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An Extended Extreme Shock Maintenance Model for a Deteriorating System Under Partial Product Process

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Abstract

In this paper, an extended extreme shock maintenance model for a deteriorating system under partial product process is studied. An explicit expression for the long run average cost per unit time under policy N is derived and an optimal policy N* for minimizing the long run average cost per unit time is determined analytically. A numerical example is also given.

Keywords: Renewal Process, Geometric Process, Partial Product Process, Replacement Policy, Shock Models.

INTRODUCTION

In reliability, the study of maintenance problem is always an important topic. In the research work of repair replacement problems, in the early stages, a common assumption is "repair is perfect", i.e., the system after repair is "as good as new". However, it is not always true for a deteriorating system. In practice, most repairable systems are deteriorating because of the ageing effect and accumulated wear. Most of the maintenance models just pay attention on the internal cause of the system failure, but not on an external cause. A system failure may be caused by some external causes, such as a shock. But shocks with a 'small' level of damage are harmless for the system, while shocks with a `large' level of damage may result in failure of the system. The system deterioration is caused by both the external shocks and the internal load. In the external, the magnitude of the shock damage, the system can bear, is decreasing with respect to the number of repairs taken. In the internal, the consecutive repair time is increasing in the number of repairs taken.

Chen and Li[10] introduced and studied an extended extreme shock maintenance model for a deteriorating system under which the consecutive repair time is geometric process. In this paper, we assume that the consecutive repair time follow an increasing partial product process[12].

We consider the system in two aspects: the internal and the external.

First, if the system is failed by one shock, it is repaired or replaced by a new and identical one. In view of the ageing time and the continuous wear, the repair time will become longer and longer and tend to infinity, i.e., finally the system is non-repairable. Repair times are not negligible. Therefore, we model the repair times after the system failures as an increasing partial product process.

The preliminary definitions and results about partial product process[12] are given below.

Definition 1.1. Let $\{X_n, n=1,2,3,...\}$ be a sequence of non-negative independent random variables and let $F(x)$ be the distribution function of X_1. Then $\{X_n, n=1,2,3,...\}$ is called a partial product process if the distribution function of X_{i+1} is $F(\beta_i x)$ $(i=1,2,3,...)$ where $\beta_i > 0$ are constants and $\beta_i = \beta_0\beta_1\beta_2...\beta_{i-1}$.

Lemma 1.1. For real $\beta_i\,(i=1,2,3,...)$, $\beta_i = \beta_0^{2^{i-1}}$.

Lemma 1.2. Given a partial product process $\{X_n, n=1,2,3,...\}$,

(i) If $\beta_0 > 1$, then $\{X_n, n=1,2,3,...\}$ is stochastically decreasing.

(ii) If $0 < \beta_0 < 1$, then $\{X_n, n=1,2,3,...\}$ is stochastically increasing.

Definition 1.2. A partial product process is called a decreasing partial product process if $\beta_0 > 1$, and is called increasing partial product process if $0 < \beta_0 < 1$.

It is clear that if $\beta_0 = 1$ then the partial product process is a renewal process.

Lemma 1.3. Let $E(X_1) = \lambda, Var(X_1) = \sigma^2$. Then for $i = 1,2,3,...$, $E(X_{i+1}) = \dfrac{\lambda}{\beta_0^{2^{i-1}}}$ and $Var(X_{i+1}) = \dfrac{\sigma^2}{\beta_0^{2^i}}$.

Theorem 1.1. (Wald's Identity) If $X_1, X_2, X_3,...$ are independent and identically distributed random variables having finite expectations, and if N is a stopping time for $X_1, X_2, X_3,...$ such that $E(N) < \infty$, then $E\left(\sum_{n=1}^{N} X_n\right) = E(N)E(X_1)$.

Next, we consider the shocks from the system's environment. A shock is called a deadly shock if the amount of damage of one shock to the system exceeds a specific threshold so that the system will fail. In practice, a deteriorating system after repair should be more weak and easier to be broken down. As a result, the threshold value, which a deadly shock exceeds, will be decreasing in n, the number of repairs taken.

Now we make the following assumptions about the model for a deteriorating system subject to shocks.

MODEL ASSUMPTIONS

A1. Initially a new system is installed. Whenever the system fails, it may be repaired or replaced by a new and identical one.

A2. Once the system is operating, the shocks from the environment arrive according to a renewal process. Let $\{W_{ni}, i=1,2,3,...\}$ be the intervals between the $(i-1)^{st}$ and the i^{th} shock after the $(n-1)^{st}$ repair. Let $E(W_{11})=\lambda$. Assume that $\{W_{ni}, i=1,2,3,...\}$ are identically independent distributed (iid) sequences for all n.

