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Asymptotic Expansions of Legendre Wavelet

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Abstract

A new construction of wavelet on the bounded interval (-1, 1) $\subset$ **R** using Legendre polynomial and the wavelet expansion in terms of Legendre polynomial is presented.

Keywords: Legendre polynomial; Legendre Wavelet transform.

INTRODUCTION

Special functions play an important role in the construction of wavelets. Pathak and Dixit [1] have constructed Bessel wavelets using Bessel functions. But the above construction of wavelets is on semi-infinite interval (0, ∞). Wavelets on finite intervals involving solution of certain Sturm-Liouville system have been studied by U. Depczynski [2]. In this chapter we describe a new construction of wavelet on the bounded interval (-1, 1) $\subset$ **R** using Legendre polynomial and the wavelet expansion in terms of Legendre polynomial is presented.

Let X denote the space $L^p(-1,1),\ 1 \le p < \infty,\ \text{or}\, C[-1,1]$ endowed with the norms

$$\| f \|_p = \left[\frac{1}{2}\int_{-1}^{1} | f(x) |^p \, dx\right]^{1/p} < \infty, \quad 1 \le p < \infty, \qquad (1.1)$$

$$\| f \|_C = \sup_{-1\le x\le 1} | f(x) |. \qquad (1.2)$$

An inner product on X is given by

$$< f, g > = \frac{1}{2}\int_{-1}^{1} f(x)\, \overline{g(x)}\, dx \qquad (1.3)$$

As usual we denote the Legendre polynomial of degree n $\in \mathbf{N}_0$ by $P_n(x)$, i.e.

$$P_n(x) = \left(2^n\ n!\right)^{-1} \left(\frac{d}{dx}\right)^n (x^2 - 1)^n\ ; \quad x \in [-1,1].$$

For these polynomials one has

(i) $| P_n(x) | \le P_n(1) = 1 \quad ; \qquad x \in [-1,1]$ (1.4)

(ii) $(1 - x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0$; (1.5)

(iii) $P_n'(1) = \dfrac{n(n+1)}{2}$. (1.6)

The Legendre transform of a function $f \in X$ is defined by

$$L[f](k) = \hat{f}(k) = \frac{1}{2}\int_{-1}^{1} f(x) P_k(x) dx;\ k \in \mathbf{N}_0 \qquad (1.7)$$

The operator L associates to each $f \in X$ sequence of real (complex) numbers $\left\{\hat{f}(k)\right\}_{k=0}^{\infty}$, called the Fourier Legendre coefficients.

The inverse Legendre transform is given by

$$L[f]^{v}(x) = f(x) = \sum_{k=0}^{\infty} (2k+1)\ \hat{f}(k)\, P_k(x). \qquad (1.8)$$

Lemma 1.1. Assume f, g $\in X$, k $\in \mathbf{N}_0$ and c $\in$ **R**, then

(i) $\left|L[f]\,(k)\right| \le \|f\|_{X;}$

(ii) $L[f + g]\,(k) = L[f]\,(k) + L[g]\,(k)$, $L[cf](k) = cL[f](k)$;

(iii) $L[f]\,(k) = 0$ for all $k \in \mathbf{N}_0$ iff f(x) = 0 a.e ;

(iv) $L[P_k]\,(j) = \begin{cases} \dfrac{1}{2k+1}, & k = j \\ 0, & k \ne j,\ (k, j) \in N_0 \end{cases}$

Let us recall the function K(x,y,z) which plays role in our investigation

$$K(x, y, z) = \begin{cases} 1 - x^2 - y^2 - z^2 + 2xyz, & z_1 < z < z_2 \\ 0 & \text{otherwise}, \end{cases} \qquad (1.9)$$

where $z_1 = xy - [(1-x^2)(1-y^2)]^{1/2}$ and $z_2 = xy + [(1-x^2)(1-y^2)]^{1/2}$.

Then the function K(x,y,z) possesses the following properties;

(i) K(x,y,z) is symmetric in all the three variables

(ii) $\int_{-1}^{1} K(x,y,z)dz = \pi$.

