Pairwise Fuzzy $\sigma$-First Category Sets

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Abstract
In this paper the concepts of pairwise fuzzy $\sigma$-first category sets are studied. Several characterizations of pairwise fuzzy $\sigma$-first category sets are established.

Keywords: Pairwise fuzzy open set, pairwise fuzzy $F_{\sigma}$-set, pairwise fuzzy $G_{\delta}$-set, pairwise fuzzy nowhere dense set, pairwise fuzzy $\sigma$-nowhere dense set, pairwise fuzzy first category set.

INTRODUCTION
In order to deal with uncertainties, the idea of fuzzy sets, fuzzy set operations was introduced by L.A.Zadeh [1] in his classical paper in the year 1965, describing fuzziness mathematically for the first time. In 1968, C.L.Chang [2] defined fuzzy topological space by using fuzzy sets introduced by Zadeh. The concept of $\sigma$-nowhere dense sets in classical topology was introduced and studied by Jiling Cao and Sina Greenwood in [3]. In 1989, A.Kandil [4] introduced and studied fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. The concept of Baire bitopological spaces have been studied extensively in classical topology in [5],[6] and [7]. The concept of pairwise fuzzy $\sigma$-first category sets are defined by authors in [8]. In this paper several characterizations of pairwise fuzzy $\sigma$-first category sets are established.

PRELIMINARIES
In order to make the exposition self-contained, some basic notations and results used in the sequel are given. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). By a fuzzy bitopological space (Kandil, 1989) we mean an ordered triple (X, T1, T2), where T1 and T2 are fuzzy topologies on the non-empty set X. Let X be a non-empty set and I the unit interval [0,1]. A fuzzy set $\lambda$ in X is a mapping from X into I.

Definition 2.1. [9] A fuzzy set $\lambda$ in a fuzzy bitopological space $(X, T_1, T_2)$ is called a pairwise fuzzy open set if $\lambda \in T_1$ (i = 1, 2). The complement of pairwise fuzzy open set in $(X, T_1, T_2)$ is called a pairwise fuzzy closed set.

Definition 2.2. [9] A fuzzy set $\lambda$ in a fuzzy bitopological space $(X, T_1, T_2)$ is called a pairwise fuzzy $G_\delta$-set if $\lambda = \wedge_{i=1}^{k} (\lambda_i)$, where $(\lambda_i)$’s are pairwise fuzzy open sets in $(X, T_1, T_2)$.

Definition 2.3. [9] A fuzzy set $\lambda$ in a fuzzy bitopological space $(X, T_1, T_2)$ is called a pairwise fuzzy $F_\sigma$-set if $\lambda = \vee_{i=1}^{k} (\lambda_i)$, where $(\lambda_i)$’s are pairwise fuzzy closed sets in $(X, T_1, T_2)$.

Lemma 2.1. [10] For a family of $\{\lambda_i\}$ of fuzzy sets of a fuzzy topological space $(X, T_1, T_2)$, if $\vee cl(\lambda_i) \leq cl(\vee \lambda_i)$ and $\lambda_i \leq \lambda_{i+1}$, then $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_k$.

Definition 2.4. [11] A fuzzy set $\lambda$ in a fuzzy bitopological space $(X, T_1, T_2)$ is called a pairwise fuzzy dense set if $\vee cl(T_1) \leq cl(\vee \lambda)$ and $\lambda \leq \lambda$. Also $\lambda \leq \lambda$.

Definition 2.5. [12] A fuzzy set $\lambda$ in a fuzzy bitopological space $(X, T_1, T_2)$ is called a pairwise fuzzy nowhere dense set if $\vee int(T_1) \leq int(\vee \lambda)$ and $\lambda \leq \lambda$. Also $\lambda \leq \lambda$.

Definition 2.6. [9] A fuzzy set $\lambda$ in a fuzzy bitopological space $(X, T_1, T_2)$ is called a pairwise fuzzy $\sigma$-nowhere dense set if $\lambda \leq \lambda_{i+1}$, where $(\lambda_i)$’s are pairwise fuzzy nowhere dense sets in $(X, T_1, T_2)$. Otherwise $(X, T_1, T_2)$ will be called a pairwise fuzzy $\sigma$-second category space.

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Definition 3.1. [8] Let $(X, T_1, T_2)$ be a fuzzy bitopological space. A fuzzy set $\lambda$ in $(X, T_1, T_2)$ is called a pairwise fuzzy $\sigma$-first category set if $\lambda = \wedge_{i=1}^{k} (\lambda_i)$, where $(\lambda_i)$’s are pairwise fuzzy $\sigma$-nowhere dense sets in $(X, T_1, T_2)$. Any other fuzzy set in $(X, T_1, T_2)$ is said to be a pairwise fuzzy $\sigma$-second category set in $(X, T_1, T_2)$. 

