence, from Fig. 9 and Fig. 10, we observed that if slip velocity increases then wall shear stress decreases during catheterization. So, drugs that have the capacity to increase slip velocity at artery wall may be beneficial for patient during the catheterization process in inclined artery.

Fig. 11: Variation of effective viscosity $\mu_e$ versus catheter radius $R_i$ for various values of stenosis shape parameter of $n$.

Fig. 11 shows the variation of effective viscosity $\mu_e$ with the variation catheter radius $R_i$ for various values of stenosis shape parameter $n = 2, 3, 4, 5$ and with fixed values of $t = \pi/4$, $\beta = \pi/3$, $u_s = 0.05$, $L_0 = 1$, $z = 8$, $d = 7.5$, $\delta = 0.2$. From the figure, it is observed that effective viscosity increases with increase of stenosis shape parameter. Further, it is also observed that with increase of catheter radius, the effective viscosity increases proportionately and after a certain limit of $R_i = 0.3$, the effective viscosity shows abnormal behaviour.

CONCLUSION

Our theoretical model study may be summarised to the following conclusions:

I. Velocity of blood increases with increase of inclination, but catheter radius should be as fine as possible, otherwise flow velocity will not reached to the level of expectations. Hence for treatment of cardiovascular disease in inclined artery by catheterization, the catheter should be as fine as possible for the safety of patient.

II. For diagnosis and treatment of coronary artery disease by means of angiography, angioplasty etc., a mild periodic body acceleration is advisable during catheterization process. We may also strongly recommend to use catheter of radius as fine as possible to keep the volumetric flow of blood in standard level.

III. Wall shear stress increases with increase of inclination, but with decrease of catheter radius, the effect of inclination on wall shear stress become negligible. Further catheterization may not be advisable for arteries with higher inclination to keep control on wall shear stress.
IV. A mild periodic body acceleration is advisable to minimize wall shear stress during angiography, angioplasty etc.

V. The drugs that may increase the wall slip velocity are advisable during catheterization to minimize the wall shear stress.

VI. As effective viscosity increases with increase of stenosis shape parameter, for the safety of the patient catheterization is not suggestive in case of inclined artery with stenosis of high shape parameter.

On the basis of above discussions, we may conclude that this model may be considered as reference model for the safety of patient with coronary artery diseases during diagnosis or treatment by means of angiography, angioplasty etc. in an inclined artery. Further, more effective models may be studied for radially non-symmetric stenosis of higher shape parameter as well as accurately calculated limiting value of inclination and body acceleration.

ACKNOWLEDGEMENT:
The authors are highly grateful to Dr. Manas Dutta, Department of Applied Science (Mathematical Science Division), Gauhati University, Guwahati, Assam, India, for his valuable suggestions and help to carry out this work.

REFERENCES


