A New Approach to Fuzzy Soft Connectedness

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Abstract
In this paper we introduce the idea of connectedness in fuzzy soft topological spaces. We define fuzzy soft connected set with examples and study some of its properties.

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INTRODUCTION AND PRELIMINARIES
The concept of fuzzy sets are initiated by L.A.Zadeh⁷ which is a general-ization of classical sets. R.Lowen⁴ defined the notion of fuzzy topology in a non-empty set. The idea of soft sets was first given by Molodtsov⁵. In[3] the concept of soft topological spaces were developed. Some of the structural properties of soft topology were seen in [2]. Some properties of fuzzy soft topological spaces was studied in[6]. In this paper we approach the notion of connected ness in fuzzy soft topological spaces via Lowen’s fuzzy topology. We investigated some theorems related to fuzzy soft connectedness.

The following definitions and theorems are in[1], which are needed for our study.

Through this paper, X be an initial universe and E be the set of all parameter sets for X, 𝑆 Params is the set of all fuzzy soft parameters on X (where, I=[0,1] and for 𝜆∈[0,1], 𝑆 Params (x)=∅, for all x∈X.)

Definition[1]. Let A⊆E. 𝑓A is called a fuzzy soft set on X, where f is a mapping from E into I^X; i.e., fe= f(e)∈fA is a fuzzy set on X for each e∈A and fA is the set of all fuzzy soft sets on X i.e., hA⊆E. That is, hA= fA∪gA, where C = A ∪ B and hA = fA ∪ gA, for each e∈A∪B.

Definition[1]. Intersection of two fuzzy soft sets fA and gA on X is the fuzzy soft set fA∩gA, where C = A ∩ B and hA = fA ∩ gA, for each e∈A∩B.

Definition[1]. The complement of a fuzzy soft set fA on X is denoted by fA^C, where f^C: E → I^X is a mapping given by f^C = I - fA, for each e∈E. Clearly, (fA^C)^C = fA.

Definition[1]. (Null fuzzy soft set) A fuzzy soft set f∅ on X is called a null fuzzy soft set and denoted by ∅ if f∅ =∅, for each e∈E.

Definition[1]. (Absolute fuzzy soft set) A fuzzy soft set fX on X is called an absolute fuzzy soft set and denoted by F̃X, if fX = X, for each e∈E. Clearly, (F̃X)^C = F̃X.

In this paper we write F̃X as T.

Proposition[1]. Let ∆ be an index set and fX, gX, hX, (fX) are FS(X,E), ∀X∈∆, then we have the following properties:

1. fX∩gX = fX, fX∪hX = fX.
2. fX∪gX = gX∪fX, hX∪gX = gX∪hX.
3. fX∪hX = (fX∪gX)∪hX.
4. fX∩gX = (fX∩gX)∩hX.
5. fX∩gX = (fX∩gX)∩(fX∩gX).
6. fX∪(gX∩hX) = (fX∪(gX∩hX)).
7. (∅∩fX) = ∅.
8. (fX∩gX) = fX.
9. (fX∩gX) = (fX∩gX).
10. (fX∩gX) = (fX∩gX).
11. (fX∩gX) = (fX∩gX).

Definition[1]. Let f:FS(X,E)→FS(Y,E') and p:E→E' be two functions, consider a fuzzy soft set hA ∈FS(X,E) where A∈E. Then the fuzzy soft set hA ∈FS(Y,E') is defined as f(hA)(β) = (∪βA(α)) if α ∈ p^−1(β) ∈ A, (3) if p^−1(β) = ∅.

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Where $\beta \in B = p(A) \subseteq E'$ and $\emptyset$ denotes the crisp empty set.

Let $h_c \in FS(Y,E')$ where $C \subseteq E'$ then the preimage of $h_c f^{-1}(h_c) \in FS(X,E)$ defined by $f^{-1}(h_c)(a) = h_c(p(a))$ for $a \in D = p^{-1}(C) \subseteq E$.