Also it has been shown in [3] that

$$P_k(x)\,P_k(y) = \frac{1}{\pi}\int_{-1}^{1} P_k(z)K(x,y,z)dz \qquad (1.10)$$

Applying (1.8) to (1.10), we have

$$K(x,y,z) = \frac{\pi}{2}\sum_{k=0}^{\infty}(2k+1)\,P_k(x)\,P_k(y)\,P_k(z) \qquad (1.11)$$

The generalized Legendre translation τ_y for $y \in [-1,1]$ of a function $f \in X$ is defined by

$$(\tau_y f)(x) = f(x,y) = \frac{1}{\pi}\int_{-1}^{1} f(z)\ K(x, y,z)dz \qquad (1.12)$$

Using Hölder's inequality it can be shown that

$$\|\tau_y f\|_X \le \|f\|_X \qquad (1.13)$$

and the map $y \to \tau_y f$ is a positive linear operator from X into itself.

As in [3], for functions f,g defined on [-1,1] the generalized Legendre convolution is given by

$$(f*g)\ (x) = \frac{1}{2}\int_{-1}^{1}(\tau_y f)(x)\ g(y)dy$$

$$= \frac{1}{2}\int_{-1}^{1}(\tau_x f)(y)\,g(y)dy$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} f(z)\,g(y)\,K(x, y,z)\,dy\,dz \qquad (1.14)$$

Lemma 1.2. If $f \in X, g \in L^1(-1,1)$, then the convolution (f*g) (x) exists (a.e.) and belongs to X. Moreover,

$$\|f*g\|_X \le \|f\|_X \|g\|_1, \qquad (1.15)$$

$$(f*g)^\wedge(k) = \hat{f}(k)\hat{g}(k). \qquad (1.16)$$

The proof can be found in [3].

For any $f \in L^2(-1,1)$ the following Parseval identity holds for Legendre transform,

$$\sum_k (2k+1)\,|\hat{f}(k)|^2 = \|f\|_2^2. \qquad (1.17)$$

LEGENDRE WAVELET

For a function $\psi \in X$, define the dilation D_a by

$$D_a\psi(t) = \psi(at), \qquad 0 < a \le 1. \qquad (2.1)$$

Using the Legendre translation and the above dilation, the wavelet $\psi_{b,a}(t)$ is defined as follows:

$$\psi_{b,a}(t) = \tau_b D_a \psi(t) = \tau_b \psi(at) \qquad (2.2)$$

$$= \frac{1}{\pi}\int_{-1}^{1} K(b,t,z)\,\psi(az)dz, \quad \psi \in X, \qquad (2.3)$$

where $-1 \le b \le 1$ and $0 < a \le 1$. The integral is convergent by virtue of (1.13).

Now, using the wavelet $\psi_{b,a}$ the Legendre wavelet transform (LWT) is defined as follows:

$$(L_\psi f)(b,a) = \langle f(t),\ \psi_{b,a}(t)\rangle. \qquad (2.4)$$

$$= \frac{1}{2}\int_{-1}^{1} f(t)\overline{\psi_{b,a}(t)}\ dt \qquad (2.5)$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} f(t)\overline{\psi(az)}\,K(b,t,z)dz\ dt \qquad (2.6)$$

Provided the integral is convergent.

Since by (1.13) and (2.2) $\psi_{b,a} \in X$ whenever $\psi \in X$, by Lemma 1.2, the integral (2.6) is convergent for $f \in L^1(-1,1)$.

The admissibility condition for the Legendre wavelet is given by

$$A_\psi = \sum_{k=0}^{\infty}\frac{|\hat{\psi}(k)|^2}{k} < \infty \qquad (2.7)$$

From (2.7) it follows that $\hat{\psi}(0) = 0$. But

$$\hat{\psi}(k) = \frac{1}{2}\int_{-1}^{1}\psi(t)\ P_k(t)dt$$

yields $2\hat{\psi}(0) = \int_{-1}^{1}\psi(t)\,P_0(t)dt = \int_{-1}^{1}\psi(t)dt = 0$.

Hence, ψ (t) changes sign in (-1,1) therefore it represents a wavelet.

THE DISCRETE TRANSFORM

The continuous Legendre wavelet transform of the function f in terms of two continuous parameters a and b can be converted into a semi-discrete Legendre wavelet transform by assuming that a = 2^{-m}; $m \in \mathbf{Z}$ and $-1 \le b \le 1$.

Now, we assume that $\psi \in L^2(-1,1)$ satisfies the so called "stability condition"

$$A \le \sum_{m=-\infty}^{\infty} |\hat{\psi}(2^{-m}k)|^2 \le B \tag{3.1}$$

for certain positive constants A and B, $0 < A \le B < \infty$. The function $\psi \in L^2(-1,1)$ satisfying (3.1) is called dyadic wavelet.