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Definition 3.2. [8] If $\lambda$ is a pairwise fuzzy $\sigma$-first category set in a fuzzy bitopological space $(X, T_1, T_2)$, then the fuzzy set $1 - \lambda$ is called a pairwise fuzzy $\sigma$-residual set in $(X, T_1, T_2)$.

Theorem 3.3. [8] If $\lambda$ is a pairwise fuzzy dense set and pairwise fuzzy $G_\delta$-set in a fuzzy bitopological space $(X, T_1, T_2)$, then $1 - \lambda$ is a pairwise fuzzy $\sigma$-nowhere dense set in $(X, T_1, T_2)$.

Proposition 3.4 If $(\lambda_k)_k$ are pairwise fuzzy dense sets and pairwise fuzzy $G_\delta$-sets in a fuzzy bitopological space $(X, T_1, T_2)$, then $1 - \lambda_k = 1 - \bigwedge_{k=1}^{\infty} \lambda_k$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$.

Proof. Let $(\lambda_k)_k$ be pairwise fuzzy dense sets and pairwise fuzzy $G_\delta$-sets in a fuzzy bitopological space $(X, T_1, T_2)$. Then, by theorem 3.3, $(1 - \lambda_k)_k$ are pairwise fuzzy $\sigma$-nowhere dense sets in $(X, T_1, T_2)$. This implies that $\bigvee_{k=1}^{\infty} (1 - \lambda_k)_k$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$. Since $\bigvee_{k=1}^{\infty} (1 - \lambda_k) = 1 - \bigwedge_{k=1}^{\infty} \lambda_k$, $(1 - \lambda_k)_k$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$.

The following proposition gives a condition for a pairwise fuzzy first category set to become a pairwise fuzzy $\sigma$-first category set in fuzzy bitopological spaces.

Theorem 3.6 [13] If the pairwise fuzzy nowhere dense set $\lambda$ is a pairwise fuzzy $F_\sigma$-set in a fuzzy bitopological space $(X, T_1, T_2)$, then $\lambda$ is a pairwise fuzzy $\sigma$-nowhere dense set in $(X, T_1, T_2)$.

Proposition 3.7 If $\lambda$ is a pairwise fuzzy first category set in a fuzzy bitopological space $(X, T_1, T_2)$ in which each pairwise fuzzy nowhere dense set is a pairwise fuzzy $F_\sigma$-set, then $\lambda$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$.

Proof. Let $\lambda$ be a pairwise fuzzy first category set in $(X, T_1, T_2)$. Then $\lambda = \bigvee_{k=1}^{\infty} \lambda_k$, where $(\lambda_k)_k$ are pairwise fuzzy nowhere dense sets in $(X, T_1, T_2)$. By hypothesis, the pairwise fuzzy nowhere dense sets are pairwise fuzzy $F_\sigma$-sets in $(X, T_1, T_2)$. By theorem 3.6, $(\lambda_k)_k$ are pairwise fuzzy $\sigma$-nowhere dense sets in $(X, T_1, T_2)$. Then $\lambda = \bigvee_{k=1}^{\infty} \lambda_k$, where $(\lambda_k)_k$ are pairwise fuzzy $\sigma$-nowhere dense sets in $(X, T_1, T_2)$ implies that $\lambda$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$.

Proposition 3.8 If $\lambda$ is a pairwise fuzzy first category set in a fuzzy bitopological space $(X, T_1, T_2)$ in which each pairwise fuzzy nowhere dense set is a pairwise fuzzy $F_\sigma$-set, then $(1 - \lambda)$ is a pairwise fuzzy $\sigma$-residual set in $(X, T_1, T_2)$.

Proof. Let $\lambda$ be a pairwise fuzzy first category set in which each pairwise fuzzy nowhere dense set is a pairwise fuzzy $F_\sigma$-set in $(X, T_1, T_2)$. By proposition 3.7, $\lambda$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$. Then $(1 - \lambda)$ is a pairwise fuzzy $\sigma$-residual set in $(X, T_1, T_2)$.

Proposition 3.9 If $(\lambda_k)_k$ are pairwise fuzzy $G_\delta$-sets in a pairwise fuzzy hyperconnected space and pairwise fuzzy $P$-space in $(X, T_1, T_2)$, then $1 - \bigwedge_{k=1}^{\infty} \lambda_k$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$.