### Fuzzy Soft Topological Spaces

**Definition.** The support of a fuzzy soft set $g E$ on $X$ is defined as $S(g E) = \left\{ e \in E \mid g_E > 0 \right\}$. A fuzzy soft set $g_E$ is said to be finite fuzzy soft set of $X$ iff $S(g_E)$ is a finite parameter set. A fuzzy soft set $g_E$ is said to be countable fuzzy soft set of $X$ iff $S(g_E)$ is a countable parameter set.

**Definition.** A non-empty family $G \subseteq FS(Y,E')$ of fuzzy soft sets is called fuzzy soft ideal on $Y$ if

i) $i \in G, j \in G$ implies that $j \in G$.

ii) $i \in G, j \in G$ implies that $i \cup j \in G$. (As $G$ is not empty, $\emptyset \in G$)

**Examples.**

1. $G_{-\infty}$ - is the fuzzy soft ideal of fuzzy soft sets of $Y$ with finite support.

2. $G_{\infty}$ - is the fuzzy soft ideal of fuzzy soft sets of $Y$ with countable support.

3. Let $h_e$ be a fixed fuzzy soft set in $Y$. Then $G(h_e) = \left\{ i \in FS(Y,E') \mid g_E \subseteq h_e \right\}$ is a fuzzy soft ideal.

**Definition.** Let $F_c$ and $g_D$ are two fuzzy soft sets of $X$. Then $F_c$ "intersection" $g_D$ is redefined as follows:

$f_c \cap g_D = max(\emptyset, f_c + g_D - 1)$ for each $e \in E$.

Now we prove a proposition which will be needed in the following part of this paper.

**Proposition.**

If $A, B$, and $C$ are the subsets of the parameter set $E$, then $g_c \cap (g_A \cup g_B) = (g_c \cap g_A) \cup (g_c \cap g_B)$.

**Proof.** For all $e \in E$,

$g_c \cap (g_A \cup g_B)(e) = max(\emptyset, g_c(e) + (g_A \cup g_B)(e) - 1) = max(\emptyset, g_c(e) + g_A(e) - 1, g_c(e) + g_B(e) - 1) = max\left(max(\emptyset, g_c(e) + g_A(e) - 1), max(\emptyset, g_c(e) + g_B(e) - 1)\right) = max\left((g_c \cap g_A)(e), (g_c \cap g_B)(e)\right) = ((g_c \cap g_A) \cup (g_c \cap g_B))(e)$.

**Proposition.** If $f : FS(X,E) \rightarrow FS(Y,E')$ and $p : E \rightarrow E'$, let $h_A h_B \in FS(Y,E')$ where $A, B \subseteq E'$, then $f^{-1}(h_a \cap h_b) = f^{-1}(h_a) \cap f^{-1}(h_b)$.

**Proof.** For all $e \in E$, we have

$f^{-1}(h_a \cap h_b)(e) = (h_a \cap h_b)(p(e)) = max(h_a(p(e)) + h_b(p(e)) - 1, 0) = max(f^{-1}(h_a)(e) + f^{-1}(h_b)(e) - 1, 0) = (f^{-1}(h_A) \cap f^{-1}(h_B))(e)$

**Definition.** A family $\tau \subseteq FS(X,E)$ is called a fuzzy soft topology on $X$, if it satisfies the following axioms.

i) For all $\lambda \in [0,1]$, $\lambda \in \tau$.

ii) $f_E, g_E \in \tau$ implies that $f_E \cap g_E \in \tau$.

iii) If $\{ f_{\lambda} \} \subseteq \tau$ is an indexed subfamily of $\tau$, then $\bigcup_{\lambda \in \Delta} f_{\lambda} \in \tau$.

The pair $(X, \tau)$ is called a fuzzy soft topological space. The elements of $\tau$ are called fuzzy soft open sets. The complement of fuzzy soft open sets are called fuzzy soft closed sets.

**Examples for fuzzy soft topological spaces**

1. $\tau = \{ \lambda \cdot \text{absolute fuzzy soft sets/0} \leq \lambda \leq 1 \}$ is a fuzzy soft topology and is called indiscrete fuzzy soft topology on $X$.