Using the definition (2.4), we define the semi-discrete Legendre wavelet transform of any $f \in L^2(-1,1)$ by

$$(L_m^{\psi} f)(b) = (L_{\psi} f)(b, \frac{1}{2^m}) = \langle f(t), \psi_{b,2^{-m}}(t) \rangle \tag{3.2}$$

$$= \frac{1}{2}\int_{-1}^{1} f(t)\tau_b \overline{\psi(2^{-m}t)}dt \tag{3.3}$$

$$= (f * \overline{\psi}_m), \tag{3.4}$$

where

$$\psi_m(z) = \psi(2^{-m}z), \qquad m \in \mathbf{Z}.$$

Now, using Parseval identity (1.17), (3.1) yields the following:

$$A\|f\|_2^2 \le \sum_{m=-\infty}^{\infty} \|L_m^{\psi} f\|_2^2 \le B\|f\|_2^2, \quad f \in L^2(-1,1). \tag{3.5}$$

Definition 3.1. A function $\tilde{\psi} \in L^2(-1,1)$ is called a dyadic dual of a dyadic wavelet ψ, if every $f \in L^2(-1,1)$ can be expressed as

$$f(t) = \sum_m \int_{-1}^{1} (L_m^{\psi} f)(b) \left(\hat{\tilde{\psi}}(2^{-m}k) P_k(t) \right)^{\vee} (b) db. \tag{3.6}$$

So far we have considered semi-discrete Legendre wavelet transform of any $f \in L^2(-1,1)$ discretizing only variable a. Now, we discrete the translation parameter b also by restricting it to the discrete set of points

$$b_{m,n} = \frac{n}{2^m} b_0, \; m \in \mathbf{Z}, \; n \in \mathbf{N_0}, \tag{3.7}$$

where $b_0 \in [-1,1]$ is a fixed constant. We write

$$\psi_{b_0;m,n}(t) = \psi_{b_{m,n};a_m}(t) = \psi(2^{-m}t, 2^{-m}n\ b_0) \tag{3.8}$$

Then the discrete Legendre wavelet transform of any $f \in L^2(-1,1)$ can be expressed as

$$(L_{\psi} f)\ (b_{m,n;} a_m) = \langle f, \psi_{b_0;m,n} \rangle \quad m \in \mathbf{Z}, n \in \mathbf{N_0}. \tag{3.9}$$

The "stability" condition for this reconstruction takes the form

$$A\|f\|_2^2 \le \sum_{\substack{m \in Z \\ n \in N_0}} \left| \langle f, \psi_{b_0;m,n} \rangle \right|^2 \le B\|f\|_2^2, \; f \in L^2(-1,1), \tag{3.10}$$

where A and B are positive constants such that $0 < A \le B < \infty$.

FRAMES AND RIESZ BASIS IN L^2 (-1,1).

In this section, using $\psi_{b_0;m,n}$ a frame is defined and Riesz basis of L^2 (-1,1) is studied.

Definition 4.1. A function $\psi \in L^2(-1,1)$ is said to generate a frame $\{\psi_{b_0;m,n}\}$ of $L^2(-1,1)$ with sampling rate b_0 if (3.10) holds for some positive constants A and B. If A = B, then the frame is called a tight frame.

Definition 4.2. A function $\psi \in L^2(-1,1)$ is said to generate a Riesz basis of $\{\psi_{b_0;m,n}\}$ with sampling rate b_0 if the following two properties are satisfied.

(i) The linear span $\langle \psi_{b_0;m,n} : m \in \mathbf{Z} \rangle$ is dense in L^2(-1,1) (4.1)

(ii) There exist positive constants A and B with $0 < A \le B < \infty$ such that

$$A\|\{c_{m,n}\}\|_2^2 \le \left\| \sum_{\substack{m \in Z \\ n \in N_0}} c_{m,n} \psi_{bo;m,n} \right\|_2^2 \le B \left\| \{c_{m,n}\} \right\|_{\ell^2}^2 \quad 4.2)$$

For all $\{c_{m,n}\} \in \ell^2(\mathbf{N}_0^2)$. Here A and B are called the Riesz bounds of $\{\psi_{b_0;m,n}\}$

ORDER OF APPROXIMATION

In this section following the technique of Depczynski [2] a discussion on the expansion of $f \in L^2(-1,1)$ in wavelet series is given and order of wavelet coefficient is obtained.

Let N_m, $m \in \mathbf{N}$, be a strictly increasing sequence of natural numbers and denote by V_m the space

$$V_m = \text{span}\{f_i : i \in \Delta\}, \tag{5.1}$$

where Δ = { 1, ---, N_m} is any index set and {f_i} is the basis of Hilbert space $L^2(-1,1)$.

