International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 16 (2018) pp. 12653-12656

Legendrian warped product submanifolds in S-Space forms

Sanjay Kumar Tiwari

Department of Applied Science and Humanities, Ajay Kumar Garg Engineering College,
27th Km. Mile stone Delhi Hapur bypass road, NH-24, Adhyatmiknagar, Ghaziabad – 201009, India.

Abstract

In the present paper we obtain a sharp relationship between the warping function of a warped product submanifold isometrically immersed in an S-space form and some application derived.

2010 Mathematics Subject Classifications: 53C25, 53C40

Keywords: Warped product, mean curvature, S-space form and Legendrian submanifolds.

INTRODUCTION

D.E. Blair [**1**] introduced the notion of S-manifolds for manifolds with an *f*-structure as the analogue of the Kaehler structure in almost Hermitian case and to the Sasakian structure in the almost contact case.

In [**3**] B.Y. Chen established sharp relationship between the warping function of a warped product submanifold isometrically immersed in a real space form and the squared mean curvature. In [**7**] Y.H. Kim and D.W. Yoon derived a similar inequality for totally real warped products in locally conformal Kaehler space forms.

In this paper, we establish similar relationship for Legendrian warped product submanifolds in S-space forms.

PRELIMINARIES

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of positive dimension n_1 and n_2, with Riemannian metrics g_1 and g_2 respectively and f_w a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold $M_1 \times_{f_w} M_2 = (M_1 \times M_2, g)$,

where $g = g_1 + f_w^2 g_2$ (see[**2**] and [**3**]).

Let $x: M_1 \times_{f_w} M_2 \to \overline{M}$ (c) be an isometric immersion of a warped product $M_1 \times_{f_w} M_2$ into a Riemannian manifold $\overline{M}$ (c) with constant sectional curvature c. We denote σ the second fundamental form of x and $H_i = \frac{1}{n_i}(trace\,\sigma_i)$, where $trace\,\sigma_i$ is the trace of σ restricted to M_i, and $n_i = dim M_i$ $(i = 1,2)$. We call H_i $(i = 1,2)$ the partial mean curvature vectors. The immersion x is said to be mixed totally geodesic if $h(X, Z) = 0$, for any vector fields X and Z tangent to M_1 and M_2, respectively.

Now, let $(\overline{M}, g)$ be a $(2m+s)$-dimensional Riemannian manifold. $\overline{M}$ is said to be a metric *f*-manifold if there exist a (1,1) tensor field f, s-global unit vector fields $\xi_1, \ldots, \xi_s$ (called structure vector fields) and s 1-forms $\eta_1, \ldots, \eta_s$ on $\overline{M}$ such that

(2.1) $$f^2 X = -X + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha, \qquad g(X, \xi_\alpha) = \eta_\alpha(X),$$

$$f\xi_\alpha = 0, \qquad \eta_\alpha o f = 0,$$

and $$g(fX, fY) = g(X,Y) - \sum_{\alpha=1}^{n} \eta_\alpha(X)\eta_\alpha(Y),$$

for any X, $Y \in T\overline{M}$, where $T\overline{M}$ denote the Lie algebra of vector fields and $\alpha = 1, \ldots, s$.

The *f*-structure f is said to be normal if

(2.2) $$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ is the Nijenhuis tensor of f. Let F denote the fundamental 2-form given by $F(X, Y) = g(X, fY)$, for any X, $Y \in T\overline{M}$. $\overline{M}$ is said to be an S-manifold if the *f*-structure is normal and

$$\eta_1 \wedge \ldots\ldots \wedge \eta_s \wedge (d\eta_\alpha) \neq 0, \qquad F = d\eta_\alpha,$$

for any $\alpha = 1, \ldots, s$. When s =1, S-manifolds are Sasakian manifolds.