2. If $\tau$ equals FS$(X,E)$ then $\tau$ is called discrete fuzzy soft topology on $X$.

3. Let $A \subseteq E$ with $|A| > 1$. Then $\tau = \{ f_E \in FS(X,E) | f_E = 0 \}$ for all $e \in A \Rightarrow f_E \neq 0$ for all $e \in A$ is a fuzzy soft topology on $X$.

4. If $\lambda = 0$, then $\lambda \in \tau$. For all $\lambda \in [0,1]$, $\lambda \neq 0$ for all $e \in A$. Hence $\lambda \in \tau$.

Let $\{ f_{\lambda} \} \subseteq \tau$ be a collection of elements of $\tau$. If $\bigcup_{\lambda \in \Delta} f_{\lambda} \neq 0$ for some $e_0 \in A$, then $f_{\lambda} \neq 0$ for at least one $k$ (say) $\in \Delta$. Therefore $f_{\lambda} e_0 \neq 0$ for all $e \in A$. Hence $\bigcup_{\lambda \in \Delta} f_{\lambda} \in \tau$.

Let $f_0, g_E \in \tau$, then $f_0 = 0$ for all $e \in E$ (or) $f_0 \neq 0$ for all $e \in E$. Similarly we have $g_E = 0$ for all $e \in E$ (or) $g_E \neq 0$ for all $e \in E$. If either $f_0 = 0$ (or) $f_0 = 0$ for all $e \in E$ then $(f \land g_E) = 0 \forall e \in E$. If $f_0 \neq 0$ (or) $g_E \neq 0 \forall e \in E$ then $(f \land g_E) = 0 \forall e \in E$. Therefore $f_0, g_E \in \tau \Rightarrow f_0 g_E \in \tau$. Thus $\tau$ is a fuzzy soft topology on $X$.

4. Let us take the parameter set $E$ by the set of all natural numbers, for all $n \in E$, consider the subset $A_n = \{ 2n-1, 2n \}$ of $E$. Clearly by above example 2.8.3 to each $n \in E$,

$\tau_n = \{ f_E \in FS(X,E) | f_E = 0 \forall e \in A_n \Rightarrow f_0 \neq 0 \forall e \in A_n \}$ is a fuzzy soft topology on $X$. Hence $\bigcap_{n=1}^\infty \tau_n$ is a fuzzy soft topology on $X$.

So $\tau = \{ f_E \in FS(X,E) | f_0 = 0 \forall e \text{if and only if} f_{2n-1} = 0 \}$

**Definition.** Let $(Y, \xi)$ be a fuzzy soft topological space. Then the closure of a fuzzy soft set $g_E$, denoted as $cl(g_E)$, and defined by

$cl(g_E) = \bigcup \{ \beta_E : g_E \subseteq \beta_E \Rightarrow g_E \cap \beta_E \neq 0 \} \subseteq B \subseteq E$.

**Definition.** Let $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy soft topological spaces and $p : E \rightarrow E'$. The function $h : FS(X,E) \rightarrow FS(Y,E')$ is fuzzy soft continuous iff $h^{-1}(g_E) \in \tau$ for all $g_E \in \sigma$ where $B \subseteq E'$. 

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**Proposition** let \((X,\tau)\) and \((Y,\sigma)\) be two fuzzy soft topological spaces let \(g:FS(X,E)\rightarrow FS(Y,E')\)be a fuzzy soft continuous function and \(p:E\rightarrow E'\)consider a fuzzy soft set \(f_A\in FS(Y,E')\) where \(A\subseteq E'\). Then \(cl(g^{-1}(f_A))(e)\subseteq g^{-1}(cl(f_A))(e)\) for all \(e\in E\).