The spaces V_m are linear and closed subspaces of $L^2(-1,1)$ and $V_m \subset V_{m+1}$.

Moreover,

$$\overline{\bigcup_{m=1}^{\infty} V_m} = L^2(-1,1). \tag{5.2}$$

The orthogonal complement of V_m in V_{m+1} is denoted by W_m, i.e. we have

$$V_{m+1} = V_m \oplus W_m \tag{5.3}$$

From (5.2) and (5.3), we have

$$V_1 \oplus W_1 \oplus W_2 \oplus --- = L^2(-1,1). \tag{5.4}$$

Now, we study the approximation properties of the spaces V_m.

Let us consider the Legendre differential equation

$$(1-x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0 \tag{5.5}$$

together with the following homogeneous boundary conditions

$$U_{-1}(f) = a_1 f(-1) + a_2 f(1) = 0,$$

$$U_1(f) = b_1 f(-1) + b_2\, f(1) = 0.$$

The eigenvalues of the above boundary value problem are given by $\lambda_n = n^2$, $n \in \mathbf{N}$.

Let P_0, P_1, P_2, --- be the corresponding eigenfunctions.

Now, we introduce the space

$$D = \{f \in L_2^2[-1,1]; U_{-1}(f) = 0 \text{ and } U_1(f) = 0\}$$

where $L^2[-1,1] = \{ f \in C'[-1,1];\, f'' \in L^2(-1,1)\}$.

We will always assume that zero is no eigenvalue of (5.5). From [2.p 236] we know that the Green's function g(x, y) of (5.5) is given by

$$g(x,y) = \sum_{n=0}^{\infty} \frac{P_n(x)\overline{P_n(y)}}{\lambda^n}, \quad (x,y) \in [-1,1]\times[-1,1]$$

and $G : L_w^2(-1,1) \longrightarrow L^2(-1,1)$ defined by

$$G(f) = \int_{-1}^{1} g(x,y)\, f(y)\, w(y)\, dy$$

is a compact operator with range D.

For $r \in \mathbf{N}$, set

$$G^r = G \circ --- \circ G(r \text{ times})$$

The iterated spaces D^r, $r \in \mathbf{N}_0$, are then defined by

$$D^r = \{G^r(f) : f \in L^2(-1,1)\}.$$

Note that $D^0 = L^2(-1,1)$ and $D^1 = D$ and $D^r \supset D^{r+1}$.

From [2, p. 237], we also note that

$$f \in D^r \Leftrightarrow \sum_{n=0}^{\infty} \lambda_n^{2\lambda} |< f, P_n >|^2 < \infty \tag{5.6}$$

We recall the following result from [2, p.237] about approximation with the spaces V_m = span {P_1, P_2 ----P_N}, $N_m < N_{m+1}$.

Theorem 5.1. Let $P_m f = \sum_{n=0}^{N_m} < f, P_n > P_n$ denote the orthogonal projection of f onto V_m. Then for $r \in \mathbf{N}$ and f∈ D^r,

$$\| f - P_m f \|_2 = O\left(N_m^{-2r+\frac{1}{2}}\right), m \longrightarrow \infty. \tag{5.7}$$

Let $\psi_{b_0;m,n}$ be the discrete Legendre wavelets defined by (3.8). Then for any $f \in L^2(-1,1)$, there exists a sequence {$c_{m,n}$} such that

$$Q_m f = \sum_{n=1}^{M_m} c_{m,n}\; \psi_{b_0;m,n}\,, \tag{5.8}$$

where $M_m = N_{m+1} - N_m$.

Suppose $Q_m : L^2(-1,1) \longrightarrow L^2(-1,1)$ be the operator such that $Q_m = P_{m+1} - P_m$, where P_m is the projection operator defined in Theorem 5.1. The range $Q_m = W_m$, where $V_{m+1} = V_m \oplus W_m$.

Following Chui [4], we have the following definition for the Riesz basis of the spaces W_m which will be used in sequel.

The discrete legendre wavelet $\psi_{b_0;m,n}$ forms a Riesz basis of the spaces W_m, if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\sum_n |c_{m,n}|^2 \le \left\| \sum_n c_{m,n}\psi_{b_0;m,n} \right\|_2^2 \le B\sum_n |c_{m,n}|^2 \tag{5.9}$$

for all sequences {$c_{m,n}$} such that $\sum_n |c_{m,n}|^2 < \infty$.