A plane section π in $T_p\overline{M}$ is called an *f*-section if it is spanned by X and fX, where X is a unit tangent vector field orthogonal to the distribution spanned by structure vector fields. The sectional curvature $K(\pi)$ of an *f*-section π is called *f*- sectional curvature. An S-manifold is said to be an S-space form if it has constant *f*- sectional curvature. We shall denote an S-manifold $\overline{M}$ with constant *f*- sectional curvature c by $\overline{M}(c)$. The curvature tensor of an S-space form $\overline{M}(c)$ is given by [**10**]

(2.3) $R(X, Y, Z, W) = g(\sigma(X, W), \sigma(Y, Z))$

$-g(\sigma(X, Z), \sigma(Y, W))$

$$+\sum_{\alpha,\beta}(g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W)$$

$$+g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z))$$

$$+\frac{c+3s}{4}(g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW))$$

$$+\frac{c-s}{4}(F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)),$$

for any X, Y, Z, W $\in T\overline{M}$.

Let M be an n-dimensional submanifold isometrically immersed in $\overline{M}(c)$ and denote by σ, ∇ and $\nabla^\perp$ the second fundamental form of M, the induced connection on M and on the normal bundle $T^\perp M$. Then the Gauss and Weingarten formulae are given by;

(2.4) $\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$,

$$\nabla_X N = -A_N X + \nabla_X^\perp N,$$

respectively, for any vector fields X, Y tangent to M and N normal to M, where A_N is the shape operator in the direction of N. The second fundamental form and the shape operator are related by

(2.5) $g(\sigma(X,Y), N) = g(A_N X, Y)$.

Let R be Riemannian curvature tensor of M, then the Gauss equation is given by

(2.6) $\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(\sigma(X, W), \sigma(Y, Z))$

$$- g(\sigma(X, Z), \sigma(Y, W)),$$

for all $X, Y, Z, W \in TM$.

Let $p \in M$ and $\{e_1,...,e_n,...,e_{2m+s}\}$ an orthonormal basis of the tangent space $T_p\overline{M}(c)$, such that $e_1,...,e_n$ are tangent to M at p. The mean curvature vector $H(p)$ is defined by

(2.7) $$H(p) = \frac{1}{n}\sum_{i=1}^{n}\sigma(e_i, e_i).$$

The submanifold is said to be minimal if H vanishes identically and it is said to totally geodesic if $\sigma(X, Y) = 0$, for any $X, Y \in TM$.

We set

(2.8) $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r),\ \ i, j \in \{1,...,n\},\ \ \ r \in \{n+1,...,2m+s\}$

and $$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)).$$

For any $X \in TM$, we put $fX = TX + NX$, where TX and NX are the tangential and normal components of fX, respectively. The submanifold is said to be invariant if N is identically zero, that is, if $fX \in TM$, for any $X \in TM$ and it is said to be anti-invariant if T is identically zero, that is, if $fX \in T^\perp M$, for any $X \in TM$.

It is well-known that

(2.9) $$\sigma(X, \xi_\alpha) = -NX,$$

for any $X \in TM$ and any $\alpha = 1,...,s$. In particular, $\sigma(\xi_\alpha, \xi_\beta) = 0$, for any $\alpha, \beta = 1,..., s$.

We recall the following Chen's Lemma for later use.

Lemma [1] . Let $n \geq 2$ and $a_1, a_2, ..., a_n$, b real numbers such that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right)$$

Then $2a_1 a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

LEGENDRIAN WARPED PRODUCT SUBMANIFOLDS

In this section, we investigate Legendrian warped product submanifolds in S-space form $\overline{M}$(c). A submanifold M normal to $\xi_\alpha, \alpha = 1, ..., s$ in an S-space form $\overline{M}$(c) is said to be an anti-invarient submanifold if f maps any tangent space of M into the normal space, that is, $f(T_pM) \subset T_p^\perp M$, for every $p \in M$. If the dimension of an anti-invariant submanifold M is maximum, then M is called a Legendrian submanifold.