Proof: \(cl(g^{-1}(f_A))=\bigcup\{\tilde{A}/f_B\in\tau, f_B\supseteq g^{-1}(f_A)\cap f_B\neq \emptyset\} \) where \(B\subseteq E\).

\[\bigcup\{\tilde{A}/f_B\in\sigma, g^{-1}(f_B)\supseteq g^{-1}(f_A)\cap g^{-1}(f_B)\neq \emptyset \} \text{ where } B'\subseteq E'\]

\[\bigcup\{\tilde{A}/f_B\in\sigma, f_B\supseteq g^{-1}(f_A)\cap g^{-1}(f_B)\neq \emptyset \} = cl(f_A) \]

That is \(cl(g^{-1}(f_A))(e)\subseteq cl(f_A)(e)\) for all \(e\in E\).

Therefore \(cl(g^{-1}(f_A))(e)\subseteq cl(f_A)(e)\).

**FUZZY SOFT CONNECTED SET**

**Definition** let \((X,\tau)\) be a fuzzy soft topological space. A fuzzy soft set \(f_A\) is said to be fuzzy soft connected set if there exists two fuzzy soft sets \(f_A\) and \(f_B\in FS(X,E)\) where \(A\) and \(B\) are subsets of \(E\) and \(C=A\cup B\) such that (i) \(f_A=f_A\cap f_B\) (ii) there exists \(e_0\in E\) such that \(f_A(e_0)=f_A(e_0)\neq 0\) and \(f_B(e_0)=f_B(e_0)\neq 0\) (iii) \(cl((f_A\cap f_B))=\emptyset \) if and only if \(f_A\) is fuzzy soft connected.

**Examples**

(i) If \((X,\tau)\) be a fuzzy soft topological space with the indiscrete fuzzy soft topology. Then the fuzzy soft set \(f_A\) is fuzzy soft connected.

(ii) If \((X,\tau)\) be a fuzzy soft topological space with the discrete fuzzy soft topology. Then the fuzzy soft set \(f_A\) is fuzzy soft connected.

**Example**

Let \((X,\tau)\) be a fuzzy soft topological space with \(|E|\geq 2\) and \(f_A\) be fuzzy soft set on \(X\) such that \(f_A\subseteq \tilde{E}^2\), then \(f_A\) is fuzzy soft connected.

Let \(B\) be the proper subset of \(E\). Take \(C=E-B\). Let \(f_1\) and \(f_2\) be two fuzzy soft sets on \(E\) defined by

\[f_1(e)=\begin{cases} 1 & \text{if } e \in B \\ 0 & \text{if } e \in C \end{cases} \quad \text{and} \quad f_2(e)=\begin{cases} 1 & \text{if } e \in C \\ 0 & \text{if } e \in B. \end{cases}\]

Let \(g_1=f_1, f_A\) (that is \(g_1(e)=f_1(e), f_A(e)\) for all \(e\in E\)) and \(g_2=f_2, f_A\). Then (i)

\[f_A=g_1\cup g_2 \quad \text{ (ii) } g_A=g_1 \text{ on } B \quad \text{ and } \quad g_A=g_2 \text{ on } C \quad \text{ (iii)} \quad g_A=g_1 \text{ on } E^2 \quad \text{ and } \quad cl(g_2(E))=g_2 \text{ (iv) } g_1(cl(g_2))=0 \quad \text{ and } \quad cl(cl(g_1))=g_1.\]

Thus \(f_A\) is the fuzzy soft disconnected.

**Proposition**

Let \((X,\tau)\) be a fuzzy soft topological space. Let \(f_A\) be fuzzy soft connected set of \(X\) and \(f_B\) is not fuzzy soft connected set of \(X\) with \(f_A\cap f_B\neq \emptyset\). If there exists two fuzzy soft sets \(f_c\) and \(f_0\) such that \(f_B=f_c\cup f_0\) with \(f_c(x_0)=\emptyset\) and \(f_0(y_0)=\emptyset\) for some \(x_0\in E\) and \(cl((f_c)(e))=\emptyset\) then \(f_c\subseteq f_0\).