A3. Let $\{D_{ni}, i=1,2,3,...\}$ be the sequence of the amount of shock damage produced by the i^{th} shock after the $(n-1)^{st}$ repair. Let $E(D_{11})=\delta$. Assume that $\{D_{ni}, i=1,2,3,...\}$ are iid sequences for all n.

A4. In the n^{th} operating stage, i.e., after the $(n-1)^{st}$ repair, the system will fail, if the amount of shock damage first exceeds $a^{n-1}M$ where $0<a\le 1$. If the system fails, it is closed, so that the random shocks have no effect on the system during the repair time.

A5. Let Y_1 be the repair time after the 1st failure and let $G(y)$ be the distribution function of Y_1. For $i=1,2,3,...$, let Y_{i+1} be the repair time after the $(i+1)^{st}$ failure. Then the distribution function of Y_{i+1} is assumed to be $G\left(\gamma_0^{2^{i-1}}x\right)$ where $0<\gamma_0\le 1$ is a constant. That is the consecutive repair times $\{Y_n, n=1,2,3,...\}$ form an increasing partial product process or a renewal process. Moreover, assume that $E(Y_1)=\mu\ge 0$. $\mu=0$ means that the repair time is negligible.

A6. Let Z be the replacement time with $E(Z)=\tau$.

A7. The processes $\{W_{ni}, i=1,2,3,...\}$, $\{D_{ni}, i=1,2,3,...\}$, $\{Y_n, n=1,2,3,...\}$ and Z are independent.

A8. The repair cost rate is c, the reward rate is r and the replacement cost is R.

THE REPLACEMENT POLICY N

Definition 3.1. A replacement policy N is a policy in which we replace the system at the N^{th} failure of the system.

Our aim is to find an optimal replacement N^* such that the long-run average cost per unit time is minimized.

By the renewal reward theorem [2], the long-run average cost per unit time under the replacement policy N is given by

$$C(N)=\frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}} \tag{1}$$

For evaluating the expected cost incurred in a cycle and the expected length of a cycle, we need to first calculate the distribution and the expectation of X_n, the real operating time of the system after the $(n-1)^{st}$ repair.

Denote $L_n=\min\left\{l: D_{nl}>a^{n-1}M\right\}$. (2)

i.e. L_n is the number of shocks until the first deadly shock occurred following the $(n-1)^{st}$ failure. Then

$$X_n=\sum_{i=1}^{L_n} W_{ni} \tag{3}$$

and L_n follows a geometric distribution $G(p_n)$, with

$$P(L_n=k)=p_n(1-p_n)^{k-1},\ k=1,2,3,... \tag{4}$$

where $p_n=P\left(D_{n1}>a^{n-1}M\right)$. (5)

By (4), we have

$$E(L_n)=\frac{1}{p_n}. \tag{6}$$

As $\{W_{ni}, i=1,2,3,...\}$ and $\{D_{ni}, i=1,2,3,...\}$ are independent, it is clear that L_n and $\{W_{ni}, i=1,2,3,...\}$ are independent. Thus, from equations (3) and (4) and by Wald's equation, we have

$$\lambda_n=E(X_n)=E(L_n)E(W_{n1})=\frac{\lambda}{p_n}. \tag{7}$$

Since $a\le 1$, from (5) and (7), we can derive that λ_n is decreasing in n.

From Equation (1),

$$C(N)=\frac{E\left(c\sum_{n=1}^{N-1}Y_n+R-r\sum_{n=1}^{N}X_n\right)}{E\left(\sum_{n=1}^{N}X_n+\sum_{n=1}^{N-1}Y_n+Z\right)}$$

$$=\frac{c\left(E(Y_1)+\sum_{n=2}^{N-1}E(Y_n)\right)+R-r\sum_{n=1}^{N}\lambda_n}{\sum_{n=1}^{N}\lambda_n+\left(E(Y_1)+\sum_{n=2}^{N-1}E(Y_n)\right)+\tau}$$

$$=\frac{c\left(\mu+\sum_{n=2}^{N-1}\frac{\mu}{\gamma_0^{2^{n-1}}}\right)+R-r\sum_{n=1}^{N}\lambda_n}{\sum_{n=1}^{N}\lambda_n+\left(\mu+\sum_{n=2}^{N-1}\frac{\mu}{\gamma_0^{2^{n-1}}}\right)+\tau} \tag{8}$$

THE OPTIMAL POLICY N*

In this section, we determine an optimal replacement policy for minimizing $C(N)$.