Theorem 3.1 *Let x:* $M_1 \times_{f_w} M_2 \to \overline{M}$ *(c) be a Legendrian isometric immersion of an n-dimensional warped product* $M_1 \times_{f_w} M_2$ *into a (2m+s)-dimensional S- space form* $\overline{M}$ *(c) of point wise constant f-sectional curvature c. Then,*

(3.1) $$\frac{\Delta f_w}{f_w} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c+3s}{4}$$

where $n_i = \dim M_i$, $i = 1,2$, *and* Δ *is the Laplacian operator of* M_1. *The equality case of (2.1) holds if and only if x is a mixed totally geodesic immersion and* $n_1H_1 = n_2H_2$, *where* H_i, $i = 1,2$, *are the partial mean curvature vectors.*

Proof: Let $M_1 \times_{f_w} M_2$ be a C-totally real warped product submanifold into a generalized Sasakian space form $\overline{M}(c)$ of constant f-sectional curvature c. Since $M_1 \times_{f_w} M_2$ is a warped product, it can be easily seen that

(3.2) $$\nabla_X Z = \nabla_Z X = \frac{1}{f_w}(X f_w)Z,$$

for any vector fields X, Z tangent to M_1, M_2 respectively. If X, Z are unit vector fields, it follows that the sectional curvature $K(X\wedge Z)$ of the plane section spanned by X and Z is given by

(3.3) $K(X\wedge Z) = g\left(\nabla_Z\nabla_X X-\nabla_X\nabla_Z X, Z\right)=\frac{1}{f_w}\left\{(\nabla_X X)f_w - X^2 f_w\right\},$

We consider local orthonormal basis $\{e_1,\ldots e_n, e_{n+1},\ldots,e_{2m+1}=\xi\}$, such that $e_1,\ldots,e_{n_1}$ are tangent to M_1, $e_{n_1+1},\ldots,e_n$ are tangent to M_2, e_{n+1} is parallel to the mean curvature vector H. Then, using (3.3) we obtain

(3.4) $$\frac{\Delta f_w}{f_w}=\sum_{j=1}^{n_1}K\left(e_j\wedge e_u\right),$$

for each $u\in\{n_1+1,\ldots,n\}$.

From equation of Gauss, we obtain

(3.5) $$2\tau=n^2\|H\|^2-\|\sigma\|^2+n(n-1)\frac{c+3s}{4},$$

where τ denotes the scalar curvature of $M_1\times_{f_w}M_2$, i.e.

$$\tau=\sum_{1\le j<u\le n}K(e_j,e_u)$$

We set

(3.6) $$\delta=2\tau-n(n-1)\frac{c+3s}{4}-\frac{n^2}{2}\|H\|^2.$$

From (3.5) and (3.6), it follows that

(3.7) $$n^2\|H\|^2=2\left(\delta+\|\sigma\|^2\right).$$

With respect to the above orthonormal basis, (3.7) takes the following form

$$\left(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\right)^2=2\left\{\begin{array}{l}\delta+\sum_{i=1}^{n}\left(\sigma_{ii}^{n+1}\right)^2\\ +\sum_{i\ne j}\left(\sigma_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2m+s}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2\end{array}\right\},$$

from above equation, we obtain

$$\left(\sum_{i=1}^{3}a_i\right)^2=2\left\{\delta+\sum_{i=1}^{3}a_i^2+\sum_{1\le i\ne j\le n}\left(\sigma_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2m+s}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2 - \sum_{2\le j\ne k\le n_1}\sigma_{jj}^{n+1}\sigma_{kk}^{n+1}-\sum_{n_1+1\le u\ne t\le n}\sigma_{uu}^{n+1}\sigma_{tt}^{n+1}\right\}$$

where $a_1=\sigma_{11}^{n+1}$, $a_2=\sum_{i=2}^{n_1}\sigma_{ii}^{n+1}$ *and* $a_3=\sum_{t=n_1+1}^{n}\sigma_{tt}^{n+1}$.

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for n =3), i.e.

$$\left(\sum_{i=1}^{3}a_i\right)^2=2\left(b+\sum_{i=1}^{3}a_i^2\right),$$

with

$$b=\delta+\sum_{1\le i\ne j\le n}\left(\sigma_{ij}^{n+1}\right)^2+\sum_{r=n+2}^{2m+s}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{r}\right)^2 -\sum_{2\le j\ne k\le n_1}\sigma_{jj}^{n+1}\sigma_{kk}^{n+1}-\sum_{n_1+1\le u\ne t\le n}\sigma_{uu}^{n+1}\sigma_{tt}^{n+1}.$$

Then $2a_1a_2\ge b$, with equality holding if and only if $a_1+a_2=a_3$.