Proof: Suppose \(f_B=f_c\cup f_0\) with \(f_c(x_0)=\emptyset\) and \(f_0(y_0)=\emptyset\) for some \(x_0\neq y_0\in E\) and \(cl((f_c)(e))=\emptyset\) then \(f_c\subseteq f_0\).

Similarly \(f_0\subseteq f_0\).

Therefore \(f_A\subseteq f_0\).

**Proposition**

Let \((f_A)\alpha\in E\) be a collection of fuzzy soft connected sets of \(X\) with \(\bigcap_{\alpha\in f_A} f_A\neq \emptyset\) then \(\bigcup_{\alpha\in f_A} f_A\) is fuzzy soft connected.

Proof. \(\bigcap_{\alpha\in f_A} f_A\neq \emptyset\) implies that there exists \(e_0\in E\) such that \(\bigcap_{\alpha\in f_A} f_A(e_0)\neq 0\). That is \(\bigcap_{\alpha\in f_A} f_A(e_0)\neq 0\) for every finite subset \(K\) of \(J\)............(1)

Obviously each, \(f_A\subseteq f_A\neq 0\).

We will prove that \(g_B\) is the fuzzy soft connected. Suppose \(g_B\) is not fuzzy soft connected, then there exists two fuzzy soft sets \(f_c\) and \(f_0\) such that \(g_B(f_c\cup f_0)\) there exists \(i_0\in E\) such that \(g_B(f_c)\neq 0\) and \(g_B(f_0)\neq 0\).

(iii) \(cl((f_c))f_0=f_0=cl(cl(f_0))\).

Then by proposition 3.4 for each \(\alpha\in f_A\), either

\(f_A\subseteq f_0\) (or) \(f_A\subseteq f_D\). If there are \(\alpha\neq \alpha_1\in f_A\) such that \(f_A\subseteq f_0\) and \(f_0\subseteq f_D\).

Then \(\emptyset \neq f_{A_{\alpha_1}}\cap f_{A_{\alpha_2}}\subseteq f_0\cap f_D\subseteq cl((f_c)\cap f_D)=\emptyset\), which is a contradiction. Therefore either \(f_A\subseteq f_0\) for all \(\alpha\) (or) \(f_A\subseteq f_D\) for all \(\alpha\).

Assume that \(f_A\subseteq f_0\) for all \(\alpha\). Then \(\bigcup_{\alpha\in f_A} f_A\neq f_0\).

Therefore \(\bigcup_{\alpha\in f_A} f_A\neq f_0\) for all \(\alpha\).

Hence \(\bigcup_{\alpha\in f_A} f_A\neq f_0\).

As \(g_B(f_0)=0\), select \(\alpha\) such that \(\frac{1}{2m}<g_B(f_0)\frac{1}{2m}<f_{A_\alpha_2}(0)\leq g_B(f_0)=f_D(f_0).

Hence \(\emptyset \neq f_{A_{\alpha_2}}(0)\neq f_{A_{\alpha_2}}(0)\).
= \{(f_{A_2} \cap f_p)(j_0) \mid \text{since}\ g_B(j_0) = f_D(j_0)\} \text{ and for all } e \neq j_0, f_{A_2}(e) = \{(f_{A_2} \cap f_p)(e) = 0 \text{ (otherwise) } f_{A_2} = (f_{A_2} \cap f_p) \cup (f_{A_2} \cap f_p)\text{ is fuzzy soft disconnection of } f_{A_2}\text{, which is a contradiction). Therefore } f_{A_2}(e) = 0\text{ for all } e \neq j_0.\text{ As } f_{A_2}(e_0) \neq 0,\text{ We get } e_0 = 0.\text{ As } f_{A_2}(e_0) \leq f_c(e_0) = \{f_c(j_0) \leq \{f_c(j_0) = f_{A_2}(e_0)\text{ (as } f_c \cap f_D = \emptyset \leq 1\). Hence } f_{A_2}(e_0) = \frac{1}{2}.\text{ Select any } \beta_1 \neq \beta_2 \in J.\text{ If } g_B(j_0) = \frac{1}{2m}, f_{A_2}(j_0) = \beta_1 \text{ then } f_{A_2}(e_0) \leq \frac{1}{2}.\text{ If } f_{A_2}(j_0) \leq g_B(j_0) = \frac{1}{2m}, f_{A_2}(j_0) \text{ then also } f_{A_2}(e_0) \leq \frac{1}{2m}.\text{ That is, } (f_{A_2} \cap f_{A_2})(e_0) = 0,\text{ Which implies a contradiction to equation (1). Thus } g_B \text{ is fuzzy soft connected.}