Proof. Let $(\lambda_k)_k$ be pairwise fuzzy $G_\delta$-sets in $(X, T_1, T_2)$. Since $(X, T_1, T_2)$ is a pairwise fuzzy $P$-space, the pairwise fuzzy $G_\delta$-sets $(\lambda_k)_k$ are pairwise fuzzy open sets in $(X, T_1, T_2)$. Since $(X, T_1, T_2)$ is a pairwise fuzzy hyperconnected space, the pairwise fuzzy open sets $(\lambda_k)_k$ are pairwise fuzzy dense sets in $(X, T_1, T_2)$. Hence $(\lambda_k)_k$ are pairwise dense and pairwise fuzzy $G_\delta$-sets in $(X, T_1, T_2)$. Then by proposition 3.4, $1 - \bigwedge_{k=1}^{\infty} \lambda_k$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$.

Proposition 3.10 If $(\lambda_k)_k$ are pairwise fuzzy $G_\delta$-sets in a pairwise fuzzy hyperconnected space and pairwise fuzzy $P$-space in $(X, T_1, T_2)$, then $\bigwedge_{k=1}^{\infty} \lambda_k$ is a pairwise fuzzy $\sigma$-residual set in $(X, T_1, T_2)$.

Proof. Let $(\lambda_k)_k$ be pairwise fuzzy $G_\delta$-sets in $(X, T_1, T_2)$. Since $(X, T_1, T_2)$ is a pairwise fuzzy hyperconnected space and pairwise fuzzy $P$-space. Since, by proposition 3.9, $1 - \bigwedge_{k=1}^{\infty} \lambda_k$ is a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$. Hence, $\bigwedge_{k=1}^{\infty} \lambda_k$ is a pairwise fuzzy $\sigma$-residual set in $(X, T_1, T_2)$.

Proposition 3.11 If $\lambda$ is a pairwise fuzzy $\sigma$-first category set in a fuzzy bitopological space $(X, T_1, T_2)$ in which pairwise fuzzy $\sigma$-nowhere dense sets are pairwise fuzzy closed sets, then $\lambda$ is a pairwise fuzzy $F_\sigma$-set in $(X, T_1, T_2)$.

Proof. Let $\lambda$ be a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$. Then $\lambda = \bigvee_{\lambda_k}$, where $(\lambda_k)_k$ are pairwise fuzzy $\sigma$-nowhere dense sets in $(X, T_1, T_2)$. By hypothesis, $(\lambda_k)_k$ are pairwise fuzzy closed sets in $(X, T_1, T_2)$. Now $(1 - \lambda) = \bigwedge_{\lambda_k}$ are pairwise fuzzy open sets in $(X, T_1, T_2)$. Then $\mu = \bigvee_{\lambda_k} = (1 - \lambda)$ is a pairwise fuzzy $G_\delta$-set in $(X, T_1, T_2)$ and $1 - \mu = 1 - \bigwedge_{\lambda_k} = \lambda$. That is $\lambda = 1 - \mu$. Hence $\lambda$ is a pairwise fuzzy $F_\sigma$-set in $(X, T_1, T_2)$.

The following proposition gives a condition for a pairwise fuzzy $\sigma$-first category set to become a pairwise fuzzy $\sigma$-first category set in fuzzy bitopological spaces.

Theorem 3.12 [8] If $\lambda$ is a pairwise fuzzy $\sigma$-nowhere dense set in a pairwise fuzzy strongly irresolvable space $(X, T_1, T_2)$, then $\lambda$ is a pairwise fuzzy first category set in $(X, T_1, T_2)$.

Proof. Let $\lambda$ be a pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$. Then $\lambda = \bigvee_{\lambda_k}$, where $(\lambda_k)_k$ are pairwise fuzzy $\sigma$-first category set in $(X, T_1, T_2)$. Since $(X, T_1, T_2)$ is a pairwise strongly irresolvable space, by theorem 3.12, the pairwise fuzzy $\sigma$-nowhere dense sets $(\lambda_k)_k$ are pairwise fuzzy nowhere dense sets in $(X, T_1, T_2)$. Hence $\lambda = \bigwedge_{\lambda_k}$, where $(\lambda_k)_k$ are pairwise fuzzy nowhere dense sets in $(X, T_1, T_2)$, implies that $\lambda$ is a pairwise fuzzy first category set in $(X, T_1, T_2)$.
Proposition 3.14. If \( \lambda \) is a pairwise fuzzy \( \sigma \)-first category set such that \( \text{int}_T(\lambda) = 0 \) in a fuzzy bitopological space \((X,T_1,T_2)\) in which pairwise fuzzy \( \sigma \)-nowhere dense sets are pairwise fuzzy closed sets, then \( \lambda \) is a pairwise fuzzy first category set in \((X,T_1,T_2)\).