In the case under consideration, we have

(3.8) $$\sum_{1\le j<k\le n_1}\sigma_{jj}^{n+1}\sigma_{kk}^{n+1}-\sum_{n_1+1\le u<t\le n}\sigma_{uu}^{n+1}\sigma_{tt}^{n+1} \ge\frac{\delta}{2}+\sum_{1\le\alpha<\beta\le n}\left(\sigma_{\alpha\beta}^{n+1}\right)^2+\frac{1}{2}\sum_{r=n+2}^{2m+s}\sum_{\alpha,\beta=1}^{n}\left(\sigma_{\alpha\beta}^{r}\right)^2.$$

Equality holds if and only if we have

(3.9) $$\sum_{i=1}^{n_1}\sigma_{ii}^{n+1}=\sum_{t=n_1+1}^{n}\sigma_{tt}^{n+1}.$$

Again, using the Gauss equation, we have

(3.10) $$n_2\frac{\Delta f_w}{f_w}=\tau-\sum_{1\le j<k\le n_1}K(e_j\wedge e_k)-\sum_{n_1+1\le u<t\le n}K(e_u\wedge e_t)$$
$$=\tau-\frac{n_1(n_1-1)(c+3s)}{8}-\sum_{r=n+1}^{2m+s}\sum_{1\le j<k\le n_1}\left(\sigma_{jj}^{r}\sigma_{kk}^{r}-(\sigma_{jk}^{r})^2\right)$$
$$-\frac{n_2(n_2-1)(c+3s)}{8}-\sum_{r=n+1}^{2m+s}\sum_{n_1+1\le u<t\le n}\left(\sigma_{uu}^{r}\sigma_{tt}^{r}-(\sigma_{ut}^{r})^2\right)$$

Combining (3.8) and (3.10) and taking account of (3.4) and (3.6), we have

(3.11) $$n_2\frac{\Delta f_w}{f_w}\le\tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{c+3s}{4}-\frac{\delta}{2} -\sum_{1\le j\le n_1;\,n_1+1\le t\le n}\left(\sigma_{jt}^{n+1}\right)^2-\frac{1}{2}\sum_{r=n+2}^{2m+s}\sum_{\alpha,\beta=1}^{n}\left(\sigma_{\alpha\beta}^{r}\right)^2$$
$$+\sum_{r=n+2}^{2m+s}\sum_{1\le j<k\le n_1}\left((\sigma_{jk}^{r})^2-(\sigma_{jj}^{r}\sigma_{kk}^{r})\right)+\sum_{r=n+2}^{2m+s}\sum_{n_1+1\le u<t\le n}\left((\sigma_{ut}^{r})^2-(\sigma_{uu}^{r}\sigma_{tt}^{r})\right)$$
$$=\tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{(c+3s)}{4}-\frac{\delta}{2} -\sum_{r=n+1}^{2m+s}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}\left(\sigma_{jt}^{r}\right)^2$$

$$-\frac{1}{2}\sum_{r=n+2}^{2m+s}\left(\sum_{j=1}^{n_1}\sigma_{jj}^{r}\right)^2-\frac{1}{2}\sum_{r=n+2}^{2m+s}\left(\sum_{t=n_1+1}^{n}\sigma_{tt}^{r}\right)^2$$

$$\leq \tau-\frac{n(n-1)(c+3s)}{8}+n_1n_2\frac{c+3s}{4}-\frac{\delta}{2}$$

$$=\frac{n^2}{4}\|H\|^2+n_1n_2\frac{(c+3s)}{4},$$

which implies the inequality (3.1).