**Proposition**

The image of a fuzzy soft connected set under a fuzzy soft continuous function is fuzzy soft connected.

Proof. Let \((X, \tau)\) and \((Y, \sigma)\) be two fuzzy soft topological spaces, \(p:X \to Y\) and \(g:FS(X,E) \to FS(Y,E')\) be a fuzzy soft continuous map. Let \(f_\lambda\) be fuzzy soft connected set in \(FS(X,E)\). We will prove that \(g(f_\lambda)\) is a fuzzy soft connected set in \(FS(Y,E')\). Suppose \(g(f_\lambda)\) is fuzzy soft disconnected. Then there exists \(f_c\) and \(f_p\) such that (i) \(g(f_\lambda) \cap f_c\) and (ii) \(g(f_\lambda) \cap f_p\) (there exists \(e_1 \neq e_2 \in E'\).

\[
g^{-1}(f_c) \cap g^{-1}(f_p) = g^{-1}(f_c \cap f_p) = g^{-1}(\emptyset) = \emptyset \text{ and } cl(g^{-1}(f_c)) \cap g^{-1}(f_p) \leq g^{-1}(cl(f_c)) \cap g^{-1}(f_p) = g^{-1}(\emptyset) = \emptyset.\]

As \(f_c \cap f_p = \emptyset\), we get \(p^{-1}(e_1) \neq \emptyset\) (empty crisp set) and \(\emptyset = g(f_\lambda(e)) = V[f_\lambda(e)]\) if \(e_1 \in p^{-1}(e_1) \cap A\). As \(\alpha_1 \in B=\{A \subseteq E'\}\). Find \(\beta_1 \in E\) such that \(p(\beta_1) = e_1\) and \(f_\lambda(\beta_1) = \bar{0}\). Similarly, find \(\beta_2 \in E\) such that \(p(\beta_2) = e_2\) and \(f_\lambda(\beta_2) > 0\). Therefore

\[
g^{-1}(f_c)(\beta_1) = f_c(p(\beta_1)) = f_c(e_1) = \emptyset \text{ and } f_c(p(\beta_2)) = f_c(e_2) \geq f_\lambda(\beta_2) > 0.
\]

(\(g^{-1}(f_c) \cap f_\lambda(\beta_1) = g^{-1}(f_c)(\beta_1) \cap A \cap f_\lambda(\beta_1) = f_c(p(\beta_1)) \cap A \cap f_\lambda(\beta_1) = f_c(e_1) \cap A \cap f_\lambda(\beta_1)\)

\[
eq \emptyset.\text{ Similarly } g^{-1}(f_p) \cap f_\lambda(\beta_2) = g^{-1}(f_p)(\beta_2) \cap A \cap f_\lambda(\beta_2) = f_p(p(\beta_2)) \cap A \cap f_\lambda(\beta_2) = f_p(e_2) \cap A \cap f_\lambda(\beta_2)\]

\[
eq \emptyset.\text{ Since } f_\lambda(p(\beta_2)) = g(f_\lambda(p(\beta_2)) \geq f_\lambda(\beta_2))\). Thus \(f_\lambda = g^{-1}(f_c) \cap f_p \cup g^{-1}(f_p) \cap f_\lambda\) which is a contradiction (as \(f_\lambda\) is fuzzy soft connected). Hence fuzzy soft continuous image of a fuzzy soft connected set is fuzzy soft connected.

**CONCLUSION**

In this paper we have investigated connectedness in fuzzy soft topological spaces and some of its properties.

**REFERENCES**