From (8), we have,

$$C(N)=\frac{(c+r)\mu\left(1+\sum_{n=2}^{N-1}\frac{1}{\gamma_0^{2^{n-1}}}\right)+R+r\tau}{h(N)}-r \tag{9}$$

where $h(N)=\sum_{n=1}^{N}\lambda_n+\mu\left(1+\sum_{n=2}^{N-1}\frac{1}{\gamma_0^{2^{n-1}}}\right)+\tau$.

In order to obtain the optimal policy N^*, we study the difference between $C(N+1)$ and $C(N)$.

$$C(N+1)-C(N)=\frac{\left[\begin{array}{c}(c+r)\mu\left(\begin{array}{c}\sum_{n=1}^{N}\lambda_n-\gamma_0^{2^{N-1}}\lambda_{N+1}+\tau\\-\lambda_{N+1}\gamma_0^{2^{N-1}}\sum_{n=2}^{N-1}\frac{1}{\gamma_0^{2^{n-1}}}\end{array}\right)\\-(R+r\tau)\left(\mu+\lambda_{N+1}\gamma_0^{2^{N-1}}\right)\end{array}\right]}{\gamma_0^{2^{N-1}}h(N)h(N+1)} \tag{10}$$

As the denominator of $C(N+1)-C(N)$ is always positive, it is clear that the sign of $C(N+1)-C(N)$ is the same as the sign of its numerator.

Thus, we introduce the auxiliary function $B(N)$ as follows:

$$B(N)=\frac{(c+r)\mu\left(\begin{array}{c}\sum_{n=1}^{N}\lambda_n-\gamma_0^{2^{N-1}}\lambda_{N+1}+\tau\\-\lambda_{N+1}\gamma_0^{2^{N-1}}\sum_{n=2}^{N-1}\frac{1}{\gamma_0^{2^{n-1}}}\end{array}\right)}{(R+r\tau)\left(\mu+\lambda_{N+1}\gamma_0^{2^{N-1}}\right)} \tag{11}$$

As a result, we have the following lemma.

Lemma 4.1.

$$\left.\begin{array}{l}C(N+1)>C(N)\Leftrightarrow B(N)>1\\C(N+1)=C(N)\Leftrightarrow B(N)=1\\C(N+1)<C(N)\Leftrightarrow B(N)<1\end{array}\right\} \tag{12}$$

Lemma 4.1 shows that the monotonicity of $C(N)$ can be determined by the value of $B(N)$. From equation (11), we have,

$$B(N+1)-B(N)=\frac{(c+r)\mu\gamma_0^{2^{N-1}}\left[\begin{array}{c}\left(\lambda_{N+1}-\lambda_{N+2}\gamma_0^{2^{N-1}}\right)\\\left(\sum_{n=1}^{N+1}\lambda_n+\tau+\mu+\mu\sum_{n=2}^{N}\frac{1}{\gamma_0^{2^{n-1}}}\right)\end{array}\right]}{(R+r\tau)\left(\mu+\lambda_{N+2}\gamma_0^{2^{N}}\right)\left(\mu+\lambda_{N+1}\gamma_0^{2^{N-1}}\right)}$$

This shows that $B(N)$ is non-decreasing in N, because λ_n is decreasing in n and $0<\gamma_0\le 1$.

Therefore, by using Lemma 4.1, we have the following theorem.

Theorem 4.1. The optimal replacement policy N^* is determined by

$$N^*=\min\{N \mid B(N)\ge 1\} \tag{13}$$

Furthermore, the optimal replacement policy N^* is unique if and only if $B(N^*)>1$.

NUMERICAL EXAMPLE

We study a numerical example with the assumption that D_{11} has an exponential distribution with expectation δ.

Then $F(x)=P(D_{11}\le x)=1-e^{-(1/\delta)x}$. From equation (5), we have,

$$p_n=P\left(D_{n1}>a^{n-1}M\right)=e^{-(1/\delta)a^{n-1}M} \tag{14}$$

Then, by Equation (7), we have

$$\lambda_n=E(X_n)=\frac{\lambda}{p_n}=\lambda e^{(1/\delta)a^{n-1}M} \tag{15}$$

Substituting Equation (15) in equations (9) and (11), the explicit expressions for $C(N)$ and $B(N)$ are

$$C(N)=\frac{(c+r)\mu\left[1+\sum_{n=2}^{N-1}\frac{1}{\gamma_0^{2^{n-1}}}\right]+R+r\tau}{\lambda\sum_{n=1}^{N}e^{(1/\delta)a^{n-1}M}+\mu\left[1+\sum_{n=2}^{N-1}\frac{1}{\gamma_0^{2^{n-1}}}\right]+\tau}-r \tag{16}$$

(16)

$$B(N)=\frac{(c+r)\mu\left(\begin{array}{c}\lambda\sum_{n=1}^{N}e^{(1/\delta)a^{n-1}M}-\lambda\gamma_0^{2^{N-1}}e^{(1/\delta)a^{N}M}\\-\lambda e^{(1/\delta)a^{N}M}\gamma_0^{2^{N-1}}\sum_{n=2}^{N-1}\frac{1}{\gamma_0^{2^{n-1}}}+\tau\end{array}\right)}{(R+r\tau)\left(\mu+\lambda e^{(1/\delta)a^{N}M}\gamma_0^{2^{N-1}}\right)} \tag{17}$$

Let $c=6, r=10, \mu=10, R=6000, \lambda=10, M=20, \delta=10,$

$a=0.98, \gamma_0=0.95, \tau=60$.