We see that the equality sign of (3.11) holds if and only if we have

(3.12) $$\sigma_{jt}^{r}=0,\quad 1\leq j\leq n_1,\quad n_1+1\leq t\leq n,\quad n+1\leq r\leq 2m+s,$$

and

(3.13) $$\sum_{i=1}^{n_1}\sigma_{ii}^{r}=\sum_{t=n_1+1}^{n}\sigma_{tt}^{r}=0,\qquad n+2\leq r\leq 2m+s,$$

From (3.12) it follows that the warped product $M_1\times_{f_w}M_2$ is mixed totally geodesic and (6.18) and (6.22) imply $n_1H_1 = n_2H_2$. The converse statement is straightforward.

Corollary 3.2 *Let $M_1\times_{f_w}M_2$ be a warped product whose warping function f is harmonic. Then*:

(i) $M_1\times_{f_w}M_2$ admits no minimal Legendrian immersion into an S-space form $\overline{M}$ (c) with $c<-3s$.

(ii) Every minimal Legendrian immersion of $M_1\times_{f_w}M_2$ in the standard Euclidian space R^{2m+s} is a warped product immersion.

Proof: Let f be a harmonic function on M_1 and $M_1\times_{f_w}M_2$ admits a minimal Legendrian immersion in an S-space form $\overline{M}$ (c). Then from (3.1), we have $c>-3s$.

If $c = -3s$, the equality case of (3.1) holds. By Theorem 3.1 it follows that $M_1\times_{f_w}M_2$ is mixed totally geodesic and $H_1 = H_2 = 0$. A well-known result of Nölker [**45**] implies that immersion is a warped product immersion.

Corollary 3.3. *If the warping function f_w of a warped product $M_1\times_{f_w}M_2$ is an eigen function of the Laplacian on M_1 with corresponding eigen value $\lambda>0$, then $M_1\times_{f_w}M_2$ does not admit a minimal Legendrian immersion in an S-space form $\overline{M}$ (c) with $c<-3s$.*

Proof: If f_w is an eigen function of the Laplacian on M_1 with eigen value $\lambda>0$, the inequality (3.1) implies that

$$\frac{n_1}{4}(c+3s)\geq\lambda>0.$$

CONCLUSION

In this paper I have studies Legendrian warped product submanifolds of S-Space form.

REFERENCES

[1] D.E.Blair, 1970, "Geometry of manifolds with structural group $U(n)\times O(s)$," J.Diff. Geometry 4, pp. 155-167.

[2] B.Y. Chen, 2002, "Geometry of warped products as Riemannian submanifolds and related problems," Soochow J. Math.28, pp. 125-156.

[3] B.Y. Chen, 2002, "On Isometric minimal immersions from warped products in to real space forms," Proc. Edinburgh Math. Soc. 45, pp. 579-587.

[4] J.L.Cabrerizo, L.M.Fernandez, and M. Fernandez, 1993, " The Curvature of Submanifolds of S-space form," Acta. Math. Hunger. 62, pp. 373-383.

[5] A. Carriazo, L.M.Fernandez, and M.B. Hans-Uber, 2005, "Some slant submanifolds of S-manifolds," Acta. Math. Hunger. 107(4), pp. 267-285.

[6] A. Carriazo, L.M.Fernandez, and M.B. Hans-Uber, 2003, " B.Y. Chen's inequality for S-space forms, Application to slant immersions," Indian J. Pure Appl. Math.,34(9), pp. 1287-1298.

[7] Y.H.Kim, and D.W. Yoon, 2004, "Inequality for totally real warped products in locally conformal Kaehler space forms," Kyungpook Math. J. 44, pp. 585-592.

[8] K.Matsumoto, and I.Mihai, 2002, "Warped product submanifolds in Sasakian space forms," Sut Journal of Mathematics, 38, No. 2, pp. 135-144.

[9] S.Nolker, 1996, " Isometric immersions of warped products," Differential Geom. Appl. 6, pp. 1-30.

[10] M.M.Tripathi, and I.Mihai, 2002, "Submanifolds of framed metric manifolds and S-manifolds," Note Di Matematica 20, No. 2, pp. 135-164.

[11] K.Yano, and M.Kon, 1984, " Structures on Manifolds," Word Scientific.

[12] K.Yano, 1963, "On a structure defined by a tensor field of type (1,1) satisfying $f^3+f=0$," Tensor 14, pp. 99-109.