Theorem 5.2. Let $\{\psi_{b0;m,n}\}$ be a Riesz basis of the spaces W_m with Riesz bound A > 0. Then for every $r \in \mathbf{N}$ and $f \in D^r$,

$$\left(\sum_n |c_{m,n}|^2\right)^{1/2} = O\left(N_m^{-2r+\frac{1}{2}}\right), \quad m \longrightarrow \infty. \qquad (5.10)$$

Proof.

$$\|Q_m f\|_2 = \| P_{m+1}f - P_m f \|_2 \le \| f - P_m f \|_2 + \| f - P_{m+1} f \|_2$$

From (5.7), it follows that

$$\| Q_m f \|_2 = O\left(N_m^{-2r+\frac{1}{2}}\right). \quad (5.11)$$

Using the Riesz stability condition (5.9) and relation (5.8), we have

$$\| Q_m f \|_2 = \left\|\sum_n c_{m,n}\psi_{b_0;m,n}\right\|_2^2 \ge A\sum_n |c_{m,n}|^2;$$

so that

$$\sum_n |c_{m,n}|^2 \le \frac{1}{A}\| Q_m f \|_2^2 .$$

From (5.11), it follows that

$$\left(\sum_n |c_{m,n}|^2\right)^{1/2} = O\left(N_m^{-2r+\frac{1}{2}}\right), (5.12)$$

which completes the proof of Theorem .

Theorem 5.3 Let $\psi_{b_0;m,n}$ be a Riesz basis of the spaces W_m with upper Riesz bound $B < \infty$. Assume that $\sum_{m=1}^{\infty} N_m^{-2\epsilon} < \infty$ for some $\epsilon > 0$. If $r \in \mathbf{N}$ and $f \in L_w^2(-1,1)$ with $\left(\sum_n |c_{m,n}|^2\right)^{1/2} = O\left(N_m^{-2r-\epsilon}\right), n \longrightarrow \infty$, then $f \in D^r$.

Proof. Let us consider the partial sum $\sum_{n=N_m+1}^{N_{m+1}} \lambda_n^{2r} |\langle f, T_n \rangle_w|^2$ of series $\sum_{n=0}^{\infty} \lambda_n^{2r} |\langle f, T_n \rangle_w|^2$.

For $\lambda_n = n^2$, we obtain

$$\sum_{n=N_m+1}^{N_{m+1}} \lambda_n^{2r} |\langle f, T_n \rangle_w|^2 \le N_{m+1}^{4r} \sum_{n=N_m+1}^{N_{m+1}} |\langle f, T_n \rangle_w|^2 . \quad (5.13)$$

Now, $Q_m f = P_{m+1} - P_m f$.

Therefore

$$\| Q_m f \|_2^2 = \| P_{m+1}f - P_m f \|_2^2$$

$$= \sum_{n=N_m+1}^{N_{m+1}} |\langle f, T_n \rangle_w|^2 \qquad (5.14)$$

Using the Riesz stability condition (5.9), we have

$$\sum_{n=N_m+1}^{N_{m+1}} |\langle f, T_n \rangle_w|^2 = \| Q_m f \|_2^2 \le B \sum_n |c_{m,n}|^2 \quad (5.15)$$

Next using the assumption $\left(\sum_n |c_{m,n}|^2\right)^{1/2} \le C_2 N_m^{-2r-\epsilon}$,

from (5.13), (5.14) and (5.15), it follows that

$$\sum_{n=N_m+1}^{N_{m+1}} \lambda_n^{2r} |\langle f, T_n \rangle_w|^2 \le \mathbf{N_{m+1}^{4r} C_2^2} \ \mathbf{N_m^{-4r-2\epsilon}} = C_2^2 \mathbf{B}\left(\frac{\mathbf{N_{m+1}}}{\mathbf{N_m}}\right)^{4r} N_m^{-2\epsilon} .$$

Let C > 0 with $C_2^2 \ B\left(\frac{N_{m+1}}{N_m}\right)^{4r} < C$ for all $m \in \mathbf{N}$.

Then from the covergence of $\sum_{m=1}^{\infty} N_m^{-2\epsilon} < \infty$, it follows that

$$\sum_{m=1}^{\infty}\sum_{n=N_m+1}^{N_{m+1}} \lambda_n^{2r} |\langle f, T_n \rangle_w|^2 \le C \sum_{m=1}^{\infty} N_m^{-2\epsilon} < \infty .$$

From (5.6) it easily follows that $f \in D^r$.

This complete the proof .

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