Proof. Let \( \lambda \) be a pairwise fuzzy \( \sigma \)-first category set in \((X,T_1,T_2)\). Then \( \lambda = \bigvee_{k=1}^{\infty} (\lambda_k) \), where, \((\lambda_k)\)'s are pairwise fuzzy \( \sigma \)-nowhere dense sets in \((X,T_1,T_2)\). By hypothesis, the pairwise fuzzy \( \sigma \)-nowhere dense sets \((\lambda_k)\)'s are pairwise fuzzy closed sets in \((X,T_1,T_2)\). By proposition 3.11, \( \lambda \) is a pairwise fuzzy \( \sigma \)-set in \((X,T_1,T_2)\). Now \( \text{int}_T(\lambda) = 0 \) implies that \( \text{int}_T(\bigvee_{k=1}^{\infty}(\lambda_k)) = 0 \). But, from lemma 2.1, \( \bigvee_{k=1}^{\infty}\text{int}_T(\lambda_k) \leq \text{int}_T(\bigvee_{k=1}^{\infty}(\lambda_k)) \) \((i=1,2)\). This implies that \( \bigvee_{k=1}^{\infty}\text{int}_T(\lambda_k) \leq 0 \), that is \( \bigvee_{k=1}^{\infty}\text{int}_T(\lambda_k) = 0 \). Hence, \( \text{int}_T(\lambda_k) = 0 \) for each \( k \). Since \((\lambda_k)\)'s are pairwise fuzzy closed, \( \text{cl}_T(\lambda_k) = \lambda_k \). Then \( \text{int}_T(\lambda_k) = \text{int}_T(\lambda_k) = 0 \). That is, \( \text{int}_T(\lambda_k) = 0 \) for each \( k \). Hence \((\lambda_k)\)'s are pairwise fuzzy nowhere dense sets in \((X,T_1,T_2)\). Therefore \( \lambda = \bigvee_{k=1}^{\infty}(\lambda_k) \), where, \((\lambda_k)\)'s are pairwise fuzzy nowhere dense sets, implies that \( \lambda \) is a pairwise fuzzy first category set in \((X,T_1,T_2)\).

Proposition 3.15 If the fuzzy bitopological space \((X,T_1,T_2)\) is a pairwise fuzzy submaximal space and if \( \lambda \) is pairwise fuzzy \( \sigma \)-first category set in \((X,T_1,T_2)\), then \( \lambda \) is a pairwise fuzzy first category set in \((X,T_1,T_2)\).

Proof. Let \( \lambda = \bigvee_{k=1}^{\infty}(\lambda_k) \) be a pairwise fuzzy \( \sigma \)-first category set in \((X,T_1,T_2)\), where the fuzzy sets \((\lambda_k)\)'s are pairwise fuzzy \( \sigma \)-nowhere dense set in \((X,T_1,T_2)\). Then we have \( \text{int}_T(\lambda_k) = 0 \) and \((\lambda_k)\)'s are pairwise fuzzy \( \sigma \)-set in \((X,T_1,T_2)\). Now \( \text{int}_T(\lambda_k) = 0 \) implies that \( 1 - \text{int}_T(\lambda_k) = 1 \) and hence \( \text{cl}_T(1 - \lambda_k) = 1 \). Since \((X,T_1,T_2)\) is a pairwise fuzzy submaximal space, the pairwise fuzzy dense sets \((1 - \lambda_k)\)'s are pairwise fuzzy open sets in \((X,T_1,T_2)\) and hence \((\lambda_k)\)'s are pairwise fuzzy closed sets in \((X,T_1,T_2)\). Then \( \text{cl}_T(\lambda_k) = (\lambda_k) \) and \( \text{int}_T(\lambda_k) = 0 \) implies that \( \text{int}_T(\lambda_k) = 0 \) \((\lambda_k)\)'s are pairwise fuzzy nowhere dense set in \((X,T_1,T_2)\). Therefore \( \lambda = \bigvee_{k=1}^{\infty}(\lambda_k) \) be a pairwise fuzzy first category set in \((X,T_1,T_2)\).

CONCLUSION
In this paper, several characterization of pairwise fuzzy \( \sigma \)-first category sets are established. The conditions for a pairwise fuzzy first category set to become a pairwise fuzzy \( \sigma \)-first category sets are established. Also the conditions under which pairwise fuzzy \( \sigma \)-first category sets to become a pairwise fuzzy first category sets are also established.

REFERENCES