The numerical results are presented in Table 1 and Table 2 and the corresponding figures are plotted in Figure 1 and Figure 2 respectively.

From the tables, it is clear that equations (16) and (17) agree with $N^*=8$ such that $C(N)$ is minimum at $N^*=8$.

Table 5.1. Values of N and $B(N)$

N	$B(N)$	N	$B(N)$
1	0.0430	7	0.9908
2	0.0485	**8**	**1.3848**
3	0.0612	9	1.5355
4	0.0927	10	1.6641
5	0.1791	11	1.7884
6	0.4329	12	1.9086

Table 5.2. Values of N and $C(N)$

N	$C(N)$	N	$C(N)$
1	39.4254	7	3.2852
2	21.4589	**8**	**3.2536**
3	13.5779	9	5.5648
4	9.1665	10	5.9991
5	6.3692	11	6.0000
6	4.4852	12	6.0000

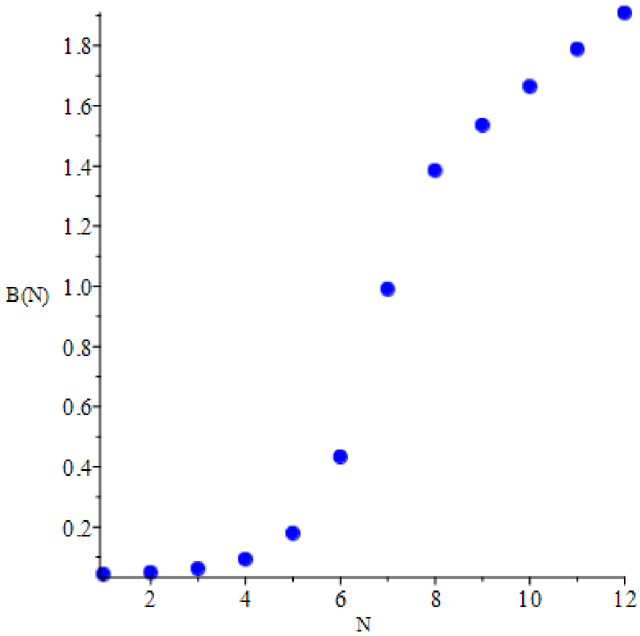

Figure 1: The graph of $B(N)$ against N

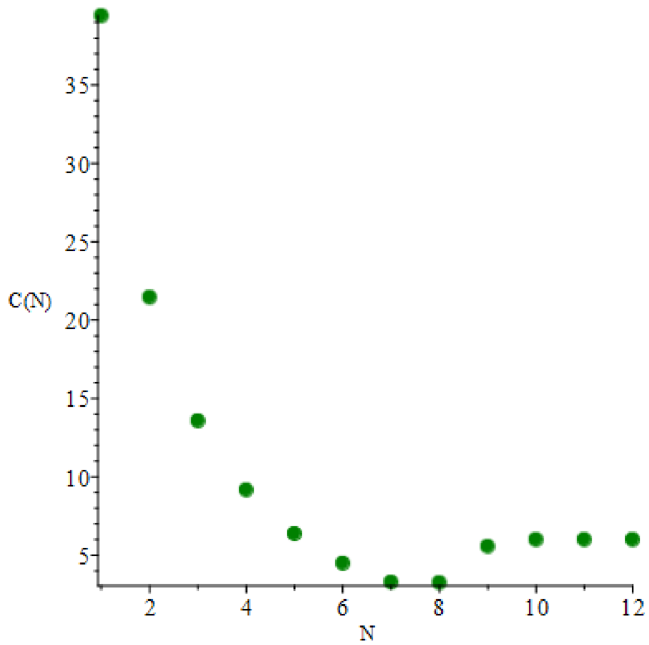

Figure 2: The graph of $C(N)$ against N

CONCLUSION

By considering an extreme shock maintenance model for a deteriorating system, an explicit expression for the long-run average cost per unit time under the replacement policy N using partial product process is derived. An optimal policy $N*$ for minimizing the long run average cost per unit time is determined analytically. A numerical example is given to illustrate the methodology developed.

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